Ultrafunctions and generalized solutions

Vieri Benci*

June 30, 2012

Abstract

The theory of distributions provides generalized solutions for problems which do not have a classical solution. However, there are problems which do not have solutions, not even in the space of distributions. As model problem you may think of

$$-\triangle u = u^{p-1}\ ,\ u > 0,\ p \geq \frac{2N}{N-2}$$

with Dirichlet boundary conditions in a bounded open star-shaped set. Having this problem in mind, we construct a new class of functions called **ultrafunctions** in which the above problem has a (generalized) solution. In this construction, we apply the general ideas of Non Archimedean Mathematics (NAM) and some techniques of Non Standard Analysis. Also, some possible applications of ultrafunctions are discussed.

Mathematics subject classification: 26E30, 26E35, 35D99, 81Q99.

Keywords. Non Archimedean Mathematics, Non Standard Analysis, ultrafunctions, distributions, generalized solutions, Sobolev critical exponent, formalism of Quantum Mechanics.

Contents

1 Introduction **2**

1.1 Notation 4

*Dipartimento di Matematica Applicata, Università degli Studi di Pisa, Via F. Buonarroti 1/c, Pisa, ITALY and Department of Mathematics, College of Science, King Saud University, Riyadh, 11451, SAUDI ARABIA. e-mail: benci@dma.unipi.it

2 Λ-theory **5**
2.1 Non Archimedean Fields 5
2.2 The Λ-limit 6
2.3 Natural extensions of sets and functions 8
2.4 Qualified sets 10

3 The abstract theory **11**
3.1 Definition of ultravectors 11
3.2 Extension of operators 13

4 The ultrafunctions **13**
4.1 Definition 13
4.2 Ultrafunctions and distributions 16

5 The Dirichlet problem **18**
5.1 Generalized solutions 18
5.2 The variational approach 21

6 The bubbling phenomenon relative to the Sobolev critical exponent **22**
6.1 Description of the problem 23
6.2 Generalized solutions 25

7 Ultrafunctions and Quantum Mechanics **28**
7.1 The axioms of Quantum Mechanics 28
7.2 Discussion of the axioms 30
7.3 The Heisenberg algebra 32

1 Introduction

We believe that Non Archimedean Mathematics (NAM), namely, mathematics based on Non Archimedean Fields is very interesting, very rich and, in many circumstances, allows to construct models of the physical world in a more elegant and simple way. In the years around 1900, NAM was investigated by prominent mathematicians such as David Hilbert and Tullio Levi-Civita, but then it has been forgotten until the '60s when Abraham Robinson presented his Non Standard Analysis (NSA). We refer to Ehrlich [9] for a historical analysis of these facts and to Keisler [10] for a very clear exposition of NSA.

In this paper we apply the general ideas of NAM and some of the techniques of NSA to a new notion of generalized functions which we have called

ultrafunctions. Ultrafunctions are a particular class of functions based on a superreal field $\mathbb{R}^* \supset \mathbb{R}$. More exactly, to any continuous function $f : \mathbb{R}^N \to \mathbb{R}$, we associate in a canonical way an ultrafunction $f_\Phi : (\mathbb{R}^*)^N \to \mathbb{R}^*$ which extends f; but the ultrafunctions are much more than the functions and among them we can find solutions of functional equations which do not have any solutions among the real functions or the distributions.

Now we itemize some of the peculiar properties of the ultrafunctions:

- the space of ultrafunctions is larger than the space of distributions, namely, to every distribution T, we can associate in a canonical way an ultrafunction T_Φ (cf. section 4.2);
- similarly to the distributions, the ultrafunctions are motivated by the need of having generalized solutions; however, while the distributions are no longer functions, the ultrafunctions are still functions even if they have larger domain and range;
- unlikely the distributions, the space of ultrafunctions is suitable for non linear problem; in fact any operator F defined for a reasonable class of functions, can be extended to the ultrafunctions; for example, in the framework of ultrafunctions δ^2 makes sense (here δ is the Dirac measure seen as an ultrafunction);
- if a problem has a unique classical solution u, then u_Φ is the only solution in the space of ultrafunctions,
- the main strategy to prove the existence of generalized solutions in the space of ultrafunction is relatively simple; it is just a variant of the Faedo-Galerkin method.

This paper is organized as follows. In Section 2 we introduce NAM via the notion of Λ-limit. This approach is quite different from the usual approach to NAM via NSA. It follows a line developed in [2], [3], [5] and [6]. In this section, we introduce all the notions necessary to understand the rest of the paper, but we omit details and most of the proofs. In sections 3 and 4, we introduce the notion of ultrafunction and the last three sections are devoted to applications. The applications are chosen as examples to show the potentiality of the theory and possible directions of study; they are not an exhaustive study of the topic treated there.

Before ending the introduction, we want to emphasize the differences by our approach to NAM and the approach of most people working in Nonstandard Analysis: there are two main differences, one in the aims and one in the methods.

Let examine the difference in the aims. We think that infinitesimal and infinite numbers should not be considered just as entities living in a parallel universe (the nonstandard universe) which are only a tool to prove some statement relative to our universe (the standard universe), but rather that they should be considered mathematical entities which have the same status of the others and can be used to build models as any other mathematical entity. Actually, the advantages of a theory which includes infinitesimals rely more on the possibility of making new models rather than in the proving techniques. Our papers [4] and [6] as well as this one, are inspired by this principle.

As far as the methods are concerned we introduce a non-Archimedean field via a new notion of limit (see section 2.2). Moreover, we make a very limited use of logic: the transfer principle (or Leibnitz Principle) is given by Th. 11 and it is not necessary to introduce a formal language. We think that this approach is closer to the way of thinking of the applied mathematician.

1.1 Notation

Let $\Omega \subset \mathbb{R}^N$ be a set not necessarily open: then

- $\mathcal{F}(\Omega, E)$ denotes the set all the functions defined in Ω with values in E;
- $\mathcal{C}(\Omega)$ denotes the set of real continuous functions defined on a set Ω;
- $\mathcal{C}_0\left(\overline{\Omega}\right)$ denotes the set of real continuous functions on $\overline{\Omega}$ which vanish on $\partial\Omega$;
- $\mathcal{C}^k(\Omega)$ denotes the set of functions defined on an $\Omega \subset \mathbb{R}^N$ which have continuous derivatives up to the order k
- $\mathcal{C}_0^k\left(\overline{\Omega}\right) = \mathcal{C}^k\left(\overline{\Omega}\right) \cap \mathcal{C}_0\left(\overline{\Omega}\right)$;
- $\mathcal{D}(\Omega)$ denotes the set of the infinitely differentiable functions with compact support defined on a set $\Omega \subset \mathbb{R}^N$; $\mathcal{D}'(\Omega)$ denotes the topological dual of $\mathcal{D}(\Omega)$, namely the set of distributions on Ω;
- $\mathcal{S}(\Omega)$ denotes the Schwartz space and $\mathcal{S}'(\Omega)$ the set of tempered distributions;
- $\mathcal{E}(\Omega) = \mathcal{C}^\infty(\Omega)$ denotes the set of the infinitely differentiable functions; $\mathcal{E}'(\Omega)$ denotes the topological dual of $\mathcal{E}(\Omega)$, namely the set of distributions with compact support in Ω;

- $H^1(\Omega)$ is the usual Sobolev space defined as the set of functions $u \in L^2(\Omega)$ such that $\nabla u \in L^2(\Omega)$;
- $H_0^1(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$;
- $H^{-1}(\Omega)$ is the topological dual of $H_0^1(\Omega)$.

2 Λ-theory

As we have already remarked in the introduction, Λ-theory can be considered as a variant of nonstandard analysis. It can be introduced via the notion of Λ-limit, and it can be easily used for the problems which we will consider in this paper.

2.1 Non Archimedean Fields

In this section, we will give the basic definitions relative to non-Archimedean fields and some of the basic facts. $\mathbb{F}$ will denote an ordered field. The elements of $\mathbb{F}$ will be called numbers. Clearly $\mathbb{F}$ contains (a set isomorphic to) the rational numbers.

Definition 1 *Let $\mathbb{F}$ be an ordered field. Let $\xi \in \mathbb{F}$. We say that:*

- *ξ is infinitesimal if for all $n \in \mathbb{N}$, $|\xi| < \frac{1}{n}$;*
- *ξ is finite if there exists $n \in \mathbb{N}$ such as $|\xi| < n$;*
- *ξ is infinite if, for all $n \in \mathbb{N}$, $|\xi| > n$ (equivalently, if ξ is not finite).*

Definition 2 *An ordered field $\mathbb{K}$ is called non-Archimedean if it contains an infinitesimal $\xi \neq 0$.*

It's easily seen that the inverse of a nonzero infinitesimal number is infinite, and the inverse of an infinite number is infinitesimal. Clearly, all infinitesimal numbers are finite.

Definition 3 *A superreal field is an ordered field $\mathbb{K}$ that properly extends $\mathbb{R}$.*

It is easy to show that any superreal field contains infinitesimal and infinite numbers. Thanks to infinitesimal numbers, in the superreal fields, we can formalize a new notion of "closeness".

Definition 4 *We say that two numbers ξ and $\zeta \in \mathbb{K}$ are infinitely close if $\xi - \zeta$ is infinitesimal. In this case, we will write $\xi \sim \zeta$.*

It is easy to see that the relation "$\sim$" of infinite closeness is an equivalence relation.

Theorem 5 *If $\mathbb{K}$ is a superreal field, every finite number $\xi \in \mathbb{K}$ is infinitely close to a unique real number $r \sim \xi$, called the **shadow** or the **standard part** of ξ. We will write $r = sh(\xi)$. If $\xi \in \mathbb{K}$ is a positive (negative) infinite number, then we put $sh(\xi) = +\infty$ ($sh(\xi) = -\infty$).*

We can also consider the relation of "finite closeness":

$$\xi \sim_f \zeta \text{ if and only if } \xi - \zeta \text{ is finite.}$$

It is readily seen that also $\sim_f$ is an equivalence relation. In the literature, the equivalence classes relative to the two relations of closeness $\sim$ and $\sim_f$, are called monads and galaxies, respectively.

Definition 6 *The monad of a number ξ is the set of all numbers that are infinitely close to it:*

$$\mathfrak{mon}(\xi) = \{\zeta \in \mathbb{K} : \xi \sim \zeta\}$$

The galaxy of a number ξ is the set of all numbers that are finitely close to it:

$$\mathfrak{gal}(\xi) = \{\zeta \in \mathbb{K} : \xi \sim_f \zeta\}$$

So, $\mathfrak{mon}(0)$ is the set of all infinitesimal numbers in $\mathbb{K}$ and $\mathfrak{gal}(0)$ is the set of all finite numbers.

2.2 The Λ-limit

$\mathcal{U}$ will denote our "mathematical universe". For our applications a good choice of $\mathcal{U}$ is given by the superstructure on $\mathbb{R}$:

$$\mathcal{U} = \bigcup_{n=0}^{\infty} \mathcal{U}_n$$

where $\mathcal{U}_n$ is defined by induction as follows:

$$\begin{aligned} \mathcal{U}_0 &= \mathbb{R} \\ \mathcal{U}_{n+1} &= \mathcal{U}_n \cup \mathcal{P}(\mathcal{U}_n) \end{aligned}$$

Here $\mathcal{P}(E)$ denotes the power set of E. If we identify the couples with the Kuratowski pairs and the functions and the relations with their graphs, clearly $\mathcal{U}$ contains almost all the mathematical objects needed in mathematics.

Given the universe $\mathcal{U}$, we denote by Λ the family of finite subsets of $\mathcal{U}$. Clearly $(\Lambda, \subset)$ is a directed set and, as usual, a function $\varphi : \Lambda \to E$ will be called *net* (with values in E).

Axioms of the Λ-limit

- (Λ-1) **Existence Axiom.** *There is a superreal field $\mathbb{K} \supset \mathbb{R}$ such that for every net $\varphi : \Lambda \to \mathbb{R}$ there exists a unique element $L \in \mathbb{K}$ called the* "Λ-limit" *of φ. The Λ-limit will be denoted by*

$$L = \lim_{\lambda \uparrow \mathcal{U}} \varphi(\lambda) \quad or \quad L = \lim_{\lambda \in \Lambda} \varphi(\lambda)$$

Moreover we assume that every $\xi \in \mathbb{K}$ is the Λ-limit *of some real function $\varphi : \Lambda \to \mathbb{R}$.*

- (Λ-2) **Real numbers axiom**. *If $\varphi(\lambda)$ is eventually constant, namely $\exists \lambda_0 \in \Lambda : \ \forall \lambda \supset \lambda_0, \ \varphi(\lambda) = r$, then*

$$\lim_{\lambda \uparrow \mathcal{U}} \varphi(\lambda) = r$$

- (Λ-3) **Sum and product Axiom**. *For all $\varphi, \psi : \Lambda \to \mathbb{R}$:*

$$\begin{aligned} \lim_{\lambda \uparrow \mathcal{U}} \varphi(\lambda) + \lim_{\lambda \uparrow \mathcal{U}} \psi(\lambda) &= \lim_{\lambda \uparrow \mathcal{U}} (\varphi(\lambda) + \psi(\lambda)) \\ \lim_{\lambda \uparrow \mathcal{U}} \varphi(\lambda) \cdot \lim_{\lambda \uparrow \mathcal{U}} \psi(\lambda) &= \lim_{\lambda \uparrow \mathcal{U}} (\varphi(\lambda) \cdot \psi(\lambda)) \end{aligned}$$

Theorem 7 *The axioms (Λ-1),(Λ-2),(Λ-3) are consistent.*

Proof. In order to prove the consistency of these axioms, it is sufficient to construct a model. Let us consider the algebra $\mathcal{F}(\Lambda, \mathbb{R})$ of the real functions defined on Λ and set

$$\mathfrak{I}_0 = \{\varphi \in \mathcal{F}(\Lambda, \mathbb{R}) \mid \varphi(\lambda) \text{ is eventually } 0\}$$

It is easy to check that $\mathfrak{I}_0$ is an ideal in the algebra Λ. By the Krull-Zorn Theorem, every ideal is contained in a maximal ideal. Let $\mathfrak{I}$ be a maximal ideal containing $\mathfrak{I}_0$. We set

$$\mathbb{K} := \frac{\mathcal{F}(\Lambda, \mathbb{R})}{\cong_{\mathfrak{I}}}$$

where the equivalence relation $\cong_{\mathfrak{I}}$ is defined as follows:

$$\varphi \cong_{\mathfrak{I}} \psi :\Leftrightarrow \varphi - \psi \in \mathfrak{I}$$

It is easy to check that $\mathbb{K}$ is an ordered field and $\mathbb{R} \subset \mathbb{K}$ if we identify $r \in \mathbb{R}$ with the equivalence class $[r]_{\cong_{\mathfrak{I}}}$. Finally, we can define the Λ-limit as

$$\lim_{\lambda \uparrow \mathcal{U}} \varphi(\lambda) = [\varphi]_{\cong_{\mathfrak{I}}}$$

Now, it is immediate to check that the Λ-limit satisfies (Λ-1),(Λ-2),(Λ-3)
□

Now we want to define the Λ-limit of any bounded net of mathematical objects in $\mathcal{U}$ (a net $\varphi : \Lambda \to \mathcal{U}$ is called bounded if there exists n such that $\forall \lambda \in \Lambda, \varphi(\lambda) \in \mathcal{U}_n$). To do this, consider a net

$$\varphi : \Lambda \to \mathcal{U}_n \tag{1}$$

We will define $\lim_{\lambda \uparrow \mathcal{U}} \varphi(\lambda)$ by induction on n. For $n = 0$, $\lim_{\lambda \uparrow \mathcal{U}} \varphi(\lambda)$ is defined by the axioms (Λ-1),(Λ-2),(Λ-3); so by induction we may assume that the limit is defined for $n-1$ and we define it for the net (1) as follows:

$$\lim_{\lambda \uparrow \mathcal{U}} \varphi(\lambda) = \left\{ \lim_{\lambda \uparrow \mathcal{U}} \psi(\lambda) \mid \psi : \Lambda \to \mathcal{U}_{n-1},\ \forall \lambda \in \Lambda,\ \psi(\lambda) \in \varphi(\lambda) \right\}$$

Definition 8 *A mathematical entity (number, set, function or relation) which is the Λ-limit of a net is called* ***internal****.*

If $E \in \mathcal{U}$, and $\varphi : \Lambda \cap \mathcal{P}(E) \to \mathcal{U}_n$, then we will use the following notation:

$$\lim_{\lambda \uparrow E} \varphi(\lambda) = \lim_{\mu \uparrow \mathcal{U}} \varphi(\mu \cap E).$$

2.3 Natural extensions of sets and functions

Definition 9 *The* ***natural extension*** *of a set $E \subset \mathbb{R}$ is given by*

$$E^* := \lim_{\lambda \uparrow \mathcal{U}} c_E(\lambda) = \left\{ \lim_{\lambda \uparrow \mathcal{U}} \psi(\lambda) \mid \psi(\lambda) \in E \right\}$$

where $c_E(\lambda)$ is the net identically equal to E.

Using the above definition we have that

$$\mathbb{K} = \mathbb{R}^*$$

In this context a function f can be identified with its graph; then the natural extension of a function is well defined. Moreover we have the following result:

Theorem 10 *The **natural extension** of a function*

$$f : E \to F$$

is a function

$$f^* : E^* \to F^*$$

and for every $\varphi : \Lambda \cap \mathcal{P}(E) \to E$, and every $f : E \to F$, we have that

$$\lim_{\lambda \uparrow \mathcal{U}} f(\varphi(\lambda)) = f^* \left(\lim_{\lambda \uparrow \mathcal{U}} \varphi(\lambda) \right)$$

When dealing with functions, when the domain of the function is clear from the context, sometimes the "$*$" will be omitted. For example, if $\eta \in \mathbb{R}^*$ is an infinitesimal, then clearly e^η is a short way to write $\exp^*(\eta)$.

The following theorem is a fundamental tool in using the Λ-limit:

Theorem 11 ***(Leibnitz Principle)*** *Let $\mathcal{R}$ be a relation in $\mathcal{U}_n$ for some $n \geq 0$ and let φ,$\psi \in \mathcal{F}(\Lambda, \mathcal{U}_n)$. If*

$$\forall \lambda \in \Lambda,\ \varphi(\lambda) \mathcal{R} \psi(\lambda)$$

then

$$\left(\lim_{\lambda \uparrow \mathcal{U}} \varphi(\lambda) \right) \mathcal{R}^* \left(\lim_{\lambda \uparrow \mathcal{U}} \psi(\lambda) \right)$$

Remark 12 *Notice that, in the above theorem, the relations "$=$" and "$\in$" do not change their "meaning", namely "$=^*$" and "$\in^*$" have the same interpretation than "$=$" and "$\in$".*

Definition 13 *An internal set is called **hyperfinite** if it is the Λ-limit of finite sets.*

All the internal finite sets are hyperfinite, but there are hyperfinite sets which are not finite. For example the set

$$\mathbb{R}^{\circ} := \lim_{\lambda \uparrow \mathcal{U}} (\mathbb{R} \cap \lambda)$$

is not finite. The hyperfinite sets are very important since they inherit many properties of finite sets via Th. 11. For example, $\mathbb{R}^{\circ}$ has the maximum and the minimum and every internal function

$$f : \mathbb{R}^{\circ} \to \mathbb{R}^{*}$$

has the maximum and the minimum as well.

Also, it is possible to add the elements of an hyperfinite set of numbers or vectors. Let

$$A := \lim_{\lambda \uparrow \mathcal{U}} A_{\lambda}$$

be an hyperfinite set; then, the hyperfinite sum is defined as follows:

$$\sum_{a \in A} a = \lim_{\lambda \uparrow \mathcal{U}} \sum_{a \in A_{\lambda}} a$$

In particular, if $A_{\lambda} = \left\{ a_1(\lambda), ..., a_{\beta(\lambda)}(\lambda) \right\}$ with $\beta(\lambda) \in \mathbb{N}$, then, setting

$$\beta = \lim_{\lambda \uparrow \mathcal{U}} \beta(\lambda) \in \mathbb{N}^{*}$$

we use the notation

$$\sum_{j=1}^{\beta} a_j = \lim_{\lambda \uparrow \mathcal{U}} \sum_{j=1}^{\beta(\lambda)} a_j(\lambda).$$

2.4 Qualified sets

Also, if $Q \subset \Lambda$ and $\varphi : \Lambda \to \mathcal{U}_n$, the following notation is quite useful

$$\lim_{\lambda \in Q} \varphi(\lambda) = \lim_{\lambda \uparrow \mathcal{U}} \widetilde{\varphi}(\lambda)$$

where

$$\widetilde{\varphi}(\lambda) = \begin{cases} \varphi(\lambda) & \text{for } \lambda \in Q \\ \varnothing & \text{for } \lambda \notin Q \end{cases}$$

We use this notation to introduce the notion of qualified set:

Definition 14 *We say that a set $Q \subset \Lambda$ is qualified if for every bounded net φ, we have that*

$$\lim_{\lambda \uparrow \mathcal{U}} \varphi(\lambda) = \lim_{\lambda \in Q} \varphi(\lambda).$$

By the above definition, we have that the Λ-limit of a net φ depends only on the values that φ takes on a qualified set. It is easy to see that (nontrivial) qualified sets exist. For example, by (Λ-2), we can deduce that, for every $\lambda_0 \in \Lambda$ the set

$$Q(\lambda_0) := \{\lambda \in \Lambda \mid \lambda_0 \subseteq \lambda\}$$

is qualified. In this paper, we will use the notion of qualified set via this Theorem

Theorem 15 *Let $\mathcal{R}$ be a relation in $\mathcal{U}_n$ for some $n \geq 0$ and let φ, $\psi \in \mathcal{F}(\Lambda, \mathcal{U}_n)$. Then the following statements are equivalent:*

- *there exists a qualified set Q such that*

$$\forall \lambda \in Q,\ \varphi(\lambda) \mathcal{R} \psi(\lambda)$$

- *we have*

$$\left(\lim_{\lambda \uparrow \mathcal{U}} \varphi(\lambda)\right) \mathcal{R}^* \left(\lim_{\lambda \uparrow \mathcal{U}} \psi(\lambda)\right)$$

Proof: It is an immediate consequence of Th. 11 and the definition of qualified set.
□

3 The abstract theory

In this section we will present a method to extend any vector space V to a larger vector space $\mathcal{B}[V]$ of hyperfinite dimension. In the next section we will apply this method to functional vector spaces.

3.1 Definition of ultravectors

Definition 16 *Let H be a separable real (or complex) Hilbert space with scalar product $(\cdot\ ,\cdot)$ and let $V \subset H$ be a dense subspace. We assume that $H \in \mathcal{U}$ and we set*

$$\mathcal{B}[V] := \lim_{\lambda \uparrow V} V_\lambda$$

where

$$V_\lambda := Sp(\lambda)$$

is the span of λ. $\mathcal{B}[V]$ is called the space of ultravectors based on V.

In order to simplify the notation, sometimes, we will set $V_{\mathcal{B}} = \mathcal{B}[V]$. Notice that $V_{\mathcal{B}}$ is a vector space of hyperfinite dimension $\beta \in \mathbb{N}^*$, were β is defined as follows:

$$\beta = \dim^*(V_{\mathcal{B}}) = \lim_{\lambda \uparrow V} (\dim V_\lambda).$$

Let $f \in V$; if we identify f and f^*, we have that $V \subset V_{\mathcal{B}}$. Now let

$$\Phi : H^* \to V_{\mathcal{B}} \tag{2}$$

be the orthogonal projector. Then, to every vector $f \in H$, we can associate the ultravector $\Phi f \in V_{\mathcal{B}}$. If $\{e_j\}_{j \leq \beta}$ is a basis for $V_{\mathcal{B}}$,then

$$\Phi f = \sum_{j=1}^{\beta} (f, e_j) e_j \tag{3}$$

Let V' denote the dual of V, namely, V' is the family of linear functionals T on V.

Definition 17 *For any $T \in V'$, we denote by ΦT the only vector in $V_{\mathcal{B}}$ such that*

$$\forall v \in V_{\mathcal{B}},\ (\Phi T, v) = \langle T^*, v \rangle;$$

*ΦT is called **dual** ultravector. Using the orthonormal basis $\{e_j\}_{j \leq \beta}$, we have that*

$$\Phi T = \sum_{j=1}^{\beta} (\Phi T, e_j) e_j = \sum_{j=1}^{\beta} \langle T^*, e_j \rangle e_j \tag{4}$$

Notice that, if we identify H as a subset of V', the operator Φ defined by (4) is the extension of the operator (3) and hence we have denoted them with the same symbol.

From our previous discussion the space of ultravectors $V_{\mathcal{B}}$ contains three types of vectors

- standard ultravectors: $u \in V_{\mathcal{B}}$ is called **standard** if $u \in V$ (or, to be more precise, if there exists $f \in V$ such that $u = f^*$);
- dual ultravectors: $u \in V_{\mathcal{B}}$ is called **dual** ultravector if $u = \Phi T$ for some $T \in V'$;
- proper ultravector: $u \in V_{\mathcal{B}}$ is called **proper** ultravector if it is not a dual ultravector.

The ultravector which are not standard will be called **ideal.**

3.2 Extension of operators

Definition 18 *Given the operator $F : D \to V'$, $D \subset V$, the map*

$$F_{\mathbf{\Phi}} : V_{\mathcal{B}} \cap D^* \to V_{\mathcal{B}}$$

defined by

$$F_{\mathbf{\Phi}} = \Phi \circ F^* \tag{5}$$

*is called **canonical** extension of F.*

By the definition of $F_{\mathbf{\Phi}}$, if $u \in V_{\mathcal{B}} \cap D^*$, we have that

$$\forall v \in V_{\mathcal{B}},\ \left(F_{\mathbf{\Phi}}\left(u\right), v\right) = \left\langle F^*\left(u\right), v\right\rangle \tag{6}$$

Using an orthonormal basis $\{e_j\}_{j\leq\beta}$ for $V_{\mathcal{B}}$,we have

$$F_{\mathbf{\Phi}}\left(u\right) = \sum_{j=1}^{\beta} \left\langle F^*(u), e_j\right\rangle e_j$$

If we identify H with its dual and we take $F : V \cap D \to H$, then equation (6) becomes:

$$\forall v \in V_{\mathcal{B}},\ \left(F_{\mathbf{\Phi}}\left(u\right), v\right) = \left(F^*\left(u\right), v\right). \tag{7}$$

4 The ultrafunctions

4.1 Definition

Definition 19 *Let Ω be a set in $\mathbb{R}^N$, and let $V\left(\Omega\right)$ be a vector space such that $\mathcal{D}(\Omega) \subseteq V\left(\Omega\right) \subseteq \mathcal{C}(\Omega) \cap L^2(\Omega)$. Then any function*

$$u \in \mathcal{B}\left[V\left(\Omega\right)\right]$$

is called ultrafunction.

So the ultrafunctions are Λ-limits of continuous functions in $V_\lambda\left(\mathbb{R}^N\right) := Sp\left(\lambda \cap V\left(\mathbb{R}^N\right)\right)$ and hence they are internal functions

$$u : \Omega^* \to \mathbb{C}^*.$$

Remark 20 *If $V(\Omega)$ is a Sobolev space such as $H^1(\Omega)$, then the elements of $V(\Omega)$ are not functions, but equivalence class of functions, so also the elements of $\mathcal{B}[V(\Omega)]$ are equivalence class of functions. In order to avoid this unpleasant fact, in the definition of ultrafunctions, we have assumed $V(\Omega) \subset \mathcal{C}(\Omega)$. Moreover, this choice has also an other motivation: as we will see in the applications, if we approach a problem via the ultrafunctions, we do not need Sobolev spaces (even if we might need the Sobolev inequalities). In some sense the ultrafunctions represent an alternative approach to problems which do not have classical solutions in some $\mathcal{C}^k(\overline{\Omega})$.*

Since $V_{\mathcal{B}}(\Omega) \subset [L^2(\Omega)]^*$, it can be equipped with the following scalar product

$$(u, v) = \int_{\Omega}^{*} u(x)\overline{v(x)}\, dx.$$

where $\int_{\Omega}^{*}$ is the natural extension of the Lebesgue integral considered as a functional.

Notice the Euclidean structure of $V_{\mathcal{B}}(\Omega)$ is the Λ-limit of the Euclidean structure of every $V_{\lambda}(\Omega)$ given by the usual $L^2(\Omega)$ scalar product.

If $f \in \mathcal{C}(\Omega)$ is a function such that,

$$\forall g \in V(\Omega),\ \int f(x)g(x)\, dx < +\infty \tag{8}$$

then it can be identified with an element of $V(\Omega)'$ and, by Def. 17, there is a unique ultrafunction f_{Φ} such that $\forall v \in V_{\mathcal{B}}(\Omega)$,

$$\int^{*} f_{\Phi}(x)v(x)\, dx = \int^{*} f^{*}(x)v(x)\, dx. \tag{9}$$

The map

$$\Phi : \mathcal{C}(\Omega) \cap V(\Omega)' \to V_{\mathcal{B}}(\Omega) \tag{10}$$

is called canonical map. Notice that $f_{\Phi} \neq f^{*}$ unless $f \in V(\Omega)$.

Now let us define a new notion which helps to understand the structure of ultrafunctions:

Definition 21 *A hyperfinite basis $\{e_j\}_{j \leq \beta}$ for $V_{\mathcal{B}}(\Omega)$ is called **regular** basis if*

- *it is an orthonormal basis,*
- *$\{e_j\}_{j \in \mathbb{N}}$ is an orthonormal Schauder basis for $L^2(\Omega)$.*

The following theorem shows that regular bases exist:

Theorem 22 *Let $\{h_j\}_{j\in\mathbb{N}} \subset V(\Omega)$ be an orthonormal Schauder basis for $L^2(\Omega)$ and let W be the space generated by **finite** linear combinations of the elements of $\{h_j\}_{j\in\mathbb{N}}$ (hence W is a dense subspace of $V(\Omega)$). Then there exists a regular basis $\{e_j\}_{j\leq\beta}$ for $V_{\mathcal{B}}(\Omega)$ such that*

$$e_j = h_j \quad \textit{for } j \leq \theta$$

where

$$\theta = \dim^* \left(V_{\mathcal{B}}(\Omega) \cap W^*\right).$$

Proof. Let $\left[\{h_j\}_{j\in\mathbb{N}}\right]^* = \{h_j\}_{j\in\mathbb{N}^*} \subset V(\Omega)^*$ be an orthonormal Schauder basis for $L^2(\Omega)^*$ and set

$$\theta = \max\{k \in \mathbb{N}^* \mid \forall j \leq k,\ h_j \in V_{\mathcal{B}}(\Omega)\}$$

Since $\{h_j\}_{j\in\mathbb{N}} \subset V(\Omega)$, θ is an infinite number in $\mathbb{N}^*$. Set $e_j = h_j$ for $j \leq \theta$. Now, we can take an orthonormal basis $\{e_j\}_{j\leq\beta}$ for $V_{\mathcal{B}}$ which contains $\{e_j\}_{j\leq\theta}$.
□

So every ultrafunction $u \in V_{\mathcal{B}}(\Omega)$ can be represented as follows:

$$u(x) = \sum_{j=1}^{\beta} u_j e_j(x) = \sum_{n=1}^{\theta} u_j h_j(x) + \sum_{j=\theta+1}^{\beta} u_j e_j(x) \tag{11}$$

with

$$u_j = \int^* u^*(x)\overline{e_j(x)}\, dx \in \mathbb{R}^*,\ j \leq \beta.$$

In particular, if $f \in L^2(\Omega)$ (or more in general if $f \in V'(\Omega)$), the numbers f_j, $j \in \mathbb{N}$, are complex numbers. The internal function $f_\Phi(x) = \sum_{j=1}^{\beta} f_j e_j$ is the orthogonal projection of $f^* \in L^2(\Omega)^*$ on $V_{\mathcal{B}}(\Omega) \subset L^2(\Omega)^*$.

Example: Let us see an example; we set

- $\Omega = [0,1]$;
- $V([0,1]) = C_0^2([0,1])$;
- $h_j(x) = \sqrt{2}\sin(j\pi x)$;

By Th. 22 there exists a regular basis $\{e_j(x)\}_{j\in J}$ which contains $\left\{\sqrt{2}\sin(j\pi x)\right\}_{j\in\mathbb{N}}$. With this assumptions, every vector $u \in V_{\mathcal{B}}([0,1])$ can be written as follows

$$u(x) = \sqrt{2}\sum_{n=1}^{\theta} u_j \sin(j\pi x) + \sum_{j=\theta+1}^{\beta} u_j e_j(x) \quad \text{with} \quad u_j = \int_0^1 u(x)e_j(x)dx.$$

4.2 Ultrafunctions and distributions

First, we will give a definition of the Dirac δ-ultrafunction concentrated in q.

Theorem 23 *Given a point $q \in \Omega$, there exists a unique function δ_q in $V_{\mathcal{B}}(\Omega)$ such that*

$$\forall v \in V_{\mathcal{B}}(\Omega),\ \int^* \delta_q(x)v(x)\ dx = v(q). \tag{12}$$

δ_q will called the Dirac ultrafunction in $V_{\mathcal{B}}(\Omega)$ concentrated in q. Moreover, we set $\delta = \delta_0$.

Proof. Let $\{e_j\}_{j\le\beta}$ be any orthonormal basis for $V_{\mathcal{B}}(\Omega)$ and set

$$\delta_q(x) = \sum_{j=1}^{\beta} e_j(q)e_j(x)$$

It is easy to check that $\delta_q(x)$ has the desired property; in fact

$$\begin{aligned}\int^* \delta_q(x)v(x)\ dx &= \int^* \sum_{j=1}^{\beta} e_j(q)e_j(x)v(x)\ dx \\ &= \sum_{j=1}^{\beta}\left(\int^* e_j(x)v(x)\ dx\right)e_j(q) = v(q).\end{aligned}$$

□

Next let us see how to associate an ultrafunction $T_\Phi = \Phi T$ to every distribution $T \in \mathcal{D}'$. Let $\{h_j\}_{j\in\mathbb{N}} \subset \mathcal{D}$ be an orthonormal Schauder basis for $L^2(\Omega)$; then, there exists an infinite number θ such that $\{h_j\}_{j\le\theta}$ is a basis for $V_{\mathcal{B}}(\Omega) \cap \mathcal{D}^*$; then, $T_\Phi(x)$ can be defined as follows:

$$T_\Phi(x) = \sum_{j=0}^{\theta} \langle T^*, h_j\rangle\, h_j(x) \tag{13}$$

Notice that this definition in independent of the choice of the basis since

$$\int^* T_\Phi(x)\overline{v(x)}\ dx = \langle T^*, v\rangle \quad \text{if} \quad v \in V_{\mathcal{B}}(\Omega) \cap \mathcal{D}^* \tag{14}$$

$$\int^* T_\Phi(x)\overline{v(x)}\ dx = 0 \quad \text{if} \quad v \in (V_{\mathcal{B}}(\Omega) \cap \mathcal{D}^*)^\perp . \tag{15}$$

where $(V_{\mathcal{B}}(\Omega) \cap \mathcal{D}^*)^\perp$ denotes the orthogonal complement of $V_{\mathcal{B}}(\Omega) \cap \mathcal{D}^*$ in $V_{\mathcal{B}}(\Omega)$.

Remark 24 *Here the reader must be careful to distinguish the Dirac ultrafunction as defined by 12 and the ultrafunction related to the distribution $\delta \in \mathcal{D}'$ which now we will call $\delta_{\mathcal{D}}$. In fact, by (13) we have that*

$$\delta_{\mathcal{D}}(x) = \sum_{j=0}^{\theta} h_j(0)h_j(x)$$

while

$$\delta(x) = \sum_{j=0}^{\theta} h_j(0)h_j(x) + \sum_{j=\theta+1}^{\beta} e_j(0)e_j(x)$$

where $\{h_j\}_{j\leq\theta} \cup \{e_j\}_{\theta+1\leq j\leq\beta}$ is a regular basis for $(V_{\mathcal{B}}(\Omega) \cap \mathcal{D}^*)^\perp$. Of course, if $\varphi \in \mathcal{D}$, we have that

$$\int^* \delta(x)\varphi(x)\ dx = \int^* \delta_{\mathcal{D}}(x)\varphi(x)\ dx = \varphi(0);$$

actually the above inequality holds for every $\varphi \in V_{\mathcal{B}}(\Omega) \cap \mathcal{D}^*$.

The above remark suggests the following definition:

Definition 25 *An ultrafunction $e_q \in V_{\mathcal{B}}(\Omega)$ is called a δ-type ultrafunction if*

$$\forall \varphi \in \mathcal{D}, \quad \int^* e_q(x)\varphi(x)\ dx \sim \varphi(q).$$

Following the classification of ultravectors, (14) and (15), the ultrafunctions can be classified as follows:

Definition 26 *An ultrafunction $u \in V_{\mathcal{B}}(\Omega)$ is called*

- ***standard*** *if $u \in V(\Omega)$ or, to be more precise, if there exists $f \in V(\Omega)$ such that $u = f^*$;*
- ***ideal*** *if it is not standard;*
- ***dual*** *ultrafunction if $u = \Phi(T)$ for some $T \in V(\Omega)'$;*
- ***distributional ultrafunction*** *if $u = \Phi(T)$ for some $T \in \mathcal{D}'$;*
- ***proper*** *ultrafunction if it is not a distributional ultrafunction.*

5 The Dirichlet problem

As first application of ultrafunctions, we will consider the following Dirichlet problem:

$$\begin{cases} u \in \mathcal{C}^2(\overline{\Omega}) \\ -\Delta u = f(x) \quad \text{for} \quad x \in \Omega \\ u(x) = 0 \qquad \text{for } x \in \partial\Omega \end{cases} \tag{16}$$

Here Ω is a bounded set in $\mathbb{R}^N$.

This problem is relatively simple and it will help to compare the Sobolev space approach with the ultrafunctions approach.

5.1 Generalized solutions

It is well known that problem (16) has a unique solution provided that $f(x)$ and $\partial\Omega$ are smooth. If they are not smooth, it is necessary to look for generalized solutions. In the Sobolev space approach, we transform problem (16) in the following one:

$$\begin{cases} u \in H_0^1(\Omega) \\ -\Delta u = f(x) \end{cases} \tag{17}$$

It is well known that this problem has a unique solution for any open set Ω and for a large class of f, namely for every $f \in H^{-1}(\Omega)$. In this approach, the boundary condition is replaced by the fact that $u \in H_0^1(\Omega)$, namely by the fact that u is the limit (in $H^1(\Omega)$) of a sequence of functions in $\mathcal{C}^2(\Omega)$ having compact support in Ω. The equation $-\Delta u = f$ is required to be satisfied in a weak sense:

$$-\int_\Omega u\Delta\varphi \; dx = \int_\Omega f\varphi \; dx \quad \forall\varphi \in \mathcal{D}(\Omega)$$

u itself is not a function but an equivalence class of functions defined $a.e.$ in Ω.

Now let us see the ultrafunctions approach. In this case we set $V_{\mathcal{B}}^{2,0}(\Omega) = \mathcal{B}\left[\mathcal{C}_0^2(\overline{\Omega})\right]$ and problem (16) can be written as follows:

$$\begin{cases} u \in V_{\mathcal{B}}^{2,0}(\Omega) \\ -\Delta_\Phi u = f(x) \quad \text{for} \quad x \in \ \Omega^* \end{cases} \tag{18}$$

where $\Delta_\Phi = \Phi \circ \Delta^* : V_{\mathcal{B}}^{2,0}(\Omega) \to V_{\mathcal{B}}^{2,0}(\Omega)$ is given by Def. 18.

The following result holds:

Theorem 27 *For any $f \in V_{\mathcal{B}}^{2,0}(\Omega)$, problem (18) has a unique solution.*

Proof. By definition, $V_{\mathcal{B}}^{2,0}(\Omega)$ is the Λ-limit of finite dimensional spaces $V_\lambda(\Omega) \subset \mathcal{C}_0^2(\overline{\Omega})$. For every $u \in \mathcal{C}_0^1(\overline{\Omega})$, by the Poincaré inequality, we have that

$$\int_\Omega \nabla u \cdot \nabla u \ dx \geq k \left\| u \right\|_{L^2(\Omega)}^2 .$$

In particular, the above inequality holds for any $u \in V_\lambda(\Omega)$. Now, let

$$\Phi_\lambda : L^2(\Omega) \to V_\lambda(\Omega),$$

be the orthogonal projection. For every $u, v \in V_\lambda(\Omega)$, we have that

$$\int_\Omega \nabla u \cdot \nabla v \ dx = \int_\Omega -\Delta u \ v \ dx$$

Then, by the Poicaré inequality,

$$-\Phi_\lambda \Delta : V_\lambda(\Omega) \to V_\lambda(\Omega)$$

is a positive definite symmetric operator. Then it is invertible. So we have that, for any $\lambda \in \Lambda$, there exists a unique $\bar{u}_\lambda \in V_\lambda(\Omega)$ such that

$$\forall v \in V_\lambda(\Omega), \ \int_\Omega -\Delta \bar{u}_\lambda v \ dx = \int_\Omega f_\lambda v \ dx \tag{19}$$

where $f_\lambda \in V_\lambda(\Omega)$ is such that $f = \lim_{\lambda \uparrow V} f_\lambda$. If we take the Λ-limit in this equality, we get

$$\forall v \in V_{\mathcal{B}}^{2,0}(\Omega), \ -\int_\Omega^* \Delta^* \bar{u} \ v \ dx = \int_\Omega^* f v \ dx \tag{20}$$

where

$$\bar{u} = \lim_{\lambda \uparrow V} \bar{u}_\lambda$$

and hence, by (7), we get

$$-\Delta_\Phi \bar{u} = f$$

The uniqueness follows from the uniqueness of $\bar{u}_\lambda$.
□

Remark 28 *This example shows quite well the general strategy to solve problems within the framework of ultrafunctions. First you solve a finite dimensional problem and then you take the Λ-limit. Since the Λ-limit exists for any sequence of mathematical objects, the solvability of the finite dimensional approximations imply the existence of a generalized solution.*

The solution is a function $\bar{u} : \overline{\Omega}^* \to \mathbb{R}^*$; $\bar{u}$ is defined for every $x \in \overline{\Omega}^*$, and we have that $u(x) = 0$ for $x \in \partial\Omega^*$. So the boundary condition can be interpreted "classically" while this is not possible in $H_0^1(\Omega)$. If problem (16) has a solution $U \in \mathcal{C}^2(\overline{\Omega})$, then

$$\bar{u} = U^*.$$

If problem (17) has a solution $U \in H_0^1(\Omega)$, then we have that

$$\int_\Omega U\varphi \, dx \sim \int_\Omega^* \bar{u}\varphi \, dx \quad \forall \varphi \in \mathcal{C}_0^2(\overline{\Omega})$$

Notice that in the above formula the left hand side integral is a Lebesgue integral while in the right hand side, $\int^*$ is the $*$-transform of the Riemann integral; the integral make sense since $\bar{u}, \varphi \in \left[\mathcal{C}_0(\overline{\Omega})\right]^*$. In the theory of ultrafunctions, the Lebesgue integral seems to be not so necessary.

There are interesting and physically relevant cases in which the generalization of the Dirichlet problem cannot be treated within the Sobolev space $H_0^1(\Omega)$. For example, consider the problem:

$$\begin{cases} -\Delta u = \delta_y & \text{for } x \in \Omega \\ u(x) = 0 & \text{for } x \in \partial\Omega \end{cases} \tag{21}$$

where δ_y is the Dirac measure concentrated at $y \in \Omega$. This problem is quite natural in potential theory; in fact u represents the potential generated by a point source (and usually it is called Green function). However this problem does not have solution in $H_0^1(\Omega)$ since $\delta \notin H^{-1}(\Omega)$. Actually, with some work, it is possible to prove that it has a "generalized solution" in $H_0^1(\Omega) + \mathcal{E}'(\Omega)$.

However, in the framework of ultrafunction, problem (21) is nothing else but a particular case of problem (18).

However, if $f \in V_{\mathcal{B}}^{2,0}(\Omega)$ is a proper ultrafunction, (namely, f cannot be associated to a distribution via (14) and (15)), problem (18) has a solution which cannot be interpreted as a distribution solution. For example, you can take $f = \delta(x)^2$. Remember that $\delta(x)^2$, in the ultrafunction theory, makes sense by Def. 18.

Remark 29 *If you take $f = \delta^2$ you get a well posed mathematical problem, but, most likely, it does not represent any "physically" relevant phenomenon. However, it is possible to choose some proper ultrafunction $f \in V_{\mathcal{B}}^{2,0}(\Omega)$ which models physical phenomena. For example*

$$f(x) = \sin \alpha(\mathbf{n} \cdot x);\ \mathbf{n} \in \mathbb{R}^N,\ |\mathbf{n}| = 1,\ \alpha \in \mathbb{R}^* \text{ infinite}, x \in K^*, K \subset\subset \Omega$$

might represent a electrostatic problem in a sort of periodic medium such as a crystal. Here K represent the support of the crystal and $f(x)$ represents its charge density; it consists of periodic layers of positive and negative charges at a distance of $\frac{1}{\pi\alpha}$. From a macroscopic point of view the solution is 0, but at the microscopic level this is not the case. In fact the solution u of problem (18) does not vanish, even if it can be proved that

$$\forall v \in \mathcal{C}^2(\overline{\Omega}),\ \int_\Omega^* \bar{u}\ v\ dx \sim 0.$$

5.2 The variational approach

Looking at problem (16) from a variational point of view, the comparison between the Sobolev space approach and the ultrafunctions approach becomes richer.

It is well known that the equation (16) is the Euler-Lagrange equation of the energy functional

$$J(u) = \int_\Omega \left(\frac{1}{2} |\nabla u|^2 - fu \right)\ dx$$

Thus a minimizer of $J(u)$ on $\mathcal{C}_0^2(\overline{\Omega})$ solves the problem. However, if $f(x)$ and $\partial\Omega$ are not smooth a minimizing sequence does not converge in $\mathcal{C}_0^2(\overline{\Omega})$ and also when it converges, it can be proved only by making hard estimates.

On the other hand, if you define $H_0^1(\Omega)$ as the closure of $\mathcal{D}(\Omega)$ with respect to the norm

$$\|u\|_{H_0^1} = \sqrt{\int_\Omega |\nabla u|^2\, dx}$$

the functional $J(u)$ becomes $\frac{1}{2}\|u\|^2_{H^1_0} - \int_\Omega fu\,dx$ and it is immediate to see that it has a minimizer provided that $f \in H^{-1}(\Omega)$.

If you consider problem (21), the trouble with the energy functional is that the energy

$$J(u) = \int_\Omega \frac{1}{2}|\nabla u|^2\,dx - u(y), \quad u \in \mathcal{C}^2_0(\overline{\Omega})$$

is not bounded below and J cannot be extended to all $H^1_0(\Omega)$.

Instead, if we use the ultrafunctions approach, the energy

$$J(u) = \int^*_{\Omega^*} \left(\frac{1}{2}|\nabla u|^2 - \delta_y u\right) dx, \quad u \in V^{2,0}_{\mathcal{B}}(\Omega)$$

is well defined and it makes sense to look for a minimizer in $V^{2,0}_{\mathcal{B}}(\Omega)$. For every $\lambda \subset \mathcal{C}^2_0(\overline{\Omega}) \cap \Lambda$, $J(u)$ has a minimizer u_λ in $V_\lambda(\Omega) \subset \mathcal{C}^2_0(\overline{\Omega})$, and hence, if you set

$$\bar{u} = \lim_{\lambda \uparrow V} u_\lambda,$$

we have that

$$J(\bar{u}) = \lim_{\lambda \uparrow V} \left[\int_\Omega \frac{1}{2}|\nabla u_\lambda|^2\ dx - u_\lambda(y)\right]$$

minimizes $J(u)$ in $V^{2,0}_{\mathcal{B}}(\Omega)$. Clearly, for some values of u, $J(u)$ may assume infinite values in $\mathbb{R}^*$, but this is not a problem, actually in my opinion, this is one of the main reason to legitimate the use non-Archimedean fields. In fact in the framework of NAM, it is possible to make models of the physical world in which there are material points with a finite charge. They have an "infinite" energy, but, nevertheless, we can make computations and if necessary to evaluate it. The epistemological (and very interesting) issue relative to the meaning of their "physical existence" should not prevent their use.

6 The bubbling phenomenon relative to the Sobolev critical exponent

The bubbling phenomenon relative to the critical Sobolev exponent is the model problem which has inspired this work. In general (at least in the simplest cases), the bubbling phenomenon consists in minimizing sequences whose *mass concentrate to some points*; however their "limit" does not exist in any Sobolev space and not even in any distribution space due to the "strong" non-linearity of the problem. Nevertheless, these problems have

been extensively studied and we know a lot of facts relative to the minimizing sequences (or more in general to non-converging Palais-Smale sequences) which, up to an equivalence relation, are called *critical points at infinity* (see [1]). The literature on this topic is huge (you can find part of it in [7]). We refer also to [1], [8] and [7] for an exposition of the utility of knowing the properties of the critical points at infinity.

Ultrafunction theory seems to be an appropriate tool to deal with these kind of problems.

6.1 Description of the problem

Let us consider the following minimization problem:

$$\min_{u\in\mathfrak{M}_p} J(u)$$

where

$$J(u) = \int_\Omega |\nabla u|^2 \ dx$$

and

$$\mathfrak{M}_p = \left\{ u \in \mathcal{C}_0^2(\overline{\Omega}) : \ \int_\Omega |u|^p \ dx = 1 \right\}$$

Here Ω is a bounded set in $\mathbb{R}^N$ with smooth boundary, $N \geq 3$ and $p > 2$. If J has a minimizer, it is a solution of the following elliptic eigenvalue problem:

$$\begin{cases} u \in \mathcal{C}_0^2(\overline{\Omega}) \\ -\Delta u = \lambda u^{p-1} \quad \text{for} \quad x \in \Omega \\ u(x) > 0 \qquad \text{for} \quad x \in \Omega \\ \int_\Omega |u|^p \, dx = 1 \end{cases} \tag{22}$$

As usual in the literature, we set

$$2^* = \frac{2N}{N-2};$$

2^* is called the critical Sobolev exponent for problem (22) (notice that this "$*$" has nothing to do with the natural extension). Moreover, we set

$$m_p := \inf_{u\in\mathfrak{M}_p} J(u)$$

The following facts are well known (see e.g. [7] and references):

- (i) if $2 < p < 2^*$, then $m_p > 0$ and it is achieved; hence problem (22) has a solution.

- (ii) if $p = 2^*$, then $m_{2^*} > 0$ and it is achieved only if $\Omega = \mathbb{R}^N$; however there are particular domains Ω such that (22) has a solution (which, of course, is not a minimizer of J, but a critical point).
- (iii) if $p > 2^*$, then $m_p = 0$ and it is not achieved.

Probably, the most interesting case is the second one (the critical exponent case) since it presents many interesting phenomena. If u_n is a minimizing sequence, it has a subsequence u_n' which concentrates to some point $x_0 \in \overline{\Omega}$; more exactly, $u_n' \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ and strongly in $H_0^1(\Omega \setminus B_\varepsilon(x_0))$; consequently, $|u_n'|^p \rightharpoonup \delta_{x_0}$ weakly in $\mathcal{D}'(\Omega)$, but $(\delta_{x_0})^{1/p}$ cannot be interpreted as a generalized solution in the framework of the distribution theory just because $(\delta_{x_0})^{1/p}$ makes no sense. This phenomenon is called "bubbling" and probably problem (22) with $p = 2^*$ is the simplest problem which presents it. Similar phenomena occur in many other variational problems such as the Yamabe problem, the Kazdan-Warner problem, in the study of harmonic maps between manifolds, in minimal surfaces theory, in the Yang-Mills equations etc.

Let us go back to discuss the concentration phenomenon of a minimizing sequence. Not all the points of $\overline{\Omega}$ have the same "dignity" as concentration points. Let us explain what do we mean.

Let

$$u_p,\ p \in (2, 2^*), \tag{23}$$

be a minimizer of $J(u)$ on the set $\mathfrak{M}_p$. If $p \to 2^*$ from the left, it is well known that

$$\lim_{p \to (2^*)^-} m_p = m_{2^*}$$

and that

$$v_p := \frac{u_p}{\int_\Omega |u_p|^{2^*}\, dx}, \tag{24}$$

is a minimizing sequence of J on $\mathfrak{M}_{2^*}$. If, for every $u \in \mathfrak{M}_{2^*}$, we set

$$\mathfrak{B}(u) = \int_\Omega x\, |u|^{2^*}\, dx$$

then we have that, in the generic case,

$$\lim_{p \to (2^*)^-} \mathfrak{B}(v_p) = \overline{x}$$

where $\overline{x}$ is an interior point of Ω. Thus, in this sense, $\overline{x}$ is a "special" concentration point. If we apply ultrafunction theory, the world "special" will get a new meaning; in fact $\overline{x}$ will be characterized as the point infinitely close to the concentration point of the generalized solution. This issue will be further discussed in the next section.

6.2 Generalized solutions

The minimization problem considered in the previous section can be studied in the framework of the ultrafunctions. In this framework the problem takes the following form:

$$\min_{u\in\widetilde{\mathfrak{M}}_p} J(u) \tag{25}$$

where

$$J(u)=\int_\Omega^* |\nabla u|^2 \ dx$$

and

$$\widetilde{\mathfrak{M}}_p=\left\{u\in V_{\mathcal{B}}^{2,0}(\overline{\Omega}) \mid \int_\Omega^* |u|^p \ dx=1\right\}$$

where $V_{\mathcal{B}}^{2,0}(\overline{\Omega})=\mathcal{B}\left[\mathcal{C}_0^2(\overline{\Omega})\right]$.

Theorem 30 *For every $p>2$, problem (25) has a solution $\tilde{u}_p$. If we set $\widetilde{m}_p=J(\tilde{u}_p)$, we have the following*

- *(i) if $2<p<2^*$, then $\widetilde{m}_p=m_p\in\mathbb{R}^+$ and there is at least one standard minimizer $\tilde{u}_p$, namely $\tilde{u}_p\in\mathcal{C}_0^2(\overline{\Omega})$;*
- *(ii) if $p=2^*$,(and $\Omega\neq\mathbb{R}^N$), then $\widetilde{m}_{2^*}=m_{2^*}+\varepsilon$ where ε is a positive infinitesimal;*
- *(iii) if $p>2^*$, then $\widetilde{m}_p=\varepsilon_p$ where ε_p is a positive infinitesimal.*

Proof. The proof of this theorem is a simple application of the nostandard methods. We will describe it with some details for the reader not acquainted with these methods.

We set

$$\tilde{u}_p=\lim_{\lambda\uparrow\mathcal{C}_0^2(\overline{\Omega})} u_{p,\lambda}$$

where $u_{p,\lambda}$ is the minimizer of $J(u)$ on the set $\mathfrak{M}_p\cap V_\lambda(\overline{\Omega})$; $V_\lambda(\overline{\Omega})=Sp(\lambda)\subset\mathcal{C}_0^2(\overline{\Omega})$. We recall that $\mathfrak{M}_p\cap V_\lambda(\overline{\Omega})\neq 0$ for λ in a qualified set and that the minimum exists since $V_\lambda(\overline{\Omega})$ is a finite dimensional vector space and hence $\mathfrak{M}_p\cap V_\lambda(\overline{\Omega})$ is compact. If we set

$$m_{p,\lambda}:=\min_{u\in\mathfrak{M}_p\cap V_\lambda(\overline{\Omega})} J(u),$$

taking the Λ-limit, we have that

$$\widetilde{m}_p := \lim_{\lambda \uparrow \mathcal{C}_0^2(\overline{\Omega})} m_{p,\lambda} = \min_{u \in \widetilde{\mathfrak{M}}_p} J(u).$$

So the existence result is proved. Now let us prove the second part of the theorem:

(i) If you take $\lambda_0 = \{u_p\}$, where u_p is given by (23) then for every $\lambda \supseteq \lambda_0$, we have that

$$m_{p,\lambda} := \min_{u \in \mathfrak{M}_p \cap V_\lambda(\overline{\Omega})} J(u) = J(u_p) = m_p$$

and hence, taking the Λ-limit, we have that $\widetilde{m}_p = m_p$.

(ii) It is well known that the value $\widetilde{m}_{2^*}$ is not achieved by any function $u \in \mathfrak{M}_{2^*} \cap V_\lambda(\overline{\Omega})$; then $m_{2^*,\lambda} > m_{2^*}$, and hence, taking the Λ-limit, we have that $\widetilde{m}_{2^*} > m_{2^*}$. On the other hand, for every $b \in \mathbb{R}^+$, there exists $u \in \mathfrak{M}_{2^*}$ such $J(u) \leq m_{2^*} + b$, and hence

$$\widetilde{m}_{2^*} = J(\tilde{u}_{2^*}) \leq J(u) \leq m_{2^*} + b,$$

and so, by the arbitrariness of b, we get that $\widetilde{m}_{2^*} \sim m_p$.

(iii) follows by the same argument used in (ii) replacing m_{2^*} with 0.

□

The next theorem shows that, for $p = 2^*$, the solution $\tilde{u}$ concentrates where it is expected to do.

Theorem 31 *Suppose that problem (22) (with $p = 2^*$) has a unique minimum $\tilde{u}$ and set*

$$\xi = \mathfrak{B}^* (\tilde{u}) := \int_\Omega^* x \left|\tilde{u}\right|^{2^*} dx \in \Omega^*.$$

Then

$$\xi \sim \lim_{p \to (2^*)^-} \mathfrak{B} (v_p).$$

where v_p is defined by (24).

Proof. Fix $r \in \mathbb{R}^+$. We want to prove that, for p sufficiently close to 2^*, we have that

$$d^*(\mathfrak{B} (v_p), \xi) \leq r$$

where d^* denotes the distance in $\left(\mathbb{R}^N\right)^*$. We have that

$$\xi = \lim_{\lambda \uparrow \mathcal{C}_0^2(\overline{\Omega})} x_\lambda \tag{26}$$

where $x_\lambda = \mathfrak{B}(u_\lambda)$ and u_λ is a minimizer of J on the manifold $\mathfrak{M}_{2^*} \cap V_\lambda$. Let $\tilde{u}$ be the minimum of J on $\widetilde{\mathfrak{M}}_{2^*}$, and apply Th. 15, to the relation $\mathcal{R}$ defined as follows:

$$u_\lambda \mathcal{R} \left(\mathfrak{M}_{2^*} \cap V_\lambda(\Omega) \right)$$

if and only if

$$u_\lambda \text{ is the unique minimum of } J \text{ on } \mathfrak{M}_{2^*} \cap V_\lambda(\Omega).$$

Then by Th. 15, there exists a qualified set $Q \subset \Lambda(V)$, such that, for every $\lambda \in Q$, u_λ is the unique minimum of J on $\mathfrak{M}_{2^*} \cap V_\lambda(\Omega)$.

Thus $\exists b \in \mathbb{R}^+, \exists \lambda_0, \forall \lambda \geq \lambda_0, \lambda \in Q, \forall u \in \mathfrak{M}_{2^*} \cap V_\lambda$

$$J(u) < m_{2^*} + b \Rightarrow d^*(\mathfrak{B}(u), x_\lambda) \leq \frac{r}{2}$$

and hence, may be taking a bigger λ_0, using (26), we get

$$J(u) < m_{2^*} + b \Rightarrow d^*(\mathfrak{B}(u), \xi) \leq r \tag{27}$$

Now, let v_p be the function defined by (24); it is well known that

$$\lim_{p \to (2^*)^-} J(v_p) = m_{2^*}$$

Then we can take p so close to 2^* so that

$$J(v_p) \leq m_{2^*} + b.$$

Since $v_p \in \mathfrak{M}_{2^*} \cap V_\lambda$, for every $\lambda \geq \lambda_0 \cup \{v_p\}$, $\lambda \in Q$, by (27), we get that

$$d^*(\mathfrak{B}(v_p), \xi) \leq r.$$

□

Remark 32 *If J does not have a unique minimum, but a set of minimizers, we set*

$$\Gamma = \{\xi \in \Omega^* : \ \xi = \mathfrak{B}(\tilde{u}) \ \textit{where } \tilde{u} \textit{ is a minimizer}\}.$$

Then, arguing as in the proof of the above theorem, it is easy to get the following result: let $p_n \to (2^*)^-$, let $x_n = \mathfrak{B}(v_{p_n})$ and let x'_n be a converging subsequence of x_n. Then there exists $\xi \in \Gamma$ such that

$$\xi \sim \lim_{n \to \infty} x'_n$$

7 Ultrafunctions and Quantum Mechanics

In this section we will describe an application of the previous theory to the formalism of Quantum Mechanics. In the usual formalism, a physical state is described by a unit vector ψ in a Hilbert space $\mathcal{H}$ and an observable by a self-adjoint operator defined on it. In the ultravectors/ultrafunctions formalism, a physical state is described by a unit vector ψ in a hyperfinite space of ultravectors $V_{\mathcal{B}}$ and an observable by a Hermitian operator defined on it.

We think that the ultravectors approach presents the following advantages:

- once you have learned the basic facts of the Λ-theory, the formalism which you get is easier to handle since it is based on the matrix theory on finite vector spaces rather than on unbounded self-adjoint operators in Hilbert spaces;
- this approach is closer to the "infinite" matrix approach of the beginning of QM before the work of von Neumann and also closer to the way of thinking of the theoretical physicists and chemists;
- all observables (hyperfinite matrices) have infinitely many eigenvectors; so the continuous spectrum can be considered as a set of eigenvalues infinitely close to each other;
- the distinction between standard and ideal ultravectors has a physical meaning;
- the dynamics does not present any difficulty since it is given by the exponential matrix relative to the Hamiltonian matrix.

Clearly it is too early to know if this formalism will lead to some new physically relevant fact; in any case we think that it is worthwhile to investigate it. In this paper we limit ourselves only to some very general remark.

7.1 The axioms of Quantum Mechanics

We start giving a list of the main axioms of quantum mechanics as it is usually given in any textbook and then we will compare it with the alternative formalism based on ultravectors.

Classical axioms of QM

Axiom C1. A physical state is described by a unit vector ψ in a Hilbert space $\mathcal{H}$.

Axiom C2. An observable is represented by a self-adjoint operator A on $\mathcal{H}$.

(a) The set of observable outcomes is given by the eigenvalues μ_j of A.

(b) After an observation/measurement of an outcome μ_j, the system is left in a eigenstate ψ_j associated with the detected eigenvalue μ_j.

(b) In a measurement the transition probability $\mathcal{P}$ from a state ψ to an eigenstate ψ_j is given by

$$\mathcal{P} = \left|\left(\psi, \psi_j\right)\right|^2.$$

Axiom C3. The evolution of a state is given by the Shroedinger equation

$$i\frac{\partial \psi}{\partial t} = H\psi$$

where H, the Hamiltonian operator, is a self-adjoint operator representing the energy of the system.

Axioms of QM based on ultravectors

Axiom U1. A physical system is described by a complex valued-ultravector space $V_{\mathcal{B}} = \mathcal{B}[V]$; a state of this system is described by a unit ultravector vector ψ in $V_{\mathcal{B}}$.

Axiom U2. An observable is represented by a Hermitian operator A on $V_{\mathcal{B}}$.

(a) The set of observable outcomes is given by $sh\left(\mu_j\right)$ where μ_j is an eigenvalue of A.

(b) After an observation/measurement of an outcome $sh\left(\mu_j\right)$, the system is left in an eigenstate ψ_j associated with the detected eigenvalue μ_j.

(b) In a measurement the transition probability $\mathcal{P}$ from a state ψ to an eigenstate ψ_j is given by

$$\mathcal{P} = \left|\left(\psi, \psi_j\right)\right|^2.$$

Axiom U3. The evolution of the state of a system is given by the Shroedinger equation

$$i\frac{\partial \psi}{\partial t} = H\psi \tag{28}$$

where H, the Hamiltonian operator, representing the energy of the system.

Axiom U4. Only the physical states represented by standard vectors (namely vectors in V) can be produced in laboratory.

7.2 Discussion of the axioms

AXIOM 1. In the classical formalism, a physical system is not described only by a given Hilbert space as axiom C1 claims, but by an Hilbert space and the domain of a self-adjoint realization of the Hamiltonian operator. On the contrary, in the ultravectors formalism the physical system is described just by the space $V_{\mathcal{B}}$. Let see an example:

A particle in a box. For simplicity, we consider a one-dimensional model and suppose that the box is modelled by the interval $[0,1]$. Clearly, the Hilbert space $L^2(0,1)$ is not sufficient to describe the system but it is necessary to give the Hamiltonian

$$H : H^2(0,1) \cap H_0^1(0,1) \to L^2(0,1)$$

defined by

$$H\psi = -\frac{1}{2m}\Delta\psi \tag{29}$$

where $\Delta\psi$ must be intended in the sense of distribution (here m denotes the mass of the particle and we have assumed $\hbar = 1$).

A particle in a ring. Now suppose that a point-particle is constrained in a ring of length 1. Also in this case any state can be represented by a vector in the Hilbert space $L^2(0,1)$, but in order to describe the system is necessary to give a different selfadjoint realization of the Hamiltonian operator, namely an operator having the form (29), but defined on the domain

$$H : H^2_{per}(0,1) \to L^2(0,1)$$

where $H^2_{per}(0,1)$ is the closure in the H^2 norm of the space

$$\mathcal{C}^2_{per}[0,1] = \left\{\psi \in \mathcal{C}^2([0,1],\mathbb{C}) \mid \psi(0) = \psi(1);\ \psi'(0) = \psi'(1)\right\}$$

Now let us see how these two cases can be described in the ultrafunctions formalism.

A particle in a box. In this case, the system is described by the space

$$V^{2,0}_{\mathcal{B}}[0,1] := \mathcal{B}\left[\mathcal{C}^2_0[0,1]\right]$$

The Hamiltonian operator H is given by the canonical extension of $-\frac{1}{2m}\Delta$ to $\mathcal{B}[\mathcal{C}^2_0[0,1]]$.

A particle in a ring. In this case, the system is described by the space

$$V^{2,per}_{\mathcal{B}}[0,1] := \mathcal{B}\left[\mathcal{C}^2_{per}[0,1]\right]$$

and the Hamiltonian operator H is given by the canonical extension of $-\frac{1}{2m}\Delta$ to $\mathcal{B}\left[\mathcal{C}^2_{per}\left[0,1\right]\right]$.

Thus in the ultrafunctions description, different physical systems give different ultrafunction spaces; on the contrary, the Hamiltonian is given by the unique canonical extension of $-\frac{1}{2m}\Delta$ in the relative spaces.

AXIOM 2. In the ultrafunction formalism, the notion of self-adjoint operator is not needed. In fact osservables can be represented by internal Hermitian operators. It follows that any observable has exactly $\beta = \dim^*(V_{\mathcal{B}})$ eigenvalues (of course, if you take account of their multiplicity). No essential distinction between eigenvalues and continuous spectrum is required. For example, consider the eigenvalues of the position operator $\hat{q}$ of a free particle. The eigenfunction relative to an eigenvalue $q \in \mathbb{R}$ is an ultrafuncion of δ-type concentrated at the point q (see Def. 25).

In general the eigenvalues μ's of an internal Hermitian operator A are hyperreal numbers, and hence, assuming that a measurement gives a real number, we have imposed in Axiom 2 that the outcome of an experiment is $sh(\mu)$. However, we think that the probability is better described by the hyperreal number $\left|\left(\psi,\psi_j\right)\right|^2$ rather than the real number $sh(\left|\left(\psi,\psi_j\right)\right|^2)$ (see [6] for a presentation and discussion of the Non Archimedean Probability). For example, let $\psi \in \mathcal{D}$ be the state of a system; the probability of finding a particle in the position q is given by

$$\left|\int \psi(x)\eta e_q(x)dx\right| = \eta\left|\psi(q)\right|$$

where e_q is a δ-type function and the normalization factor

$$\eta = \left\|e_q\right\|^{-1}_{(L^2)^*} \sim 0$$

is an infinitesimal number.

AXIOM 3. Since H is an internal operator defined on a hyperfinite vector space it can be represented by an Hermitian hyperfinite matrix and hence the evolution operator of (28) is the exponential matrix e^{tH}.

AXIOM 4. In ultrafunction theory, the mathematical distinction between the standard states and the ideal states is intrinsic and it does not correspond to anything in the usual formalism. The point is to know if it corresponds to something physically meaningful. Basically, we can say that the standard states can be prepared in a laboratory, while the ideal states represent "extreme" situations useful in the foundations of the theory and in thought

experiments (gedankenexperiment). For example the Dirac δ-measure is not a standard state but an ideal state and it represents a situation in which the position of a particle is perfectly determined. Clearly this situation cannot be produced in a laboratory, but nevertheless it is useful in our description of the physical world. The standard states are represented by functions in V which is chosen depending on the model of the physical system. The other states (namely, the states in $V_{\mathcal{B}} \backslash V$) are the ideal states. This situation makes more explicit something which is already present in the classical approach. For example, in the Shroedinger representation of a free particle in $\mathbb{R}^3$, consider the state

$$\psi(x) = \frac{\varphi(x)}{|x|}, \ \varphi \in \mathcal{D}(\mathbb{R}^3), \ \varphi(0) > 0.$$

We have that $\psi(x) \in L^2(\mathbb{R}^3)$ but this state cannot be produced in a laboratory, since the expected value of its energy

$$(H\psi, \psi) = \frac{1}{2m} \int |\nabla \psi|^2 \, dx$$

is infinite. In other words, Axiom 4 makes formally precise something which is already present (but hidden) in the classical theory. This point will be discussed also in the next section.

7.3 The Heisenberg algebra

In this section we will apply ultrafunction theory to the description of a quantum particle via the algebraic approach. For simplicity here we consider the one-dimensional case. The states of a particle are defined by the observables q and p which represent the position and the momentum respectively. A quantum particle is described by the algebra of observables generated by p and q according to the following commutation rules:

$$[p, q] = i, \quad [p, p] = 0, \quad [q, q] = 0$$

The algebra generated by p and q with the above relations is called the Heisenberg algebra and denoted by $\mathfrak{A}_H$. The Heisenberg algebra does not fit in the general theory of C^*-algebras since both p and q are not bounded operator. The usual technical solution to this problem is done via the Weyl operators and the Weyl algebra (for more details and a discussion on this point we refer to [11]).

Let us see an alternative approach via ultrafunction theory. First of all we take a representation of $\mathfrak{A}_H$, namely an algebra homomorphism

$$J : \mathfrak{A}_H \to \mathfrak{L}(V)$$

where $\mathfrak{L}(V)$ is the algebra of the linear operators on a complex vector space $V \subset H \in \mathcal{U}$ where H is an Hilbert space and $\mathcal{U}$ is our universe (see section 2.2). To fix the ideas, we can consider the following "classical example":

$$H = L^2(\mathbb{R}); \quad V = \mathcal{S};$$

$$J(p) = -i\partial; \quad J(q) = x.$$

The quantum system of a particle will be described by the ultravector space $V_{\mathcal{B}} = \mathcal{B}[V]$. The operators $J(p)$ and $J(q)$ can be extended to the space $V_{\mathcal{B}}$ according to definition (18); such extensions will be called $\hat{p}$ and $\hat{q}$ respectively. $\hat{p}$ and $\hat{q}$ are Hermitian operators and hence $V_{\mathcal{B}}$ has an othonormal basis generated by the eigenfunctions of $\hat{p}$ or $\hat{q}$. Let $\{e_a\}_{a\in\Sigma}$ be the eigenfunctions of $\hat{q}$ corresponding to the eigenvalue $a \in \Sigma \subset \mathbb{R}^*$. A very interesting fact is that the eigenfunctions violate the Heisenberg relation $[\hat{p}, \hat{q}] = i$.

To see this fact we argue indirectly. Assume that the Heisenberg relation holds; then

$$([\hat{p}, \hat{q}]\, e_a, e_a) = i \left\| e_a \right\|^2.$$

On the other hand, by a direct computation, we get:

$$\begin{aligned} ([\hat{p}, \hat{q}]\, e_a, e_a) &= ((\hat{p}\hat{q} - \hat{q}\hat{p})\, e_a, e_a) = (\hat{p}\hat{q}e_a, e_a) - (\hat{q}\hat{p}e_a, e_a) \\ &= (\hat{q}e_a, \hat{p}e_a) - (\hat{p}e_a, \hat{q}e_a) = a\,(e_a, \hat{p}e_a) - a\,(\hat{p}e_a, e_a) = 0. \end{aligned}$$

This fact is consistent with the Axiom **U4** which establishes that the ideal states cannot be produced in laboratory. According to this description of QM, the uncertainty relations hold only for the limitation of the experimental apparatus. In a laboratory you can prepare a state corresponding to a function ψ in the space $V = \mathcal{S}$, but you cannot prepare a state such as $e_a \in V_{\mathcal{B}} \backslash \mathcal{S}$ which corresponds to a particle which is exactly in the position a.

References

[1] Bahri, A., *Critical points at infinity in the variational calculus*, in: Partial differential equations (Rio de Janeiro, 1986), 1–29, Lecture Notes in Math., **1324**, Springer, Berlin, (1988), p. 1-29.

[2] Benci, V., *A construction of a nonstandard universe*, in: Advances of Dynamical Systems and Quantum Physics (S. Albeverio et al., eds.), World Scientific, Singapore, (1995), p. 11-21.

[3] Benci, V., *An algebraic approach to nonstandard analysis,* in: Calculus of Variations and Partial differential equations, p. 285-307, (G.Buttazzo, et al., eds.), Springer, Berlin, (1999), p.285-326.

[4] Benci, V., Galatolo, S., Ghimenti, M., *An elementary approach to Stochastic Differential Equations using the infinitesimals*, in Contemporary Mathematics, **530**, Ultrafilters across Mathematics, American Mathematical Society, (2010), p. 1-22.

[5] Benci, V., Di Nasso, M., *Alpha-theory: an elementary axiomatic for nonstandard analysis*, Expositiones Mathematicae **21** (2003) p. 355–386.

[6] Benci, V., Horsten, H., Wenmackers, S., *Non-Archimedean probability*, submitted, URL: http://arxiv.org/abs/1106.1524.

[7] Chabrowski, J., *Variational methods for potential operator equations, with applications to nonlinear elliptic equations.* Walter de Gruyter & Co., Berlin, 1997.

[8] H. Brezis, L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving criticalSobolev exponents,* Comm. Pure Appl. Math., 36 (1983), p. 437–477

[9] Ehrlich, Ph., *The Rise of non-Archimedean Mathematics and the Roots of a Misconception I: The Emergence of non-Archimedean Systems of Magnitudes,* Arch. Hist. Exact Sci. 60 (2006) 1–121, Identifier (DOI) 10.1007/s00407-005-0102-4.

[10] Keisler, H.J., *Foundations of Infinitesimal Calculus*, Prindle, Weber & Schmidt, Boston 1976. [This book is now freely downloadable at: http://www.math.wisc.edu/~keisler/foundations.html]

[11] Strocchi, F., *An introduction to the mathematical structure of quantum mechanics*, World Scientific, Singapore, 2005.