Models of wealth distributions: a perspective

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I. ABSTRACT

A class of conserved models of wealth distributions are studied where wealth (or money) is assumed to be exchanged between a pair of agents in a population just like the elastically colliding molecules of a gas exchanging energy. All sorts of distributions from exponential (Boltzmann-Gibbs) to something like Gamma distributions and to that of Pareto's law (power law) are obtained out of such models with simple algorithmic exchange processes. Numerical investigations, analysis through transition matrix and a mean field approach are employed to understand the generative mechanisms. A general scenario is examined wherefrom a power law and other distributions can emerge.

II. INTRODUCTION

Wealth is accumulated in various forms and factors. The continual exchange of wealth (a value assigned) among the agents in an economy gives rise to interesting and many often universal statistical distributions of individual wealth. Here the word 'wealth' is used in a general sense for the purpose and the spirit of the review (inspite of separate meanings attached to the terms 'money', 'wealth' and 'income'). Econophysics of wealth distributions [1] is an emerging area where mainly the ideas and techniques of statistical physics are used in interpreting real economic data of wealth (available mostly in terms of income) of all kinds of people or other entities (*e.g.*, companies) for that matter, pertaining to different societies and nations. Literature in this area is growing steadily (see an excellent website [2]). The prevalence of income data and apparent interconnections of many socio-economic problems with physics have inspired a wide variety of statistical models, data analysis and other works in econophysics [3], sociophysics and other emerging areas [4] in the last one decade or more (see an enjoyable article by Stauffer [5]).

Simple approach of *agent based models* have been able to bring out all kinds of wealth distributions that open up a whole new way of understanding and interpreting empirical data. One of the most important and controversial issues has been to understand the emergence of *Pareto's law*:

$$P(w) \propto w^{-\alpha}, \tag{1}$$

where $w \geq w_0$, w_0 being some value of wealth beyond which the power law is observed (usually towards the tail of the distributions). Pareto's law has been observed in income distributions among the people of almost all kinds of social systems across the world in a robust way. This phenomenon is now known for more than a century and has been discussed at a great length in innumerable works in economics, econophysics, sociophysics and physics dealing with power law distributions. In many places, while mentioning *Pareto's law*, the power law is often written in the form: $P(w) \propto w^{-(1+\nu)}$, where ν is referred to as 'Pareto index'. This index is usually found between 1 and 2 from empirical data fitting. Power laws in distributions appear in many other cases [6, 7, 8] like that of computer file sizes, the growth of sizes of business firms and cities *etc*. Distributions are often referred to as 'heavy tailed' or 'fat tailed' distributions [9]. Smaller the value of α, fatter the tail of the distribution as it may easily be understood (the distribution is more spread out).

Some early attempts [10] have been made to understand the income distributions which follow Pareto's law at the tail of the distributions. Some of them are stochastic logistic equations or some related generalized versions of that which have been able to generate power laws. However, the absence of direct interactions of one agent with any other often carries less significance in terms of interpreting real data.

Some part of this review is centered around the concept of emergence of Pareto's law in the wealth distributions, especially in the context of the models that are discussed here. However, a word of caution is in order. In the current literature and as well as in the historical occurrences, the power law distribution has often been disputed with a closely related lognormal distribution [6]. It is often not easy to distinguish between the two. Thus a brief discussion is made here on this issue. Let us consider the probability density function of a lognormal distribution:

$$p(w) = \frac{1}{\sqrt{2\pi}\sigma w} \exp[-(\ln w - \overline{w})^2/2\sigma^2], \tag{2}$$

The logarithm of the above can be written as:

$$\ln p(w) = -\ln w - \ln \sqrt{2\pi}\sigma - \frac{(\ln w - \overline{w})^2}{2\sigma^2}. \tag{3}$$

If now the variance σ^2 in the lognormal distribution is large enough, the last term on the right hand side can be very small so that the distribution may appear linear on a log-log plot. Thus the cause of concern remains, particularly when one deals with real data.

In the literature, sometimes one calculates a *cumulative distribution* function (to show the power law in a more convincing way) instead of plotting ordinary distribution from simple histogram (probability density function). The cumulative probability distribution function $P(\geq w)$ is such that the argument has a value greater than or equal to w:

$$P(\geq w) = \int_w^{\infty} P(w')dw'. \tag{4}$$

If the distribution of data follows a power law $P(w) = Cw^{-\alpha}$, then

$$P(\geq w) = C\int_w^{\infty} w'^{-\alpha}dw' = \frac{C}{\alpha-1}w^{-(\alpha-1)}. \tag{5}$$

When the ordinary distribution (found from just histogram and binning) is a power law, the cumulative distribution thus also follows a power law with the exponent 1 less: $\alpha - 1$, which can be seen from a log-log plot of data. An extensive discussion on power laws and related matters can be found in [7].

Besides power laws, a wide variety of wealth distributions from exponential to something like Gamma distributions are all reported in recent literature in econophysics. Exchange of wealth is considered to be a primary mechanism behind all such distributions. In a class of wealth exchange models [11, 12, 13, 14] that follow, the economic activities among agents have been assumed to be analogous to random elastic collisions among molecules as considered in kinetic gas theory in statistical physics. Analogy is drawn between wealth (w) and Energy (E), where the average individual wealth ($\overline{w}$) at equilibrium is equivalent to temperature (T). Wealth (w) is assumed to be exchanged between two randomly selected economic agents like the exchange of energy between a pair of molecules in kinetic gas theory. The interaction is such that one agent wins and the other loses the same amount so that the sum of their wealth remains constant before and after an interaction (trading): $w_i(t+1)+w_j(t+1) = w_i(t)+w_j(t)$; each trading increases time t by one unit. Therefore, it is basically a process of zero sum exchange between a pair of agents; amount won by one agent is equal to the amount lost by another. This way wealth is assumed to be redistributed among a fixed number of agents (N) and the local conservation ensures the total wealth ($W = \sum w_i$) of all the agents to remain conserved.

Random exchange of wealth between a randomly selected pair of agents may be viewed as a *gambling process* (with zero sum exchange) which leads to Boltzmann-Gibbs type exponential distribution in individual wealth ($P(w) \propto \exp(-w/\overline{w})$. However, a little variation in the mode of wealth exchange can lead to a distribution distinctly different from exponential. A number of agent based conserved models [12, 15, 16, 17, 18, 19, 20, 21], invoked in recent times, are essentially variants of a gambling process. A wide variety of distributions evolve out of these models. There has been a renewed interest in such two-agent exchange models in the present scenario while dealing with various problems in social systems involving complex interactions. A good insight can be drawn by looking at the 2×2 transition matrices associated with the process of wealth exchange [22].

In this review, the aim would be to arrive at some understanding of how wealth exchange processes in a simple working way may lead to a variety of distributions within the framework of the conserved models A fixed number of N agents in a system are allowed to interact (trade) stochastically and thus wealth is exchanged between them. The basic steps of such a wealth exchange model are as follows:

$$w_i(t+1) = w_i(t) + \Delta w, \tag{6}$$

$$w_j(t+1) = w_j(t) - \Delta w,$$

where $w_i(t)$ and $w_j(t)$ are wealths of i-th and j-th agents at time t and $w_i(t+1)$ and $w_j(t+1)$ are that at the next time step (t+1). The amount Δw (to be won or to be lost by an agent) is determined by the nature of interaction. If the agents are allowed to interact for a long enough time, a steady state equilibrium distribution for individual wealth is achieved. The equilibrium distribution does not depend on the initial configuration (initial distribution of wealth among the agents). A single interaction between a randomly chosen pair of agents is referred here as one 'time step'. In some simulations, N such interactions are considered as one time step. This, however, does not matter as long as the system is evolved through enough time steps to come to a steady state and then data is collected for making equilibrium probability distributions. For all the numerical results presented here, data have been produced following the available models, conjectures and conclusions. Systems of $N = 1000$ agents have been considered in each case. In each numerical investigation, the system is allowed to equilibrate for a sufficient time that ranges between 10^5 to 10^8 time steps. Configuration averaging has been done over 10^3 to 10^5 initial configurations in most cases. The average wealth (averaged over the agents) is kept fixed at $\overline{w} = 1$ (by taking total wealth, $W = N$) for all the cases. The wealth distributions, that are dealt here in this review, are ordinary distributions (probability density function) and not the cumulative ones.

III. PURE GAMBLING

In a pure gambling process (usual kinetic gas theory), entire sum of wealths of two agents is up for gambling. Some random fraction of this sum is shared by one agent and the rest goes to the other. The randomness or stochasticity is introduced into the model through a parameter ϵ which is a random number drawn from a uniform distribution in [0, 1]. (Note that ϵ is independent of a pair of agents *i.e.*, a pair of agents is not likely to share the same fraction of aggregate wealth the same way when they interact repeatedly). The interaction can be seen through:

$$w_i(t+1) = \epsilon[w_i(t) + w_j(t)], \tag{7}$$

$$w_j(t+1) = (1-\epsilon)[w_i(t) + w_j(t)],$$

where the pair of agents (indicated by i and j) are chosen randomly. The amount of wealth that is exchanged is $\Delta w = \epsilon[w_i(t) + w_j(t)] - w_i(t)$. The individual wealth distribution ($P(w)$ vs. w) at equilibrium emerges out to be Boltzmann-Gibbs distribution like exponential. Exponential distribution of personal income has in fact been shown to appear in real data [13, 14]. In the kinetic theory model, the exponential distribution is found by standard formulation of master equation or by entropy maximization method, the latter has been discussed later in brief in section VIII. A normalized exponential distribution obtained numerically out of this pure gambling process is shown in Fig. 1 in semi logarithmic plot. The high end of the distribution appears noisy due to sampling of data. The successive bins on the right hand side of the graph contain less and less number of samples in them so the fractional counts in them are seen to fluctuate more (finite size effect). One way to get rid of this sampling error in a great extent is by way of taking logarithmic binning [7]. Here it is not important to do so as the idea is to show the nature of the curve only. (In the case of power law distribution, an even better way to demonstrate and extract the power law exponent is to plot the cumulative distribution as discussed already.)

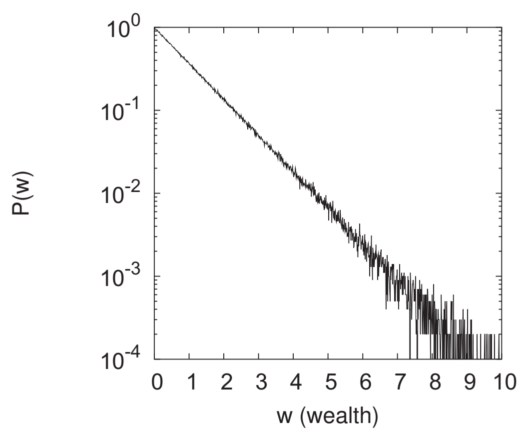

FIG. 1: Distribution of wealth for the case of Pure Gambling: the linearity in the semi-log plot indicates exponential distribution.

If one intends to take time average of wealth of a single agent over a sufficient time, it comes out to be equal for all the agents. Therefore, the distribution of individual *time averaged wealth turns out to be a delta function* which is checked from numerical data. This is because the fluctuation of wealth of any agent over time is statistically no different from that of any other. The same is true in case of the distribution of wealth of a single agent over time. However, when the average of wealth of any agent is calculated over a short time period, the delta function broadens and the right end part of which decays exponentially. The distribution of individual wealth at a certain time turns out to be purely exponential as mentioned earlier. This may be thought of as a 'snap shot' distribution.

IV. UNIFORM SAVING PROPENSITY

Instead of random sharing of their aggregate wealth during each interaction, if the agents decide to save (keep aside) a uniform (and fixed) fraction (λ) of their current individual wealth, then the wealth exchange equations look like the following:

$$w_i(t+1) = \lambda w_i(t) + \epsilon(1-\lambda)[w_i(t) + w_j(t)], \tag{8}$$

$$w_j(t+1) = \lambda w_j(t) + (1-\epsilon)(1-\lambda)[w_i(t) + w_j(t)],$$

where the amount of wealth that is exchanged is $\Delta w = (\epsilon - 1)(1-\lambda)[w_i(t) + w_j(t)]$. The concept of saving as introduced by Chakrabarti and group [12] in an otherwise gambling kind of interactions brings out distinctly different distributions. A number of numerical works followed [23, 24, 25] in order to understand the emerging distributions to some extent. Saving induces accumulation of wealth. Therefore, it is expected that the probability of finding agents with zero wealth may be zero unlike in the previous case of pure gambling where due to the unguarded nature of exchange many agents are likely to go nearly bankrupt! (It is to be noted that for an exponential distribution, the peak is at zero.) In this case the most probable value of the distribution (peak) is somewhere else than at zero (the distribution is right skewed). The right end, however, decays exponentially for large values of w. It has been claimed through heuristic arguments (based on numerical results) that the distribution is a close approximate form of the Gamma distribution [23]:

$$P(w) = \frac{n^n}{\Gamma(n)} w^{n-1} e^{-nw}, \tag{9}$$

where the Gamma function $\Gamma(n)$ and the index n are understood to be related to the saving propensity parameter λ through the following relation:

$$n = 1 + \frac{3\lambda}{1-\lambda}. \tag{10}$$

The emergence of probable Gamma distribution is also subsequently supported through numerical results in [24].

However, it has later been shown in [26] by considering moments' equation that moments up to third order agree with that obtained from the above form of distribution subject to the condition stated in eqn. (10). Discrepancies start showing only from 4th order onwards. Therefore, the actual form of distribution still remains an open question.

In Fig. 2, two distributions are shown for two different values of saving propensity factor: $\lambda = 0.4$ and $\lambda = 0.8$. Smaller the value of λ, lesser the amount one is able to save. This in turn means more wealth is available in the market for gambling. In the limit of zero saving ($\lambda = 0$) the model reduces to that of pure gambling. In the opposite extent of large saving, only a small amount of wealth is up for gambling. Then the exchange of wealth will not be able to drastically change the amount of individual wealth. This means the width of distribution of individual wealth will be narrow. In the limit of $\lambda = 1$, all the agents save all of their wealth and thus the distribution never evolves. The concept of 'saving' here is of course a little different from that in real life where people do save some amount to be kept in a bank or so and the investment (or business or gambling) is done generally not with the entire amount (or a fraction) of wealth that one holds at a time.

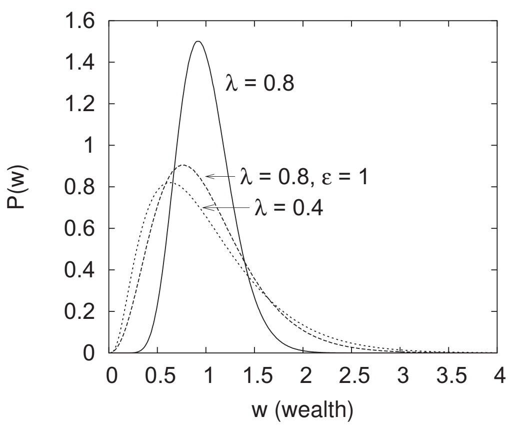

FIG. 2: Wealth distribution for the model of uniform and fixed saving propensity. Two distributions are shown with $\lambda = 0.4$ and $\lambda = 0.8$ where the stochasticity parameter ϵ is drawn randomly and uniformly in [0, 1]. Another distribution is plotted with $\lambda = 0.8$ but with fixed value of the stochasticity parameter, $\epsilon = 1$.

Stochastic evolution of individual wealth is also examined without the inclusion of stochastic parameter ϵ. The stochasticity seems to be automatically introduced anyhow through the random selection of a pair of agents (and the random choice of the winner or loser as well) at each time. Therefore, it is interesting to see how the distributions evolve with a fixed value of ϵ. As an example, the equations in (8) reduce to the following with $\epsilon = 1$:

$$w_i(t+1) = w_i(t) + (1-\lambda)w_j(t), \qquad (11)$$

$$w_j(t+1) = \lambda w_j(t).$$

The above equations indicate that the randomly selected agent j keeps (saves) an amount $\lambda w_j(t)$ which is proportional to the wealth he currently has and transfers the rest to the other agent i. This is indeed a stochastic process and is able to produce Gamma type distributions in wealth as observed. However, a distribution with random ϵ and that with a fixed ϵ are different. Numerically, it has been observed, the distribution with $\lambda = 0.8$ and with $\epsilon = 1$ is very close to that with $\lambda = 0.5$ and with random ϵ. In Fig. 2 the distribution with fixed $\lambda = 0.8$ and fixed $\epsilon = 1$ is plotted along with other two distributions with random ϵ. It should also be noted that while with fixed ϵ, one does not get Gamma type distributions for all values of λ; especially for low values of λ the distributions become close to exponential as observed. This is not clearly understood though.

It has recently been brought to notice in [27] that a very similar kind of agent based model was proposed by Angle [28] (see other references cited in [27]) in sociological journals quite some years ago. The pair of equations in Angle's model are as follows:

$$w_i(t+1) = w_i(t) + D_t\omega w_j(t) - (1-D_t)\omega w_i(t), \qquad (12)$$

$$w_j(t+1) = w_j(t) + (1-D_t)\omega w_i(t) - D_t\omega w_j(t),$$

where ω is a fixed fraction and the winner is decided through a random toss D_t which takes a value either 0 or 1. Now, the above can be seen as the more formal way of writing the pair of equations (11) which can be arrived at by choosing $D_t = 1$ and identifying $\omega = (1-\lambda)$.

It can in general be said, within the framework of this kind of (conserved) models, different ways of incorporating wealth exchange processes may lead to drastically different distributions. If the gamble is played in a *biased way*, then this may lead to a distinctly different situation than the case when it is played in a normal unbiased manner. Since in this class of models negative wealth or debt is not allowed, it is desirable that in each wealth exchange, the maximum that any agent may invest is the amount that he has at that time. Suppose, the norm is set for an 'equal amount invest' where the amount to be deposited by an agent for gambling is decided by the amount the poorer agent can afford and consequently the same amount is agreed to be deposited by the richer agent. Let us suppose $w_i > w_j$. Now, the poorer agent (j) may invest a certain fraction of his wealth, an amount λw_j and the rest $(1-\lambda)w_j$ is saved by him. Then the total amount $2\lambda w_j$ is up for gambling and as usual a fraction of this, $2\epsilon\lambda w_j$ may be shared by the richer agent i where the rest $(1-\epsilon)\lambda w_j$ goes to the poorer agent j. This may appear fine, however, this leads to 'rich gets richer and poor gets poorer' way. The richer agent draws more and more wealth in his favour in the successive encounters and the poorer agents are only able to save less and less and finally there is a condensation of wealth at the hands of the richest person. This is more apparent when one considers an agent with $\lambda = 1$ where it can be easily checked that the richer agent automatically saves an amount equal to the difference of their wealth $(w_i - w_j)$ and the poorer agent ends up saving zero amount. Eventually, poorer agents get extinct. This is 'minimum exchange model' [21].

V. DISTRIBUTED SAVING PROPENSITY

The distributions emerge out to be dramatically different when the saving propensity factor (λ) is drawn from a uniform and random distribution in [0,1] as introduced in a model proposed by Chatterjee, Chakrabarti and Manna [15]. Randomness in λ is assumed to be quenched (*i.e.*, remains unaltered in time). Agents are indeed heterogeneous. They are likely to have different (characteristic) saving propensities. The pair of wealth exchange equations are now written as:

$$w_i(t+1) = \lambda_i w_i(t) + \epsilon[(1-\lambda_i)w_i(t) + (1-\lambda_j)w_j(t))], \quad (13)$$

$$w_j(t+1) = \lambda_j w_j(t) + (1-\epsilon)[(1-\lambda_i)w_i(t) + (1-\lambda_j)w_j(t))].$$

A power law with exponent $\alpha = 2$ (Pareto index $\nu = 1$) is observed at the right end of the wealth distribution for several decades. Such a distribution is plotted in Fig. 3 where a straight line is drawn in the log-log plot with slope = -2 to illustrate the power law and the exponent. Extensive numerical results with different distributions in the saving propensity parameter λ are reported in [18]. Power law (with exponent $\alpha = 2$) is found to be robust. The value of Pareto index obtained here ($\nu = 1$), however, differs from what is generally extracted (1.5 or above) from most of the empirical data of income distributions (see discussions and analysis on real data by various authors in [1]). The present model is not able to resolve this discrepancy and it is not expected at the outset either. introducing random waiting time in the interactions of agents in order to have a justification for a larger value of the exponent ν [29].

The distributed saving gives rise to an additional interesting feature when a special case is considered where the saving parameter λ is assumed to have only two fixed values, λ_1 and λ_2 (preferably widely separated). A bimodal distribution in individual wealth results in [22]. This can be seen from the Fig. 4. The system evolves towards a robust and distinct two-peak distribution as the difference in λ_1 and λ_2 is increased systematically. Later it is seen that one still gets a two-peak distribution even when λ_1 and λ_2 are drawn from narrow distributions centered around two widely separated values (one large and one small). Two economic classes seem to persist until the distributions in λ_1 and λ_2 have got sufficient widths. A population can be imagined to have two distinctly different kinds of people: some of them tend to save a very large fraction (fixed) of their wealth and the others tend to save a relatively small fraction (fixed) of their wealth. Bimodal distributions (and a polymodal distribution, in general) are, in fact, reported with real data for the income distributions in Argentina [30]. The distributions were derived at a time of political crisis and thus they may not be regarded as truly equilibrium distributions though. However, it remains an interesting possibility out of a simple model of wealth exchange.

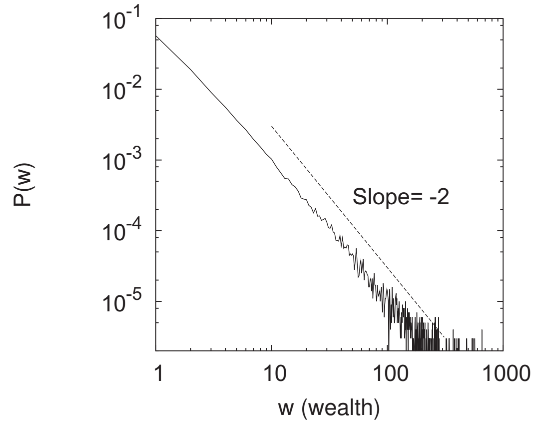

FIG. 3: Wealth distribution for the model of random saving propensity plotted in log-log scale. A straight line with slope = -2 is drawn to demonstrate that the power law exponent is $\alpha = 2$.

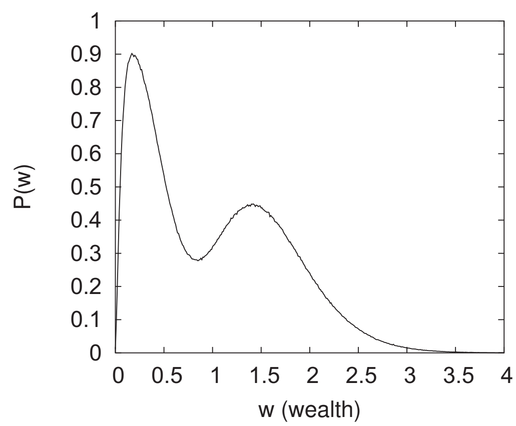

FIG. 4: Bimodal distribution of wealth (w) with fixed values of saving propensities, λ_1=0.2 and λ_2=0.8. Emergence of two economic classes are apparent.

A. Power law from mean field analysis

One can have an estimate of ensemble averaged value of wealth [31] using one of the above equations (13) in section V. Emergence of a power law in the wealth distribution can be established through a simple consideration as follows. Taking ensemble average of all the terms on both sides of the first eqn. (13), one may write:

$$\langle w_i\rangle = \lambda_i\langle w_i\rangle+\langle\epsilon\rangle[(1-\lambda_i)\langle w_i\rangle+\langle\frac{1}{N}\sum_{j=1}^{N}(1-\lambda_j)w_j\rangle] \quad (14)$$

The last term on the right hand side is replaced by the average over agents where it is assumed that any agent (here the i-th agent), on an average, interacts with all other agents of the system, allowing sufficient time to interact. This is basically a *mean field approach*. If ϵ is assumed to be distributed randomly and uniformly between 0 and 1 then $\langle\epsilon\rangle = \frac{1}{2}$. Wealth of each individual keeps on changing due to interactions (or wealth exchange processes that take place in a society). No matter what the personal wealth one begins with, the time evolution of wealth of an individual agent at the steady state is independent of that initial value. This means the distribution of wealth of a single agent over time is stationary. Therefore, the time averaged value of wealth of any agent remains unchanged whatever the amount of wealth one starts with. In course of time, an agent interacts with all other agents (presumably repeatedly) given sufficient time. One can thus think of a number of ensembles (configurations) and focus attention on a particular tagged agent who eventually tends to interact with all other agents in different ensembles. Thus the time averaged value of wealth is equal to the ensemble averaged value in the steady state.

Now if one writes

$$\langle\overline{(1-\lambda)w}\rangle \equiv \langle\frac{1}{N}\sum_{j=1}^{N}(1-\lambda_j)w_j\rangle, \quad (15)$$

the above equation (14) reduces to:

$$(1-\lambda_i)\langle w_i\rangle = \langle\overline{(1-\lambda)w}\rangle), \quad (16)$$

The right hand side of the above equation is independent of any agent-index and the left hand side is referred to any arbitrarily chosen agent i. Thus, it can be argued that the above relation can be true for any agent (for any value of the index i) and so it can be equated to a constant. Let us now recognize $C = \langle\overline{(1-\lambda)w}\rangle$, a constant which is found by averaging over all the agents in the system and which is further averaged over ensembles. Therefore, one arrives at a unique relation for this model:

$$w = \frac{C}{(1-\lambda)}, \quad (17)$$

where one can get rid of the index i and may write $\langle w_i\rangle = w$ for brevity. The above relation is also verified numerically which is obtained by many authors in their numerical simulations and scaling of data [18, 24]. One can now derive $dw = \frac{w^2}{C}d\lambda$ from the above relation (17). An agent with a (characteristic) saving propensity factor (λ) ends up with wealth (w) such that one can in general relate the distributions of the two:

$$P(w)dw = g(\lambda)d\lambda. \quad (18)$$

If now the distribution in λ is considered to be uniform then $g(\lambda) =$ constant. Therefore, the distribution in w is bound to be of the form:

$$p(w) \propto \frac{1}{w^2}. \quad (19)$$

This may be regarded as Pareto's law with index $\alpha = 2$ which is already numerically verified for this present model. The same result is also obtained recently in [32] where the treatment is argued to be exact.

B. Power law from reduced situation

From numerical investigations, it seems that the stochasticity parameter ϵ is irrelevant as long as the saving propensity parameter λ is made random. It has been tested that the model is still able to produce power law (with the same exponent, $\alpha = 2$) for any fixed value of ϵ. As an example, the case for $\epsilon = 1$ is considered. The pair of wealth exchange equations (refakg:eqn:ransave) now reduce to the following:

$$w_i(t+1) = w_i(t)+(1-\lambda_j)w_j(t) = w_i(t)+\eta_j w_j(t), \quad (20)$$

$$w_j(t+1) = w_j(t) - (1-\lambda_j)w_j(t) = (1-\eta_j)w_j(t).$$

The exchange amount, $\Delta w = (1-\lambda_j)w_j(t) = \eta_j w_j(t)$ is now regulated by the parameter $\eta = (1-\lambda)$ only. If λ is drawn from a uniform and random distribution in [0, 1], then η is also uniformly and randomly distributed in [0, 1]. *To achieve a power law in the wealth distribution it seems essential that randomness in η has to be quenched.* For 'annealed' type disorder (*i.e.*, when the distribution in η varies with time) the power law gets washed away (which is observed through numerical simulations). It has also been observed that power law can be obtained when η is uniformly distributed between 0 and some value less than or equal to 1. As an example, η is taken in the range between 0 and 0.5, a power law is obtained with the exponent around $\alpha = 2$. However, when η is taken in the range $0.5 < \eta < 1$, the distribution clearly deviates from power law which is evident from the log-log plot in Fig. 5. *Thus there seems to be a crossover from power law to some distribution with exponentially decaying tail as one tunes the range in the quenched parameter η.*

At this point, *two important criteria may be identified for achieving power law* within this reduced situation:

- The disorder in the controlling parameter η has to be quenched (fixed set of η's for a configuration of agents),
- It is required that η, when drawn from a uniform distribution, the lower bound of that should be 0.

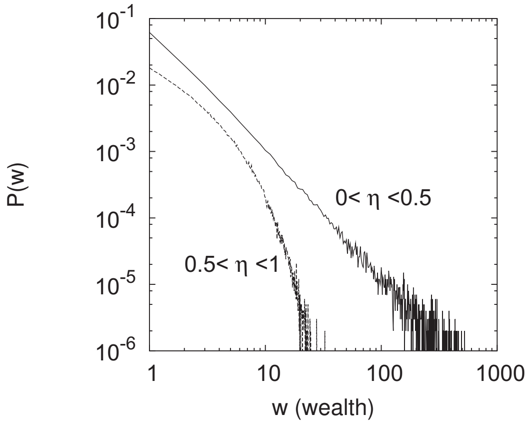

FIG. 5: Wealth distributions (plotted in the log-log scale) for two cases of the 'reduced situation': (i) $0 < \eta < 0.5$ and (ii) $0.5 < \eta < 1$ plotted in log-log scale. In one case, the distribution follows a power law (with exponent around $\alpha = 2$) and in the other case, it is seen to be clearly deviating from a power law.

The above criteria may appear ad hoc, nevertheless have been checked by extensive numerical investigations. It is further checked that the power law exponent does not depend on the width of the distribution in η (as long as it is between 0 and something less than 1). This claim is substantiated by taking various ranges of η in which it is uniformly distributed. Systematic investigations are made for the cases where η is drawn in [0, 0.2], [0, 0.4], . . . ,[0, 1]. Power laws result in in all the cases with the exponent around $\alpha = 2$.

VI. UNDERSTANDING THROUGH TRANSITION MATRIX

The evolution of wealth in the kind of two-agent wealth exchange process can be described through the following 2×2 transition matrix (T) [22]:

$$\begin{pmatrix} w_i' \\ w_j' \end{pmatrix} = T \begin{pmatrix} w_i \\ w_j \end{pmatrix},$$

where it is written, $w_i' \equiv w_i(t+1)$ and $w_i \equiv w_i(t)$ and so on. The transition matrix (T) corresponding to *pure gambling* process (in section III) can be written as:

$$T = \begin{pmatrix} \epsilon & \epsilon \\ 1-\epsilon & 1-\epsilon \end{pmatrix}.$$

In this case the above matrix is *singular* (determinant, $|T| = 0$) which means the inverse of this matrix does not exit. This in turn indicates that an evolution through such transition matrices is bound to be *irreversible*. This property is connected to the emergence of exponential (Boltzmann-Gibbs) wealth distribution. The same may be perceived in a different way too. When a product of such matrices (for successive interactions) are taken, the left most matrix (of the product) itself returns:

$$\begin{pmatrix} \epsilon & \epsilon \\ 1-\epsilon & 1-\epsilon \end{pmatrix} \begin{pmatrix} \epsilon_1 & \epsilon_1 \\ 1-\epsilon_1 & 1-\epsilon_1 \end{pmatrix} = \begin{pmatrix} \epsilon & \epsilon \\ 1-\epsilon & 1-\epsilon \end{pmatrix}.$$

The above signifies the fact that during the repeated interactions of the same two agents (via this kind of transition matrices), the last of the interactions is what matters (the last matrix of the product survives) $[T^{(n)}.T^{(n-1)}\ldots T^{(2)}.T^{(1)} = T^{(n)}]$. This 'loss of memory' (random history of collisions in case of molecules) may be attributed here to the path to irreversibility in time.

The singularity can be avoided if one considers the following general form:

$$T_1 = \begin{pmatrix} \epsilon_1 & \epsilon_2 \\ 1-\epsilon_1 & 1-\epsilon_2 \end{pmatrix},$$

where ϵ_1 and ϵ_2 are two different random numbers drawn uniformly from [0, 1] (This ensures the transition matrix to be nonsingular.). The significance of this general form can be seen through the wealth exchange equations in the following way: the ϵ_1 fraction of wealth of the 1st agent (i) added with ϵ_2 fraction of wealth of the 2nd agent (j) is retained by the 1st agent after the trade. The rest of their total wealth is shared by the 2nd agent. This may happen in a number of ways which can be related to the detail considerations of a model. The general matrix T_1 is nonsingular as long as $\epsilon_1 \neq \epsilon_2$ and then the two-agent interaction process remains reversible in time. Therefore, it is expected to have a steady state equilibrium distribution of wealth which may deviate from exponential distribution (as in the case with pure gambling model). When one considers $\epsilon_1 = \epsilon_2$, one again gets back the pure exponential distribution. A trivial case is obtained for $\epsilon_1 = 1$ and $\epsilon_2 = 0$. The transition matrix then reduces to the identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which trivially corresponds to no interaction and no evolution.

It may be emphasized that any transition matrix $\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$, for such conserved models is bound to be of the form such that the sum of two elements of either of the columns has to be *unity by design*: $t_{11} + t_{21} = 1$, $t_{12} + t_{22} = 1$. It is important to note that whatever extra parameter, no matter, one incorporates within the framework of the conserved model, the transition matrix has to retain this property.

In Fig. 6 three distributions (with $\epsilon_1 \neq \epsilon_2$) are plotted where ϵ_1 and ϵ_2 are drawn randomly from uniform distributions with different ranges. It is demonstrated that qualitatively different distributions are possible as the parameter ranges are tuned appropriately.

Now let us compare the above situation with the model of equal saving propensity as discussed in section IV.

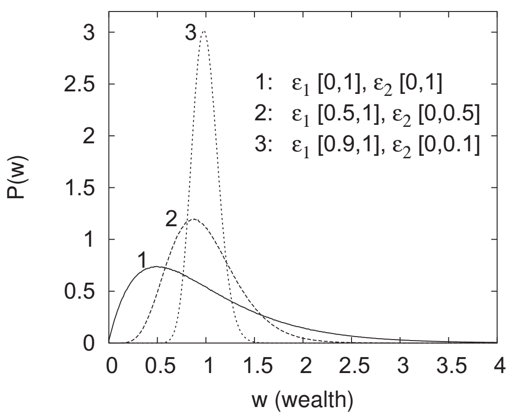

FIG. 6: Three normalized wealth distributions are shown corresponding to the general matrix T_2 (in general with $\epsilon_1 \neq \epsilon_2$) as discussed in the text. Curves are marked by numbers (1, 2 and 3) and the ranges of ϵ_1 and ϵ_2 are indicated within which they are drawn uniformly and randomly.

With the incorporation of saving propensity factor λ, the transition matrix now looks like:

$$\begin{pmatrix} \lambda + \epsilon(1-\lambda) & \epsilon(1-\lambda) \\ (1-\epsilon)(1-\lambda) & \lambda + (1-\epsilon)(1-\lambda) \end{pmatrix}.$$

The matrix elements can now be rescaled by assuming $\tilde{\epsilon}_1 = \lambda + \epsilon(1-\lambda)$ and $\tilde{\epsilon}_2 = \epsilon(1-\lambda)$ in the above matrix. Therefore, the above transition matrix reduces to

$$T_2 = \begin{pmatrix} \tilde{\epsilon}_1 & \tilde{\epsilon}_2 \\ 1-\tilde{\epsilon}_1 & 1-\tilde{\epsilon}_2 \end{pmatrix}.$$

Thus the matrix T_2 is of the same form as T_1. The distributions due to above two matrices of the same general form can now be compared if one can correctly identify the ranges of the rescaled elements. In the model of uniform saving: $\lambda < \tilde{\epsilon}_1 < 1$ and $0 < \tilde{\epsilon}_2 < (1-\lambda)$ as the stochasticity parameter ϵ is is drawn from a uniform and random distribution in [0, 1]. As long as $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$ are different, the determinant of the matrix is nonzero ($|T_2| = \tilde{\epsilon}_1 - \tilde{\epsilon}_2 = \lambda$). Therefore, the incorporation of the saving propensity factor λ brings *two effects*:

- The transition matrix becomes nonsingular,
- The matrix elements t_{11} $(= \tilde{\epsilon}_1)$ and t_{12} $(= \tilde{\epsilon}_2)$ are now drawn from truncated domains (somewhere in [0, 1]).

Hence it is clear from the above discussion that the wealth distribution with uniform saving is likely to be qualitatively no different from what can be achieved with general transition matrices having different elements, $\epsilon_1 \neq \epsilon_2$. The distributions obtained with different λ may correspond to that with appropriately chosen ϵ_1 and ϵ_2 in T_1.

In the next stage, when the saving propensity factor λ is distributed as in section V, the transition matrix between any two agents having different λ's (say, λ_1 and λ_2) now looks like:

$$\begin{pmatrix} \lambda_1 + \epsilon(1-\lambda_1) & \epsilon(1-\lambda_2) \\ (1-\epsilon)(1-\lambda_1) & \lambda_2 + (1-\epsilon)(1-\lambda_2) \end{pmatrix}.$$

Again the elements of the above matrix can be rescaled by putting $\tilde{\epsilon}_1' = \lambda_1 + \epsilon(1-\lambda_1)$ and $\tilde{\epsilon}_2' = \epsilon(1-\lambda_2)$. Hence the transition matrix can again be reduced to the same form as that of T_1 or T_2:

$$T_3 = \begin{pmatrix} \tilde{\epsilon}_1' & \tilde{\epsilon}_2' \\ 1-\tilde{\epsilon}_1' & 1-\tilde{\epsilon}_2' \end{pmatrix}.$$

The determinant here is $|T_3| = \tilde{\epsilon}_1' - \tilde{\epsilon}_2' = \lambda_1(1-\epsilon) + \epsilon\lambda_2$. Here also the determinant is ensured to be nonzero as all the parameters ϵ, λ_1 and λ_2 are drawn from the same positive domain: [0, 1]. This means that each transition matrix for two-agent wealth exchange remains nonsingular which ensures the interaction process to be reversible in time. Therefore, it is expected that *qualitatively different distributions are possible when one appropriately tunes the two independent elements in the general form of transition matrix* (T_1 or T_2 or T_3). However, the emergence of power law tail (*Pareto's law*) in the distribution can not be explained by this. Later it is examined that to obtain a power law in the framework of present models, it is essential that the distribution in λ has to be quenched (frozen in time) which means the matrix elements in the general form of any transition matrix have to be quenched. In the section V B, it has been shown that the model of distributed saving (section V) is equivalent to a reduced situation where one needs only one variable η. The this case the corresponding transition matrix looks even simpler:

$$T_4 = \begin{pmatrix} 1 & \eta \\ 0 & 1-\eta \end{pmatrix},$$

where a nonzero determinant ($|T_4| = 1-\eta \neq 0$) is ensured among other things.

A. Distributions from generic situation

From all the previous discussions, it is clear that the the transition matrix (for zero sum wealth exchange) is bound to be of the following general form:

$$\begin{pmatrix} \epsilon_1 & \epsilon_2 \\ 1-\epsilon_1 & 1-\epsilon_2 \end{pmatrix}.$$

The matrix elements, ϵ_1 and ϵ_2 can be appropriately associated with the relevant parameters in a model. A generic situation arrives where one can generate all sorts of distributions by controlling ϵ_1 and ϵ_2.

As long as $\epsilon_1 \neq \epsilon_2$, the matrix remains nonsingular and one achieves Gamma type distributions. In a special case, when $\epsilon_1 = \epsilon_2$, the transition matrix becomes singular and Boltzmann-Gibbs type exponential distribution results in. It has been numerically checked that a power law with exponent $\alpha = 2$ is obtained with the general matrix when the elements ϵ_1 and ϵ_2 are of the same set of quenched random numbers drawn uniformly in [0, 1]. The matrix corresponding to the reduced situation in the section V B, as discussed, is just a special case with $\epsilon_1 = 1$ and $\epsilon_2 = \eta$, drawn from a uniform and (quenched) random distribution. Incorporation of any parameter in an actual model (saving propensity, for example) results in the adjustment or truncation of the full domain [0, 1] from which the element ϵ_1 or ϵ_2 is drawn. Incorporating distributed λ's in section V is equivalent to considering the following domains: $\lambda_1 < \epsilon_1 < 1$ and $0 < \epsilon_2 < (1-\lambda_2)$.

A more general situation arrives when the matrix elements ϵ_1 and ϵ_2 are of two sets of random numbers drawn separately (one may identify them as $\epsilon_1^{(1)}$ and $\epsilon_2^{(2)}$ to distinguish) from two uniform and random distributions in the domain: [0, 1]. In this case a power law is obtained with the exponent $\alpha = 3$ which is, however, distinctly different from that obtained in 'distributed saving model' in section V. To test the robustness of the power law, the distributions in the matrix elements are taken in the following truncated ranges: $0.5 < \epsilon_1 < 1$ and $0 < \epsilon_2 < 0.5$ (widths are narrowed down). A power law is still obtained with the same exponent (α close to 3). These results are plotted in Fig. 7.

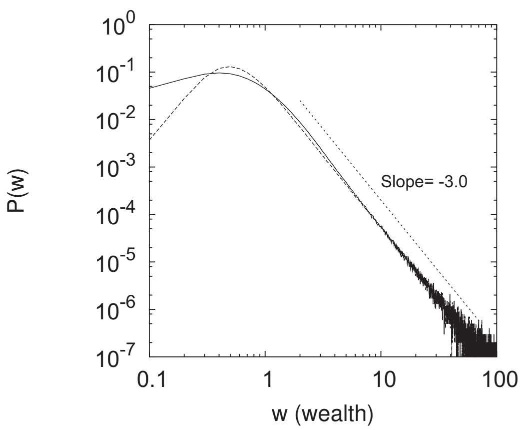

FIG. 7: Distribution of individual wealth (w) for the most general case with random and quenched ϵ_1 and ϵ_2: The elements are drawn from two separate distributions where $0 < \epsilon_1 < 1$ and $0 < \epsilon_2 < 1$ in one case and in the other case, they are chosen from the ranges, $0.5 < \epsilon_1 < 1$ and $0 < \epsilon_2 < 0.5$. Both show power laws with the same exponent around 3.0 (the two distributions almost superpose). A straight line (with slope -3.0) is drawn to demonstrate the power law in the log-log scale.

It is possible to achieve distributions other than power laws as one draws the matrix elements, ϵ_1 and ϵ_2 from different domains within the range between 0 and 1. There is indeed a *crossover from power law to Gamma like distributions* as one tunes the elements. It appears from extensive numerical simulations that power law disappears when both the parameters are drawn from some ranges that do not include the lower limit 0. For example, when it is considered, $0.8 < \epsilon_1 < 1.0$ and $0.2 < \epsilon_2 < 0.4$, the wealth distribution does not follow a power law. In contrast, when ϵ_1 and ϵ_2 are drawn from the ranges, $0.8 < \epsilon_1 < 1.0$ and $0 < \epsilon_2 < 0.1$, the power law distribution is back again.

It now appears that *to achieve a power law in such a generic situation, the following criteria are to be fulfilled*:

- It is essential to have the randomness or disorder in the elements ϵ_1 and ϵ_2 to be quenched,
- In the most general case, ϵ_1 should be drawn from a uniform distribution whose upper bound has to be 1 and for ϵ_2 the lower bound has to be 0. Then a power law with higher exponent $\alpha = 3$ is achieved. To have a power law with exponent $\alpha = 2$, the matrix elements are to be drawn from the same distribution. (These choices automatically make the transition matrices to be nonsingular.)

The above points are not supported analytically at this stage. However, the observation seems to bear important implications in terms of generation of power law distributions.

When the disorder or randomness in the elements ϵ_1 and ϵ_2 change with time (*i.e.*, not quenched) unlike the situation just discussed above, the problem is perhaps similar to the mass diffusion and aggregation model by Majumdar, Krishnamurthy and Barma [33]. The mass model is defined on a one dimensional lattice with periodic boundary condition. A fraction of mass from a randomly chosen site is assumed to be continually transported to any of its neighbouring sites at random. The total mass between the two sites then is unchanged (one site gains mass and the other loses the same amount) and thus the total mass of the system remains conserved. The mass of each site evolves as

$$m_i(t+1) = (1-\eta_i)m_i(t) + \eta_j m_j(t). \tag{21}$$

Here it is assumed that η_i fraction of mass m_i is dissociated from that site i and joins either of its neighbouring sites $j = i \pm 1$. Thus $(1-\eta_i)$ fraction of mass m_i remains at that site whereas a fraction η_j of mass m_j from the neighbouring site joins the mass at site i. Now if we identify $\epsilon_1 = (1-\eta_i)$ and $\epsilon_2 = \eta_j$ then this model is just the same as described by the general transition matrix as discussed so far. If η_i's are drawn from a random and uniform distribution in [0, 1] then a mean field calculation (which turns out to be exact in the thermodynamic limit), as shown in [33], brings out the stationary mass distribution $P(m)$ to be a Gamma distribution:

$$P(m) = \frac{4m}{\overline{m}^2} e^{-2m/\overline{m}}, \tag{22}$$

where $\overline{m}$ is the average mass of the system. It has been numerically checked that there seems to be no appreciable change in the distribution even when the lattice is not considered. Lattice seems to play no significant role in the case of kinetic theory like wealth distribution models as well. Incidentally, this distribution with $\overline{m} = 1$ is exactly the same as the Gamma distribution [eqn. (9)], mentioned in section IV when one considers $n = 2$ in that. The index n equals to 2 when one puts $\lambda = \frac{1}{4}$ in the relation (10).

In the general situation ($\epsilon_1 \neq \epsilon_2$), when both the parameters are drawn from a random and uniform distribution in [0, 1], the emerging distribution very nearly follows the above expression (22). Only when the randomness in them is quenched (fixed in time), there is a possibility of getting a power law as it is already mentioned. The Gamma distribution [eqn. (22)] and the numerically obtained distributions for different cases (as discussed in the text) are plotted in Fig. 8 in support of the above discussions.

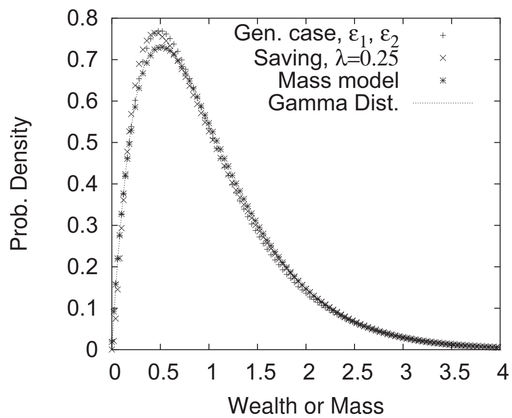

FIG. 8: Normalized probability distribution functions obtained for three different cases: (i) Wealth distribution with random and uniform ϵ_1 and ϵ_2 in [0, 1], (ii) Wealth distribution with uniform and fixed saving propensity, $\lambda = \frac{1}{4}$, (iii) Mass distribution for the model [33] in one dimensional lattice (as discussed in text). The theoretical Gamma distribution [the eqn. (22)] is also plotted (line draw) to have a comparison.

VII. ROLE OF SELECTIVE INTERACTION

So far the models of wealth exchange processes have been considered where a pair of agents is selected randomly. However, interactions or trade among agents in a society are often guided by personal choice or some social norms or some other reasons. Agents may like to interact selectively and it would be interesting to see how the Individual wealth distribution is influenced by selection [19]. The concept of selective interaction is already there when one considers the formation of a family. The members of a same family are unlikely to trade (or interact) among each other. It may be worth to examine the role played by the concept of 'family' in wealth distributions of families: 'family wealth distribution' for brevity. A family in a society usually consists of more than one agent. In computer simulation, the agents belonging to a same family are coloured to keep track of. To find wealth distributions of families, the contributions of the same family members are added up. In Fig. 9 family wealth distributions are plotted for three cases: (i) families consist of 2 members each, (ii) families consist of 4 members each and (iii) families of mixed sizes between 1 and 4. The distributions are clearly not pure exponential, but modified exponential distributions (Gamma type distributions) with different peaks and different widths. This is quite expected as the probability of zero income of a family is zero. Modified exponential distribution for family wealth is also supported by fitting real data [14].

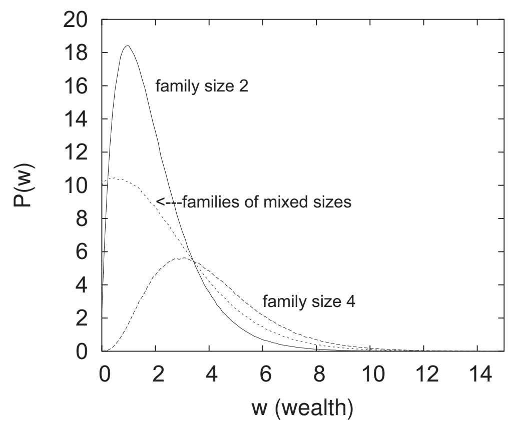

FIG. 9: Family wealth distributions: two curves are for families consisting of all equal sizes of 2 and 4. One curve is for a system of families consisting of various sizes between 1 and 4. The distributions are not normalized.

Some special way of incorporating selective interaction is seen to have a drastic effect in the individual wealth distribution. To implement the idea of 'selection', a 'class' of an agent is defined through an index ϵ. The class may be understood in terms of some sort of efficiency of accumulating wealth or some other closely related property. Therefore, ϵ's are assumed to be quenched. It is assumed that during the interactions, the agents may convert an appropriate amount of wealth proportional to their efficiency factor in their favour or against. Now, the model can be understood in terms of the general form of equations:

$$w_i(t+1) = \epsilon_i w_i(t) + \epsilon_j w_j(t), \qquad (23)$$

$$w_j(t+1) = (1-\epsilon_i) w_i(t) + (1-\epsilon_j) w_j(t),$$

where ϵ_i's are quenched random numbers between 0 and 1 (randomly assigned to the agents at the beginning). Now the agents are supposed to make a choice to whom not to trade with. This option, in fact, is not unnatural in the context of a real society where individual or group opinions are important. There has been a lot of works on the process and dynamics of opinion formations [4, 34] in model social systems. In the present model it may be imagined that the 'choice' is simply guided by the relative class index of the two agents. It is assumed that an interaction takes place when the ratio of two class factors remain within certain upper limit. The requirement for interaction (trade) to happen is then $1 < \epsilon_i/\epsilon_j < \tau$, where $\epsilon_i > \epsilon_j$. Wealth distributions for various values of τ are numerically investigated. Power laws in the tails of the distributions are obtained in all cases. In Fig. 10 the distributions for $\tau = 2$ and $\tau = 4$ are shown. Power laws are clearly seen with an exponent, $\alpha = 3.5$ (a straight line with slope around -3.5 is drawn) which means the Pareto index ν is close to 2.5. It is not further investigated whether the exponent (α) actually differs in a significant way for different choices of τ. It has been shown that preferential behaviour [20] generates power law in money distribution with some imposed conditions which allows the rich to get higher probability of getting richer. The rich is also favoured in a model with some kind of asymmetric exchange rules as proposed in [21] where a power law results in. The dice seems to be loaded in favour of the rich otherwise the rich can not be the rich!

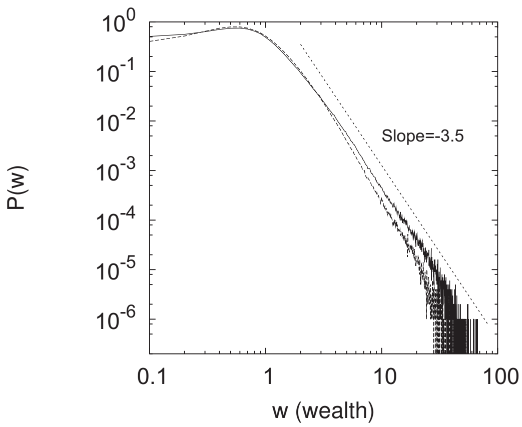

FIG. 10: Distribution of individual wealth with selective interaction. Power law is evident in the log-log plot where a straight line is drawn with slope = -3.5 for comparison.

VIII. MEASURE OF INEQUALITY

Emergence of Pareto's law signifies the existence of inequality in wealth in a population. Inequality or disparity in wealth or that of income is known to exist in almost all societies. To have a quantitative idea of inequality one generally plots Lorenz curve and then calculates Gini coefficient. Here the entropy approach [35] is considered. The time evolution of an appropriate quantity is examined which may be regarded as a measure of wealth-inequality.

Let us consider $w_1, w_2, \ldots, w_N$ be the wealths of N agents in a system. Let $W = \sum_{i=1}^{N} w_i$ be the total wealth of all the agents. Now $p_i = w_i/W$ can be considered as the fraction of wealth the i-th agent shares. Thus each of $p_i > 0$ and $\sum_{i=1}^{N} p_i = 1$. Thus the set of $p_1, p_2, \ldots, p_N$ may be regarded as a probability distribution. The well known Shannon entropy is defined as the following:

$$S = -\sum_{i=1}^{N} p_i \ln p_i. \tag{24}$$

From the maximization principle of entropy it can be easily shown that the entropy (S) is maximum when

$$p_1 = p_2 = \cdots = p_N = \frac{1}{N}, \tag{25}$$

giving the maximum value of S to be $\ln N$ where it is a limit of equality (everyone possesses the same wealth). A measure of inequality should be something which measures a deviation from the above ideal situation. Thus one can have a measure of wealth-inequality to be

$$H = \ln N - S = \ln N + \sum_{i=1}^{N} p_i \ln p_i = \sum_{i=1}^{N} p_i \ln(N p_i). \tag{26}$$

The greater the value of H, the greater the inequality is.

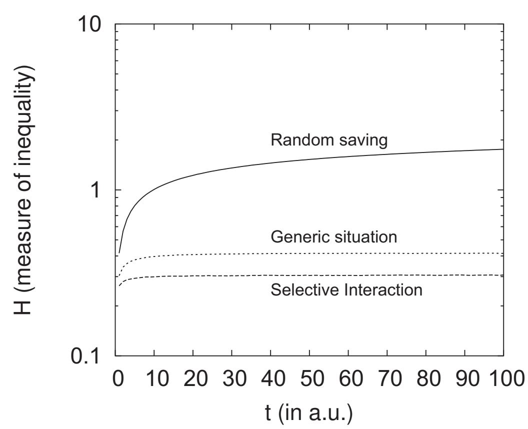

FIG. 11: Comparison of time evolution of the measure of inequality (H) in wealth for different models. Each 'time step' (t) is equal to a single interaction between a pair of agents. Data is taken after every 10^4 time steps to avoid clumsiness and each data point is obtained by averaging over 10^3 configurations. Y-axis is shown in log-scale to have a fair comparison.

It is seen that the wealth exchange algorithms are so designed that the resulting disparity or variance (or measure of inequality), in effect, increases with time. Wherever power law in distribution results in, the distribution naturally broadens which indicates that the variance (σ^2) or the inequality measure [H in eqn. (26)] should increase. In the Fig. 11 and in Fig. 12 time evolution of inequality measure H and variance σ^2 respectively are

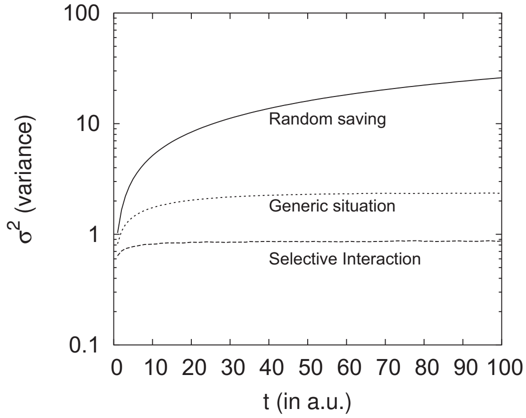

FIG. 12: Evolution of variance (σ^2) with 'time' (t) for different models. Y-axis is shown in log scale to accommodate three sets of data in a same graph. Data is taken after every 10^4 time steps to avoid clumsiness and each data point is obtained by averaging over 10^3 configurations.

plotted with time for three models to have a comparison. It is apparent that the measure of inequality in steady state attains different levels due to different mechanisms of wealth exchange processes, giving rise to different power law exponents. The growth of variance is seen to be different for different models considered, which is responsible for power laws with different exponents as discussed in the text. The power law exponents (α) appear to be related to the magnitudes of variance that are attained in equilibrium in the finite systems.

IX. DISTRIBUTION BY MAXIMIZING INEQUALITY

It is known that probability distribution of wealth of majority is different from that of handful of minority (rich people). Disparity is more or less a reality in all economies. A wealth exchange process can be thought of within the present framework where the interactions among agents eventually lead to increasing variance. It is numerically examined [19] whether the process of forcing the system to have ever increasing variance (measure of disparity) leads to a power law as it is known that power law is usually associated with infinite variance. Evolution of variance, $\sigma^2 = \langle w^2 \rangle - \langle w \rangle^2$ is calculated after each interaction in the framework of pure gambling model [the pair of equations (7)] and it is then forced to increase monotonically by comparing this to the previously calculated value (the average value $\overline{w}$ is fixed by virtue of the model). This results in a very large variance under this imposed condition. The inequality factor H also likewise increases monotonically and attains a high value. A power law distribution is obtained with the exponent, α close to 1. None of the available models does bring out such a low value of the exponent. The variance in any of the usual models generally settles at a level much lower than that is obtained in such a way. The resulting distribution of wealth is plotted in a log-log scale in Fig. 13.

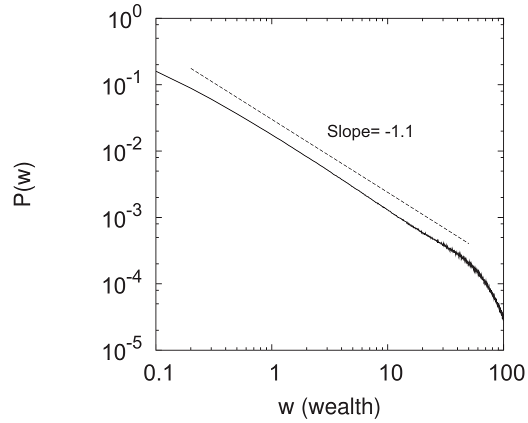

FIG. 13: Wealth distribution by maximizing the variance in the pure gambling model. Power law is clearly seen (in the log-log plot) and a straight line is drawn with slope = -1.1 to compare.

Power law, however, could not be obtained by the same way in the case of a non-conserved model like the following: $w_i(t+1) = w_i(t) \pm \delta$, where the increase or decrease (δ) in wealth (w) of any agent is independent of any other.

It has also been noted, considering some of the available models, larger the variance, smaller the exponent one gets. For example, the variance is seen to attain higher values (with time) in the model of distributed (random) saving propensities [18] compared to the model of selective interaction [19] and the resulting power law exponent α is close to 2.0 in the former case whereas it is close to 3.5 in the later. In the present situation the variance attains even higher values and the exponent α seems to be close to 1, the lowest among all.

X. CONFUSIONS AND CONCLUSIONS

As it is seen, the exchange of wealth in various modes generates a wide variety of distributions within the framework of simple wealth exchange models as discussed. In this review, some basic structures and ideas of interactions are looked at which seem to be fundamental to bring out the desired distributions. In this kind of agent based models (for some general discussions, see [36]) the division of labour, demand and supply and the human qualities (selfish act or altruism) and efforts (like investments, business) which are essential ingredients in classical economics are not considered explicitly. What causes the exchange of wealth of a specific kind among agents is not important in this discussion. Models are considered to be conserved (no inflow or outflow of money/ wealth in or from the system). It is not essential to look for inflation, taxation, debt, investment returns etc. of and in an economic system at the outset for the kind of questions that are addressed here. The essence of complexities of interactions leading to distributions can be understood in terms of the simple (microscopic) exchange rules much the same way the simple logistic equations that went on to construct 'roads to Chaos' and opened up a new horizon of thinking of a complex phenomenon like turbulence [37].

Some models of zero sum wealth exchange are examined here in this review. One may start thinking in a fresh way how the distributions emerge out of the kind of algorithmic exchange processes that are involved. The exchange processes can be understood in a general way by looking at the structure of associated 2×2 transition matrices. Wealths of individuals evolve to have a specific distribution in a stead state through the kind of interactions which are basically stochastic in nature. The distributions shift away from Boltzmann-Gibbs like exponential to Gamma type distributions and in some cases distributions emerge with power law tails known as Pareto's law ($P(w) \propto w^{-\alpha}$). It is also seen that the wealth distributions seem to be influenced by personal choice. In a real society, people usually do not interact arbitrarily rather do so with purpose and thinking. Some kind of personal preference is always there which may be incorporated in some way or other. Power law with distinctly different exponent ($\alpha = 3.5$, Pareto exponent $\nu = 2.5$) is achieved through a certain way of selective interaction. The value of Pareto index ν does not correspond to what is generally obtained empirically. However, the motivation is not to attach much importance to the numerical value at the outset rather than to focus on the fact of how power laws emerge with distinctly different exponents governed by the simple rules of wealth exchange.

The fat tailed distributions (power laws) are usually associated with large variance, which can be a measure of disparity. Economic disparity usually exists among a population. The detail mechanism leading to disparity is not always clear but it can be said to be associated with the emergence of power law tails in wealth distributions. Monotonically increasing variance (with time) can be associated with the emergence of power law in individual wealth distributions. The mean and variance of a power law distribution can be analytically derived [7] to see that they are finite when the power law exponent α is greater than 3. For $\alpha \leq 3$, the variance diverges but then the mean is finite. In case of the models discussed here in this review, mean is kept fixed but large or enhanced variance is observed in different models whenever there results in a power law. It remains a question of what can be the mechanisms (in the kind of discrete and conserved models) that generate large variance and power law tails. Large and increasing variance is also associated with lognormal distributions. A simple multiplicative stochastic process like $w(t+1) = \epsilon(t)w(t)$ can be used to explain the emergence of lognormal distribution and indefinite increase in variance. However, empirical evidence shows that the Pareto index and some other appropriate indices (Gibrat index, for example), generally dwindle within some range [38] indicating that the variance (or any other equivalent measure of inequality) does not increase forever. It seems to attain a saturation, given sufficient time. This is indeed the case the numerical results suggest. Normally there occurs simultaneous increase of variance and mean in statistical systems (in fact, the relationship between mean and variance goes by a power law as $\sigma^2 \propto \overline{w}^b$ know as Taylor's power law [39] as curiously observed in many natural systems). In this conserved model the mean is not allowed to vary as it is fixed by virtue of the model. It may be the case that σ^2 then has to have a saturation. The limit of σ^2 is tested through an artificial situation where the association of power law with large variance is tested in a reverse way.

Understanding the emergence of power law [7, 8] itself has been of great interest for decades. There is usually no accepted framework which may explain the origin and wealth of varieties of its appearance. It is often argued that the dynamics which generate power laws is dominated by multiplicative processes. It is true that in an economy wealth (or money) of an agent multiplies and that is coupled to the large number of interacting agents. The generic stochastic Lotka-Volterra systems like $w_i(t+1) = \epsilon w_i(t) + a\overline{w}(t) - bw_i(t)\overline{w}(t)$ have been studied [34, 40] to achieve power law distributions in wealth. However, these kinds of models are not discussed in this review as the basic intention had been to understand the ideas behind models of conserved wealth which the above is not.

In a twist of thinking, let us imagine a distribution curve which can be stretched in any direction as one wishes to have, keeping the area under this to be invariant. If now the curve is pulled too high around the left then the right hand side is to fall off too quickly, exponential decay is a possible option then. On the other hand, if the width of it is to be stretched too far (distribution becomes fat) at the right hand side, it should then decay fairly slowly giving rise to a possible power law fall at the right end while keeping the area under the curve preserved. What makes such a stretching possible? This review has been an attempt to integrate some ideas regarding models of wealth distributions and to reinvent things with a fresh outlook. In the way, some confusions, conjectures and conclusions emerged where many questions possibly have been answered with further questions and doubts. At the end of the day, the usefulness of this (review) may be measured by further curiosities and enhanced attention on the subject if at all this may generate.

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