Classical Mechanics
(Particles and Biparticles)

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Abstract

This paper considers the existence of biparticles and presents a general equation of motion, which can be applied in any non-rotating reference frame (inertial or non-inertial) without the necessity of introducing fictitious forces.

Universal Reference Frame

The universal reference frame $\mathring{\text{S}}$ is a reference frame in which the acceleration $\mathring{\mathbf{a}}$ of any particle is given by the following equation:

$$\mathring{\mathbf{a}} = \frac{\mathbf{F}}{m}$$

where $\mathbf{F}$ is the net force acting on the particle, and m is the mass of the particle.

The universal reference frame $\mathring{\text{S}}$ is an inertial reference frame. Therefore, it can be stated that the universal reference frame $\mathring{\text{S}}$ is also a non-rotating reference frame.

General Equation of Motion

The general equation of motion for two particles A and B, is as follows:

$$m_a m_b(\mathbf{r}_a - \mathbf{r}_b) = m_a m_b(\mathring{\mathbf{r}}_a - \mathring{\mathbf{r}}_b)$$

where m_a and m_b are the masses of particles A and B, $\mathbf{r}_a$ and $\mathbf{r}_b$ are the positions of particles A and B relative to a non-rotating reference frame S, $\mathring{\mathbf{r}}_a$ and $\mathring{\mathbf{r}}_b$ are the positions of particles A and B relative to the universal reference frame $\mathring{\mathrm{S}}$.

If $m_a m_b = m_{ab}$, $(\mathbf{r}_a - \mathbf{r}_b) = \mathbf{r}_{ab}$ and $(\mathring{\mathbf{r}}_a - \mathring{\mathbf{r}}_b) = \mathring{\mathbf{r}}_{ab}$, then the above equation reduces to:

$$m_{ab}\,\mathbf{r}_{ab} = m_{ab}\,\mathring{\mathbf{r}}_{ab}$$

The general equation of motion for a system of N particles, is as follows:

$$\sum_i \sum_{j>i} m_i m_j(\mathbf{r}_i - \mathbf{r}_j) = \sum_i \sum_{j>i} m_i m_j(\mathring{\mathbf{r}}_i - \mathring{\mathbf{r}}_j)$$

where m_i and m_j are the masses of the i-th and j-th particles, $\mathbf{r}_i$ and $\mathbf{r}_j$ are the positions of the i-th and j-th particles relative to a non-rotating reference frame S, $\mathring{\mathbf{r}}_i$ and $\mathring{\mathbf{r}}_j$ are the positions of the i-th and j-th particles relative to the universal reference frame $\mathring{\mathrm{S}}$.

If $m_i m_j = m_{ij}$, $(\mathbf{r}_i - \mathbf{r}_j) = \mathbf{r}_{ij}$ and $(\mathring{\mathbf{r}}_i - \mathring{\mathbf{r}}_j) = \mathring{\mathbf{r}}_{ij}$, then the above equation reduces to:

$$\sum_i \sum_{j>i} m_{ij}\,\mathbf{r}_{ij} = \sum_i \sum_{j>i} m_{ij}\,\mathring{\mathbf{r}}_{ij}$$

A system of particles forms a system of biparticles. For example, the system of particles A, B, C and D forms the system of biparticles AB, AC, AD, BC, BD and CD.

Particles and Biparticles

From the general equation of motion for two particles A and B (underlined blue equation) the following equations are obtained:

BIPARTICLE PARTICLE

$$
\begin{tikzcd}
& \boxed{\tfrac{1}{2}\mathbf{a}_{ab}^2 = \tfrac{1}{2}\mathring{\mathbf{a}}_{ab}^2} & \boxed{\mathbf{a}_{ab} = \mathring{\mathbf{a}}_{ab}} \arrow[l] \arrow[r] \arrow[dl] & \boxed{\mathbf{a}_a = \mathring{\mathbf{a}}_a} \arrow[r] \arrow[dr] & \boxed{\tfrac{1}{2}\mathbf{a}_a^2 = \tfrac{1}{2}\mathring{\mathbf{a}}_a^2} & \\
\text{KINEMATICS} & \boxed{\tfrac{1}{2}\mathbf{v}_{ab}^2 = \tfrac{1}{2}\mathring{\mathbf{v}}_{ab}^2} \arrow[u] & \boxed{\mathbf{v}_{ab} = \mathring{\mathbf{v}}_{ab}} \arrow[l] \arrow[r] \arrow[u] & \boxed{\mathbf{v}_a = \mathring{\mathbf{v}}_a} \arrow[r] \arrow[u] & \boxed{\tfrac{1}{2}\mathbf{v}_a^2 = \tfrac{1}{2}\mathring{\mathbf{v}}_a^2} \arrow[u] & \text{KINEMATICS} \\
& \boxed{\tfrac{1}{2}\mathbf{r}_{ab}^2 = \tfrac{1}{2}\mathring{\mathbf{r}}_{ab}^2} \arrow[u] & \boxed{\mathbf{r}_{ab} = \mathring{\mathbf{r}}_{ab}} \arrow[l] \arrow[r] \arrow[u] & \boxed{\mathbf{r}_a = \mathring{\mathbf{r}}_a} \arrow[r] \arrow[u] & \boxed{\tfrac{1}{2}\mathbf{r}_a^2 = \tfrac{1}{2}\mathring{\mathbf{r}}_a^2} \arrow[u] & \\
& \boxed{\tfrac{1}{2}m_{ab}\mathbf{r}_{ab}^2 = \tfrac{1}{2}m_{ab}\mathring{\mathbf{r}}_{ab}^2} \arrow[u] \arrow[d] & \boxed{\underline{m_{ab}\mathbf{r}_{ab} = m_{ab}\mathring{\mathbf{r}}_{ab}}} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \boxed{m_a\mathbf{r}_a = m_a\mathring{\mathbf{r}}_a} \arrow[r] \arrow[u] \arrow[d] & \boxed{\tfrac{1}{2}m_a\mathbf{r}_a^2 = \tfrac{1}{2}m_a\mathring{\mathbf{r}}_a^2} \arrow[u] \arrow[d] & \\
\text{DYNAMICS} & \boxed{\tfrac{1}{2}m_{ab}\mathbf{v}_{ab}^2 = \tfrac{1}{2}m_{ab}\mathring{\mathbf{v}}_{ab}^2} \arrow[d] & \boxed{m_{ab}\mathbf{v}_{ab} = m_{ab}\mathring{\mathbf{v}}_{ab}} \arrow[l] \arrow[r] \arrow[d] & \boxed{m_a\mathbf{v}_a = m_a\mathring{\mathbf{v}}_a} \arrow[r] \arrow[d] & \boxed{\tfrac{1}{2}m_a\mathbf{v}_a^2 = \tfrac{1}{2}m_a\mathring{\mathbf{v}}_a^2} \arrow[d] & \text{DYNAMICS} \\
& \boxed{\tfrac{1}{2}m_{ab}\mathbf{a}_{ab}^2 = \tfrac{1}{2}m_{ab}\mathring{\mathbf{a}}_{ab}^2} & \boxed{m_{ab}\mathbf{a}_{ab} = m_{ab}\mathring{\mathbf{a}}_{ab}} \arrow[l] \arrow[r] \arrow[ul] & \boxed{m_a\mathbf{a}_a = m_a\mathring{\mathbf{a}}_a} \arrow[r] \arrow[ur] & \boxed{\tfrac{1}{2}m_a\mathbf{a}_a^2 = \tfrac{1}{2}m_a\mathring{\mathbf{a}}_a^2} &
\end{tikzcd}
$$

BIPARTICLE PARTICLE

The blue equations are valid in any non-rotating reference frame, since $(\mathbf{r}_{ab} = \mathring{\mathbf{r}}_{ab})$, $(\mathbf{v}_{ab} = \mathring{\mathbf{v}}_{ab})$ and $(\mathbf{a}_{ab} = \mathring{\mathbf{a}}_{ab})$

The red equations are valid in any inertial reference frame, since $(\mathbf{a}_{a} = \mathring{\mathbf{a}}_{a})$

The kinematic equations are obtained from the dynamic equations if we consider that all particles have the same mass. Therefore, the kinematic equations are a special case of the dynamic equations.

The dynamics of particles is obtained from the dynamics of biparticles if we only consider biparticles that have the same particle.

For example:

If we consider a system of biparticles AB, AC and BC, we have:

$$\mathrm{AB} + \mathrm{AC} + \mathrm{BC} = \mathrm{A\mathring{B}} + \mathrm{A\mathring{C}} + \mathrm{B\mathring{C}}$$

Considering only the biparticles that have particle C, it follows:

$$\mathrm{AC} + \mathrm{BC} = \mathrm{A\mathring{C}} + \mathrm{B\mathring{C}}$$

Applying the general equation of motion, we obtain:

$$m_a m_c(\mathbf{r}_a - \mathbf{r}_c) + m_b m_c(\mathbf{r}_b - \mathbf{r}_c) = m_a m_c(\mathring{\mathbf{r}}_a - \mathring{\mathbf{r}}_c) + m_b m_c(\mathring{\mathbf{r}}_b - \mathring{\mathbf{r}}_c)$$

Differentiating twice with respect to time, yields:

$$m_a m_c(\mathbf{a}_a - \mathbf{a}_c) + m_b m_c(\mathbf{a}_b - \mathbf{a}_c) = m_a m_c(\mathring{\mathbf{a}}_a - \mathring{\mathbf{a}}_c) + m_b m_c(\mathring{\mathbf{a}}_b - \mathring{\mathbf{a}}_c)$$

Dividing by m_c, using a reference frame C fixed to particle C ($\mathbf{a}_c = 0$ relative to reference frame C) and assuming that reference frame C is inertial $(\mathbf{a}_c = \mathring{\mathbf{a}}_c)$, we obtain:

$$m_a \mathbf{a}_a + m_b \mathbf{a}_b = m_a \mathring{\mathbf{a}}_a + m_b \mathring{\mathbf{a}}_b$$

Substituting $\mathring{\mathbf{a}} = \mathbf{F}/m$ and rearranging, finally yields:

$$\mathbf{F}_a + \mathbf{F}_b = m_a \mathbf{a}_a + m_b \mathbf{a}_b$$

Equation of Motion

From the general equation of motion it follows that the acceleration $\mathbf{a}_a$ of a particle A relative to a reference frame S (non-rotating) fixed to a particle S, is given by the following equation:

$$\mathbf{a}_a = \frac{\mathbf{F}_a}{m_a} - \frac{\mathbf{F}_s}{m_s}$$

where $\mathbf{F}_a$ is the net force acting on particle A, m_a is the mass of particle A, $\mathbf{F}_s$ is the net force acting on particle S, and m_s is the mass of particle S.

In contradiction with Newton's first and second laws, from the above equation it follows that particle A can have non-zero acceleration even if there is no force acting on particle A, and also that particle A can have zero acceleration (state of rest or of uniform linear motion) even if there is an unbalanced force acting on particle A.

On the other hand, from the above equation it also follows that Newton's first and second laws are valid in the reference frame S only if the net force acting on particle S equals zero. Therefore, the reference frame S is an inertial reference frame only if the net force acting on particle S equals zero.

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Appendix

Transformations

The universal reference frame $\mathring{S}$ is an inertial reference frame.

Any inertial reference frame is a non-rotating reference frame.

Any central reference frame S^{cm} (reference frame fixed to the center of mass of a system of particles) is a non-rotating reference frame.

A change of coordinates x, y, z, t from a reference frame S (non-rotating) to coordinates x', y', z', t' from another reference frame S' (non-rotating) whose origin O' has coordinates $x_{o'}, y_{o'}, z_{o'}$ measured from reference frame S, can be carried out by means of the following equations:

$$x' = x - x_{o'}$$

$$y' = y - y_{o'}$$

$$z' = z - z_{o'}$$

$$t' = t$$

From the above equations, the transformation of position, velocity and acceleration from reference frame S to reference frame S' may be carried out, and expressed in vector form as follows:

$$\mathbf{r}' = \mathbf{r} - \mathbf{r}_{o'}$$

$$\mathbf{v}' = \mathbf{v} - \mathbf{v}_{o'}$$

$$\mathbf{a}' = \mathbf{a} - \mathbf{a}_{o'}$$

where $\mathbf{r}_{o'}$, $\mathbf{v}_{o'}$ and $\mathbf{a}_{o'}$ are the position, velocity and acceleration respectively, of reference frame S' relative to reference frame S.

Definitions

	Particles	Biparticles
Mass	$M_i = \sum_i \; m_i$	$M_{ij} = \sum_i \sum_{j>i} \; m_{ij}$
Vector position	$\mathbf{R}_i = \sum_i \; m_i \mathbf{r}_i \,/M_i$	$\mathbf{R}_{ij} = \sum_i \sum_{j>i} \; m_{ij} \mathbf{r}_{ij} \,/M_{ij}$
Vector velocity	$\mathbf{V}_i = \sum_i \; m_i \mathbf{v}_i \,/M_i$	$\mathbf{V}_{ij} = \sum_i \sum_{j>i} \; m_{ij} \mathbf{v}_{ij} \,/M_{ij}$
Vector acceleration	$\mathbf{A}_i = \sum_i \; m_i \mathbf{a}_i \,/M_i$	$\mathbf{A}_{ij} = \sum_i \sum_{j>i} \; m_{ij} \mathbf{a}_{ij} \,/M_{ij}$
Scalar position	$\mathrm{R}_i = \sum_i \; \frac{1}{2} m_i \mathbf{r}_i^2 /M_i$	$\mathrm{R}_{ij} = \sum_i \sum_{j>i} \; \frac{1}{2} m_{ij} \mathbf{r}_{ij}^2 \,/M_{ij}$
Scalar velocity	$\mathrm{V}_i = \sum_i \; \frac{1}{2} m_i \mathbf{v}_i^2 /M_i$	$\mathrm{V}_{ij} = \sum_i \sum_{j>i} \; \frac{1}{2} m_{ij} \mathbf{v}_{ij}^2 \,/M_{ij}$
Scalar acceleration	$\mathrm{A}_i = \sum_i \; \frac{1}{2} m_i \mathbf{a}_i^2 /M_i$	$\mathrm{A}_{ij} = \sum_i \sum_{j>i} \; \frac{1}{2} m_{ij} \mathbf{a}_{ij}^2 \,/M_{ij}$
Work	$W_i = \sum_i \; \int m_i \mathbf{a}_i \cdot d\mathbf{r}_i$	$W_{ij} = \sum_i \sum_{j>i} \; \int m_{ij} \mathbf{a}_{ij} \cdot d\mathbf{r}_{ij}$
	$W_i = \Delta \left(\sum_i \; \frac{1}{2} m_i \mathbf{v}_i^2 \right)$	$W_{ij} = \Delta \left(\sum_i \sum_{j>i} \; \frac{1}{2} m_{ij} \mathbf{v}_{ij}^2 \right)$

Relations

$$M_{ij}\,\mathrm{R}_{ij} = M_i^2 \left(\mathrm{R}_i - \tfrac{1}{2} \mathbf{R}_i^2 \right)$$

$$M_{ij}\,\mathrm{V}_{ij} = M_i^2 \left(\mathrm{V}_i - \tfrac{1}{2} \mathbf{V}_i^2 \right)$$

$$M_{ij}\,\mathrm{A}_{ij} = M_i^2 \left(\mathrm{A}_i - \tfrac{1}{2} \mathbf{A}_i^2 \right)$$

If $M_i^2/M_{ij} = k$, then the above equations relative to the central reference frame S^{cm} reduces to:

$$\mathrm{R}_{ij}^{cm} = k\,\mathrm{R}_i^{cm}$$
$$\mathrm{V}_{ij}^{cm} = k\,\mathrm{V}_i^{cm}$$
$$\mathrm{A}_{ij}^{cm} = k\,\mathrm{A}_i^{cm}$$

Principles

The positions, velocities and accelerations (vector and scalar) of a system of biparticles are invariant under transformations between non-rotating reference frames.

$$\mathbf{R}_{ij} = \mathring{\mathbf{R}}_{ij} = \mathbf{R}_{ij}^{cm} = \mathbf{R}'_{ij} \qquad \mathrm{R}_{ij} = \mathring{\mathrm{R}}_{ij} = \mathrm{R}_{ij}^{cm} = \mathrm{R}'_{ij}$$
$$\mathbf{V}_{ij} = \mathring{\mathbf{V}}_{ij} = \mathbf{V}_{ij}^{cm} = \mathbf{V}'_{ij} \qquad \mathrm{V}_{ij} = \mathring{\mathrm{V}}_{ij} = \mathrm{V}_{ij}^{cm} = \mathrm{V}'_{ij}$$
$$\mathbf{A}_{ij} = \mathring{\mathbf{A}}_{ij} = \mathbf{A}_{ij}^{cm} = \mathbf{A}'_{ij} \qquad \mathrm{A}_{ij} = \mathring{\mathrm{A}}_{ij} = \mathrm{A}_{ij}^{cm} = \mathrm{A}'_{ij}$$

From this principle it follows that the acceleration $\mathbf{a}_a$ of a particle A relative to a non-rotating reference frame S fixed to a particle S, is given by the following equation:

$$\mathbf{a}_a = \frac{\mathbf{F}_a}{m_a} - \frac{\mathbf{F}_s}{m_s}$$

where $\mathbf{F}_a$ is the net force acting on particle A, m_a is the mass of particle A, $\mathbf{F}_s$ is the net force acting on particle S, and m_s is the mass of particle S.

The accelerations (vector and scalar) of a system of particles are invariant under transformations between inertial reference frames.

$$\mathbf{A}_i = \mathring{\mathbf{A}}_i = \mathbf{A}'_i \qquad \mathrm{A}_i = \mathring{\mathrm{A}}_i = \mathrm{A}'_i$$

From this principle it follows that the acceleration $\mathbf{a}_a$ of a particle A relative to an inertial reference frame S, is given by the following equation:

$$\mathbf{a}_a = \frac{\mathbf{F}_a}{m_a}$$

where $\mathbf{F}_a$ is the net force acting on particle A, and m_a is the mass of particle A.

Work and Force

The work W_{ij} done by the forces acting on a system of biparticles relative to a non-rotating reference frame, is given by:

$$W_{ij} = \sum_{i} \sum_{j>i} \int m_i m_j \left(\frac{\mathbf{F}_i}{m_i} - \frac{\mathbf{F}_j}{m_j} \right) \cdot d\left(\mathbf{r}_i - \mathbf{r}_j\right)$$

The work W_i done by the forces acting on a system of particles relative to the central reference frame, is given by:

$$W_i = \sum_{i} \int \mathbf{F}_i \cdot d\mathbf{r}_i$$

The work W_i done by the forces acting on a system of particles relative to an inertial reference frame, is given by:

$$W_i = \sum_{i} \int \mathbf{F}_i \cdot d\mathbf{r}_i$$

Conservation of Kinetic Energy

If the forces acting on a system of particles do not perform work relative to the central reference frame, then the kinetic energy of the system of particles is conserved relative to the central reference frame.

If the kinetic energy of the system of particles is conserved relative to the central reference frame, then the kinetic energy of the system of biparticles is conserved relative to any non-rotating reference frame.

If the forces acting on the system of particles do not perform work relative to an inertial reference frame, then the kinetic energy and linear momentum (magnitude) of the system of particles are conserved relative to the inertial reference frame; even if Newton’s third law were not valid.