THE n-ARY ADDING MACHINE AND SOLVABLE GROUPS

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Abstract. We describe under a various conditions abelian subgroups of the automorphism group $\mathrm{Aut}(T_n)$ of the regular n-ary tree T_n, which are normalized by the n-ary adding machine $\tau = (e, ..., e, \tau)\sigma_\tau$ where σ_τ is the n-cycle $(0, 1, ..., n-1)$. As an application, for $n = p$ a prime number, and for $n = p^2$ when $p = 2$, we prove that every finitely generated soluble subgroup of $\mathrm{Aut}(T_n)$, containing τ is an extension of a torsion-free metabelian group by a finite group.

Contents

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1. Introduction

Adding machines have played an important role in dynamical systems, and in the theory of groups acting on trees : see [1, 2, 5, 4, 10].

An element α in the automorphism group $\mathcal{A}_n = \mathrm{Aut}(T_n)$ of the n-ary tree T_n, is represented as $\alpha = \alpha|_\phi = (\alpha|_0, ..., \alpha|_{n-1})\,\sigma_\alpha$ where ϕ is the empty sequence from the free monoid $\mathcal{M}$ generated by $Y = \{0, 1, .., n-1\}$, where $\alpha|_i \in \mathcal{A}_n$ ($i \in Y$)-called 1st level states of α- and where σ_α (the activity of α) is a permutation in the symmetric group Σ_n on Y extended 'rigidly' to act on the tree; if $= e$, we say α inactive. In applying the same representation to $\alpha|_0$ we produce $\alpha|_{0i}$ where $i \in Y$ and in general we produce $\{\alpha|_u \mid u \in \mathcal{M}\ \}$ the set of *states* of α. Following this notation, the n-ary adding machine is represented as $\tau = (e, ..., e.\tau)\sigma_\tau$ where e is the identity automorphism an σ_τ is the regular permutation $\sigma = (0, 1, ..., n-1)$. In this sense the adding machine may be viewed as an infinite variant of the regular permutation which often appears in geometric and combinatorial contexts.

A characteristic feature of τ is that its n-th power τ^n is the diagonal automorphism of the tree $(\tau, ..., \tau)$. This fact implies that the centralizer of the cyclic group $\langle\tau\rangle$ in $\mathcal{A}_n$ is equal to its topological closure $\overline{<\tau>}$ in $\mathcal{A}_n$ seen as a topological group with respect to the the natural topology induced by the tree.

A large variety of subgroups of $\mathcal{A}_n$ which contain τ have been constructed, including finitely generated groups which are torsion-free and just non-solvable, yet without free subgroups of rank 2 [3, 6], and generalizations thereof [9], as well as constructions of free groups of rank 2 [11]. Yet solvable groups which contain τ are expected to have restricted structure [2]. For nilpotent groups we show

Proposition. *Let G be a nilpotent subgroup of $\mathcal{A}_n$ which contains the n-adic adding machine τ. Then G is a subgroup of $\overline{<\tau>}$.*

Let $\mathbb{Z}_n$ be the ring of n-adic integers and $U(\mathbb{Z}_n)$ its subgroup of units. The normalizer of $\overline{<\tau>}$ in $\mathcal{A}_n$ is isomorphic to the holomorph of $\mathbb{Z}_n$, the semi-direct product $\mathbb{Z}_n \rtimes U(\mathbb{Z}_n)$, and is therefore metabelian.

The most visible examples of finitely generated solvable groups containing τ are conjugate to subgroups of those belonging to the sequence of groups

$$\Gamma_0 = N_{\mathcal{A}_n}\overline{<\tau>}, \Gamma_1 = (\times_n\Gamma_0) \rtimes G_1, ..., \Gamma_{i+1} = (\times_n\Gamma_i) \rtimes G_{i+1}, ...$$

where $\times_n\Gamma_i$ is a direct product of n copies of Γ_i (seen as a subgroup of the 1st level stabilizer of the tree) and where G_i is a solvable subgroup of Σ_n in its canonical action on the tree, containing the cycle σ_τ. We note that for all i, the groups Γ_i are metabelian by 'finite solvable subgroups of Σ_n'. It was shown by the second author that for $n = 2$, that finitely generated solvable groups which contain the binary adding machine are conjugate to some subgroups of Γ_i acting on the binary tree [7].

The description for degrees $n > 2$ requires a classification of solvable subgroups of Σ_n which contain the cycle $\sigma = (0, 1, ..., n-1)$[8]. This is an open problem, even for metabelian groups. On the other hand, the answer for primitive solvable subgroups of Σ_n is simple and classical. For then, n is a prime number p or $n = 4$. In case $n = p$, the solvable subgroups G_i can all be taken to be the normalizer $F = N_{\Sigma_n}(\langle\sigma\rangle)$ of order $p\,(p-1)$ and in case $n = 4$, the G_i's can all be taken to be the symmetric group Σ_4.

Given this background, the main theorem of this paper is

Theorem A. *Let $n = p$, a prime number, or $n = 4$. Then any finitely generated solvable subgroup of $\mathcal{A}_n$,which contains the n-ary machine τ is conjugate to a subgroup of Γ_i for some i.*

The result follows first from general analysis of the conditions $[\beta, \beta^{\tau^x}] = e$ (for some $\beta \in \mathcal{A}_n$ and all $x \in \mathbb{Z}$), their impact on the 1st level states of the subgroup $\langle\beta, \tau\rangle$ and then how these in turn translate successively to conditions on states at lower levels. It is somewhat surprising that the process converges to a clear global description for trees of degrees p and 4.

If σ_β is either a power of σ_τ or a transposition, we describe abelian subgroups normalized by τ.

Theorem B. *Let B be an abelian subgroup of $\mathcal{A}_n$ normalized by τ, let $\beta = (\beta|_0, \beta|_1, \cdots, \beta|_{n-1})\sigma_\beta \in B$ and define the subgroup $H = \langle\beta|_i \ (i \in Y), \tau\rangle$ generated by the states of β* and *τ.*
(I) Suppose $\sigma_\beta = (\sigma_\tau)^s$ for some integer s and set $m = \frac{n}{\gcd(n,s)}$. Then, H is metabelian-by-finite. Indeed,on defining the subgroup

$$K = \left\langle [\beta|_i, \tau^k],\ \beta|_i\beta|_{\overline{i+s}}\beta|_{\overline{i+2s}} \cdots \beta|_{\overline{i+(m-1)s}} \mid k \in \mathbb{Z},\ i \in Y \right\rangle$$

(the bar notation means 'modulo m') then K is a normal subgroup of H and $O = K\langle\tau\rangle$ is a metabelian normal subgroup of H where $\frac{H}{O}$ is a homomorphic image of a subgroup of the wreath product $C_m \wr C_n$ of the cyclic groups C_m, C_n.
(II) Let n be an even number. Then H is a metabelian group if $s = \frac{n}{2}$ or σ_β is a transposition.

Let P be a subgroup of Σ_n. The *layer closure* of P in $\mathcal{A}_n$ is the group $L(P)$ formed by elements of $\mathcal{A}_n$ all of whose states lie in P. The following result is yet another characterization of the adding machine.

Theorem C. *Let n be an odd number, $\sigma = (0, \cdots, n-1) \in \Sigma_n$ and let $L = L(\langle\sigma\rangle)$, the layer closure of $\langle\sigma\rangle$ in A_n. Let s be an integer relatively prime to n and let $\beta = (\beta|_0, \beta|_1, \cdots, \beta|_{n-1})\sigma^s \in L$ be such* that $[\beta, \beta^{\tau^x}] = e$ *for all $x \in Z$. Then β is a conjugate of τ in L.*

2. Preliminaries

We start by introducing definitions and notation. The n-ary tree T_n can be identified with the free monoid $\mathcal{M} =< 0, 1, .., n-1 >^*$ of finite sequences from $Y = \{0, 1, ..., n-1\}$, ordered by $v \leq u$ provided u is an initial subword of v.

The identity element of $\mathcal{M}$ is the empty sequence ϕ. The level function for T_n, denoted by $|m|$ is the length of $m \in \mathcal{M}$; the root vertex ϕ has level 0.

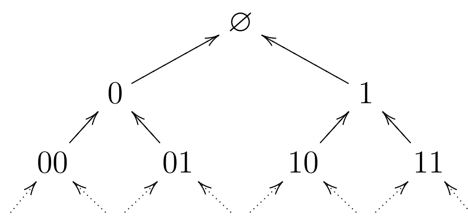

Figure 1. The Binary Tree

The action $\rho : i \to j$ of a permutation $\rho \in \Sigma_n$ will be from the right and written as $(i)\,\rho = j$ or as $i^{\rho} = j$. If i, j are integers then the action of ρ on i is to be identified with its action on its representatives $\bar{i}$ in Y, modulo n . Permutations σ in Σ_n are extended 'rigidly' to automorphisms of $\mathcal{A}_n$ by

$$(y.u)\rho = (y)\rho.u, \ \forall\, y \in Y, \ \forall\, u \in \mathcal{M}.$$

An automorphism $\alpha \in \mathcal{A}_n$ induces a permutation σ_α on the set Y. Consequently, α affords the representation $\alpha = \alpha' \sigma_\alpha$ where α' fixes Y point-wise and for each $i \in Y$, α' induces $\alpha|_i$ on the subtree whose vertices form the set $i \cdot \mathcal{M}$. If j is an integer the $\alpha|_j$ will be understood as $\alpha|_{\bar{j}}$ where $\bar{j}$ is the representative of j in Y modulo n.

Given i in Y, we use the canonical isomorphism $i \cdot u \mapsto u$ between $i \cdot \mathcal{M}$ and the tree T_n, and thus identify $\alpha|_i$ with an automorphism of T_n; therefore, $\alpha' \in \mathcal{F}(Y, \mathcal{A}_n)$, the set for functions from Y into $\mathcal{A}_n$, or what is the same, the 1st level stabilizer $Stab(1)$ of the tree. This provides us with the factorization $\mathcal{A}_n = \mathcal{F}(Y, \mathcal{A}_n) \cdot \Sigma_n$.

Let $\alpha, \beta, \gamma \in \mathcal{A}_n$. Then following formulas hold

$$\sigma_{\alpha^{-1}} = \left(\sigma_\alpha\right)^{-1}, \ \sigma_\alpha \sigma_\beta = \sigma_{\alpha\beta}, \tag{1}$$

$$(\alpha^{-1})|_u = \alpha|_{(u)^{\alpha^{-1}}}, \tag{2}$$

$$(\alpha\beta)|_u = (\alpha|_u)\,(\gamma|_u) \ \text{ where } \gamma|_u = \beta|_{(u)^\alpha} \tag{3}$$

$$\gamma = \alpha^{-1}\beta\alpha \Leftrightarrow \sigma_\gamma = \sigma_\alpha^{-1}\sigma_\beta\sigma_\alpha, \tag{4}$$

$$\gamma|_{(i)\sigma_\alpha} = \alpha|_i^{-1}\beta|_i\alpha|_{(i)\sigma_\beta}, \forall i \in Y. \tag{5}$$

$$\theta = [\beta, \alpha] = \beta^{-1}\beta^\alpha \Rightarrow \sigma_\theta = [\sigma_\beta, \sigma_\alpha], \tag{6}$$

$$\theta|_{(i)\sigma_{\alpha\beta}} = \left(\beta|_{(i)\sigma_\alpha}\right)^{-1}(\alpha|_i)^{-1}\,(\beta|_i)\left(\alpha|_{(i)\sigma_\beta}\right), \forall i \in Y. \tag{7}$$

(8) $$(\alpha^m)\,|_i = (\alpha|_i)\left(\alpha|_{(i)\sigma_\alpha}\right)\left(\alpha|_{(i)\sigma_\alpha^2}\right)\cdots\left(\alpha|_{(i)\sigma_{\alpha^{m-1}}}\right)$$

(9) $$(\beta^\alpha)\,|_u = \left(\beta|_{(u)\alpha^{-1}}\right)^{\alpha|_{(u)\alpha^{-1}}}, \text{where } \beta \in Stab(k) \text{ and } |u| \leq k.$$

An automorphism $\alpha \in \mathcal{A}_n$ corresponds to an input-output automaton with alphabet Y and with set of states $\mathrm{Q}(\alpha) = \{\alpha|_u \mid u \in \mathcal{M}\}$. The automaton α transforms the letters as follows: if the automaton is in state $\alpha|_u$ and reads a letter $i \in Y$ then it outputs the letter $j = (i)\,\alpha|_u$ and its state changes to $\alpha|_{ui}$; these operations can be best described by the labeled edge $\alpha|_u \xrightarrow{i|j} \alpha|_{ui}$. Following terminology of automata theory, every automorphism $\alpha|_u$ is called the *state* of α at u.

The tree T_n is a topological space which is the direct limit of its truncations at the n-th levels. Thus the group $\mathcal{A}_n$ is the inverse limit of the permutation groups it induces on the n-th level vertices. This transforms $\mathcal{A}_n$ into a topological group. An infinite product of elements $\mathcal{A}_n$ is a well-defined element of $\mathcal{A}_n$ provided for any given level l, only finitely many of the elements in the product have non-trivial action on vertices at level l. Thus, if $\alpha \in \mathcal{A}_n$ and $\xi = \sum_{i\geq 0} a_i n^i \in \mathbb{Z}_n$ then $\alpha^\xi = \alpha^{a_0}.\alpha^{na_1}..\alpha^{n^i a_i}...$ is a well define element of $\mathcal{A}_n$. The topological closure of a subgroup H in $\mathcal{A}_n$ will be indicated by $\overline{H}$. We note that if H is abelian then

$$\overline{H} = \{h^\xi | h \in H, \xi \in \mathbb{Z}_n\}.$$

One of the characterizing aspects of the n-ary adding machine is that the centralizer of τ is a pro-cyclic group; namely,

$$C_{\mathcal{A}_n}(\tau) = \overline{\langle\tau\rangle} = \{\tau^\xi \mid \xi \in \mathbb{Z}_n\}.$$

Let $v = yu$ where $y \in Y, u \in \mathcal{M}$. The image of v under the action of α is

$$(v)\alpha = (yu)\alpha = (y)\,\sigma_\alpha.(u)\alpha|_y.$$

The action extends to infinite sequences (or boundary points of the tree) in the same manner. A boundary point of the tree $c = c_0c_1c_2\ldots$where $c_i \in Y$ for all i, corresponds also to the n-adic integer $\xi = \sum\{c_i n^i | i \geq 0\} \in \mathbb{Z}_n$. Thus the action of the tree automorphism α can thus be translated to an action on the ring of n-adic integers. We will indicate c_0 by $\overline{\xi}$ which is ξ modulo n. In the case of the automorphism $\tau = (e, e, ..., e, \tau)\sigma$, the action of τ on c is

$$(c)\,\tau = \begin{cases} (c_0+1)\,c_1c_2\ldots & \text{if } 0 \leq c_0 \leq n-2, \\ 0(c_1c_2,\ldots)^\tau, & \text{if } c_0 = n-1, \end{cases}$$

which translates to the n-ary addition

$$\xi^\tau = 1 + \xi.$$

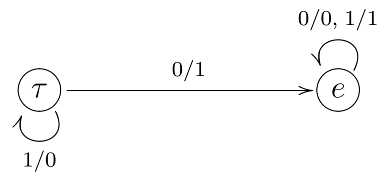

FIGURE 2. The binary adding machine

3. The holomorph of the n-adic integers

The holomorph of $\mathbb{Z}_n$ is the extension $\mathbb{Z}_n$ by the its group of units $U(\mathbb{Z}_n)$ in its natural action on $\mathbb{Z}_n$. An element ξ is a unit in $\mathbb{Z}_n$ if and only if $\overline{\xi}$ is a unit in $\mathbb{Z}$ modulo n. The subgroup of $U(\mathbb{Z}_n)$ consisting of elements ξ with $\overline{\xi} = 1$ is denoted by by $\mathbb{Z}_n^1$. This subgroup has the transversal $\{j \mid 1 \leq j \leq n-1, \gcd(j,n) = 1\}$ in $\mathbb{Z}_n$ and therefore has index $[U(\mathbb{Z}_n) : \mathbb{Z}_n^1] = \varphi(n)$ where φ is the Euler function. The normalizer of $\overline{\langle\tau\rangle}$ in the group of automorphisms of the tree is the holomorph of $\mathbb{Z}_n$.

Given $\alpha \in \mathcal{A}_n$ we denote the diagonal automorphism $(\alpha, ..., \alpha)$ by $\alpha^{(1)}$ and define inductively $\alpha^{(i+1)} = \left(\alpha^{(i)}\right)^{(1)}$ for all $i \geq 1$.

3.1. **Powers of τ.** Let $\xi = \sum_{i \geq 0} a_i n^i \in \mathbb{Z}_n$. Then $a_0 = \overline{\xi}$ and $\sum_{i\geq 1} a_i n^{i-1} = \frac{\xi - \overline{\xi}}{n}$.

Lemma 1. *Let $\xi \in \mathbb{Z}_n$. Then*

$$\tau^{\xi} = (\tau^{\frac{\xi - a_0}{n}}, \cdots, \tau^{\frac{\xi - a_0}{n}}, \underbrace{\tau^{\frac{\xi - a_0}{n}+1}, \cdots, \tau^{\frac{\xi - a_0}{n}+1}}_{a_0 \text{ terms}})\sigma_{\tau}^{a_0}.$$

Proof. For j an integer with $1 \leq j \leq n-1$, we have

$$\tau^j = \left(e, ..., e, \underbrace{\tau, \cdots, \tau}_{j \text{ terms}}\right)\sigma_{\tau}^{j}$$

and $\tau^n = (\tau, ..., \tau) = \tau^{(1)}$.

Given $\xi = \sum_{i\geq 0} a_i n^i$, then

$$\tau^{a_0} = (e, \cdots, e, \underbrace{\tau, \cdots, \tau}_{a_0 \text{ terms}})\sigma_\tau^{a_0}, \tag{10}$$

$$\tau^{a_j n^j} = \tau^{(a_j n^{j-1})n} = \left(\tau^{a_j n^{j-1}}\right)^{(1)}, \tag{11}$$

$$\tau^{\xi} = (\tau^{\frac{\xi - a_0}{n}}, \cdots, \tau^{\frac{\xi - a_0}{n}}, \underbrace{\tau^{\frac{\xi - a_0}{n}+1}, \cdots, \tau^{\frac{\xi - a_0}{n}+1}}_{a_0 \text{ terms}})\sigma_\tau^{a_0} \tag{12}$$

$$= (\tau^{\frac{\xi - \overline{\xi}}{n}}, \cdots, \tau^{\frac{\xi - \overline{\xi}}{n}}, \underbrace{\tau^{\frac{\xi - \overline{\xi}}{n}+1}, \cdots, \tau^{\frac{\xi - \overline{\xi}}{n}+1}}_{\overline{\xi} \text{ terms}})\sigma_\tau^{\overline{\xi}}. \tag{13}$$

□

As we have seen, the description of τ^ξ involves the partition of the interval $[0, ..., n-1]$ into two subintervals. Therefore we introduce the step function $\delta : \frac{\mathbb{Z}}{n\mathbb{Z}} \times \frac{\mathbb{Z}}{n\mathbb{Z}} \to \{0, 1\}$ given by

$$\delta(i, j) = \frac{i + j - \overline{i + j}}{n} = \begin{cases} 0, & \text{if } 0 \le i \le n - j \\ 1, & \text{otherwise} \end{cases} .$$

which we will call the *Polarizer Function.* With this,

$$\tau^\xi = \left(\tau^{\frac{\xi - \overline{\xi}}{n} + \delta(i, \xi)}\right)_{0 \le i \le n-1} \sigma_\tau^{\overline{\xi}}.$$

The function δ extends to $\mathbb{Z}_n \times \mathbb{Z}_n$, simply by $\delta(\eta, \kappa) = \delta(i, k)$ where $i = \overline{\eta}, k = \overline{\kappa}$. Note that

$$\sum_{i=0}^{n-1} \delta(i, j) = j.$$

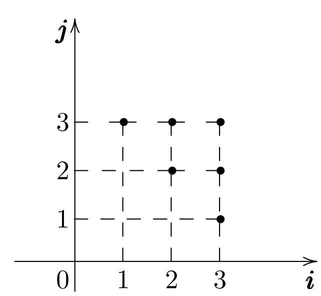

FIGURE 3. Polarizer Function for $n = 4$.

3.2. **Centralizer of τ.**

Lemma 2. $C_{\mathcal{A}_n}(\tau) = \overline{\langle\tau\rangle}$.

Proof. Let $\alpha \in \mathcal{A}_n$ commute with τ. Then, $[\sigma_\alpha, \sigma_\tau] = e$ and therefore $\sigma_\alpha = (\sigma_\tau)^{s_0}$ for some integer $0 \le s_0 \le n-1$. Therefore, $\beta = \alpha\tau^{-s_0} = (\beta|_0, ..., \beta|_{n-1})$ commutes with τ and $\sigma_\beta = e$. Now,

$$\beta^\tau = \left((\beta|_{n-1})^\tau, \beta|_0, ..., \beta|_{n-1}\right) = \beta$$

implies $\beta|_i = \beta|_0$ for all $0 \le s_0 \le n-1$ and $\beta|_0$ commutes with τ. Therefore $\beta = (\beta|_0)^{(1)}$ and $\beta|_0$ replaces α in previous argument. Hence,

there exists an integer $0 \le s_1 \le n-1$ such that $\gamma = \beta|_0\tau^{-s_1} = (\gamma|_0)^{(1)}$. From which we conclude

$$\begin{aligned}\alpha &= \beta\tau^{s_0} = (\beta|_0)^{(1)}\tau^{s_0} \\ &= \left((\gamma|_0)^{(1)}\tau^{s_1}, .., (\gamma|_0)^{(1)}\tau^{s_1}\right)\tau^{s_0} \\ &= (\gamma|_0)^{(2)}\tau^{ns_1}\tau^{s_0} = (\gamma|_0)^{(2)}\tau^{ns_1+s_0}.\end{aligned}$$

Inductively then, we obtain the desired form $\alpha = \tau^\xi$ where $\xi = s_0 + ns_1 + ...$ ☐

A characterization of nilpotent groups which contain τ follows.

Proposition 1. *Let G be a nilpotent subgroup of $\mathcal{A}_n$ which contains the n-adic adding machine. Then G is a subgroup of $\overline{<\tau>}$.*

Proof. Suppose G is a nilpotent group of class $k > 1$ which contains τ. Then, the center $Z(G)$ is contained in $\overline{\langle\tau\rangle}$. Let j be the maximum index such that $Z_j(G) \le \overline{\langle\tau\rangle}$; therefore $j < k$. Let $\alpha \in Z_{j+1}(G) \setminus Z_j(G)$; then $[\tau, \alpha] = \tau^\xi$ and $\xi \ne 0$. Now, $[\tau, \alpha, \alpha] = \left[\tau^\xi, \alpha\right] = e$. Yet $\left[\tau^\xi, \alpha\right] = [\tau, \alpha]^\xi = \tau^{\xi^2} = e$ and so, $\xi = 0$ and $[\tau, \alpha] = e$; a contradiction. ☐

3.3. **Normalizer of the topological closure $\overline{\langle\tau\rangle}$.**

Lemma 3. *The group $\Gamma_0 = N_{\mathcal{A}_n}\left(\overline{\langle\tau\rangle}\right)$ is metabelian. Indeed, the derived subgroup Γ_0' is contained in $\overline{\langle\tau\rangle}$.*

Proof. Let $\alpha, \beta \in \Gamma_0$, then $\tau^\alpha = \tau^\xi$ and $\tau^\beta = \tau^\eta$ for some $\eta, \xi \in U(\mathbb{Z}_n)$. Therefore,

$$\tau^\alpha = \tau^\xi, \tau = (\tau^\xi)^{\alpha^{-1}} = (\tau^{\alpha^{-1}})^\xi,$$
$$\tau^{\alpha^{-1}} = \tau^{\xi^{-1}}.$$

Likewise, $\tau^{\beta^{-1}} = \tau^{\eta^{-1}}$. Thus, $\tau^{[\alpha,\beta]} = \tau$ and $\Gamma_0' \le C_{\mathcal{A}_n}(\tau) = \overline{\langle\tau\rangle}$ follows. ☐

We present a property of the polarizer function δ which we will use in the sequel.

Lemma 4. *For all $i, j \in \mathbb{Z}$, $\xi \in \mathbb{Z}_n$ we have*

$$\frac{j\xi - \overline{j\xi}}{n} - j\left(\frac{\xi - \overline{\xi}}{n}\right) + \delta(i, j\xi) = \sum_{k=0}^{j-1} \delta(i + k\xi, \xi).$$

Proof. Since

$$\begin{aligned}(\tau^{\xi})^{j}|_i &= (\tau^{\xi})|_i.(\tau^{\xi})|_{\overline{i+\xi}} \cdots (\tau^{\xi})|_{\overline{i+(j-1)\xi}},\\ (\tau^{\xi})|_i &= \tau^{\frac{\xi-\overline{\xi}}{n}+\delta(i,\xi)}\end{aligned}$$

the assertion follows from

$$\tau^{\frac{j\xi-\overline{j\xi}}{n}+\delta(i,j\xi)} = \tau^{j\left(\frac{\xi-\overline{\xi}}{n}\right)+\sum_{k=0}^{j-1}\delta(i+k\xi,\xi)}.$$

□

Proposition 2. *Suppose $\alpha \in \mathcal{A}_n$ satisfies $\tau^{\alpha} = \tau^{\xi}$ for some $\xi \in U(\mathbb{Z}_n)$. Then:*

(i)

$$\alpha|_i = \alpha|_0 \tau^{\mu_i}, (1 \le i \le n-1)\};$$

where

$$\mu_i = i\frac{(\xi - \overline{\xi})}{n} + \sum_{k=0}^{i-1} \delta((v(\alpha) + k)\xi, \xi)$$

and $0 \le v(\alpha) \le n-1$ is such that

$$(0)\,\sigma_{\alpha} = \overline{v(\alpha)\xi};$$

(ii) (recursion) $\tau^{\alpha|_0} = \tau^{\xi}$;

(iii)

$$(j)\sigma_{\alpha} = \overline{(v(\alpha) + j)\xi}, (0 \le j \le n-1)\}.$$

If $\xi \in \mathbb{Z}_n^1$ then $v(\alpha) = 0$, $(j)\sigma_{\alpha} = \overline{j\xi} = j, \mu_i = i\frac{\xi-1}{n}$.

Proof. Since $\sigma_{\tau}^{\sigma_{\alpha}} = \sigma_{\tau}^{\xi}$, we have

$$((0)\,\sigma_{\alpha}, (1)\,\sigma_{\alpha}, \cdots, (n-1)\sigma_{\alpha}) = (0, \overline{\xi}, \overline{2\xi}, \cdots, \overline{(n-1)\xi}).$$

Therefore, there exists $v(\alpha) \in Y$ such that $(0)\,\sigma_{\alpha} = \overline{v(\alpha)\xi}$ and so,

$$(j)\sigma_{\alpha} = \overline{(v(\alpha) + j)\xi},\ \forall j \in Y.$$

Now, $\tau^{\alpha} = \tau^{\xi}$ is equivalent to

$$\left(\begin{array}{l}\sigma_{\tau}^{\sigma_{\alpha}} = \sigma_{\tau}^{\xi} \quad \text{and} \quad \alpha|_{(i)\sigma_{\tau}^{s}} = ((\tau^{s})|_i)^{-1}\,\alpha|_i(\tau^{\xi s})|_{(i)\sigma_{\alpha}},\\ \forall i \in Y, \forall s \in \mathbb{Z},\ \text{by...}\end{array}\right).$$

The latter conditions are equivalent to

$$\left(\begin{array}{l}\alpha|_0 = \alpha|_{(0)\sigma_{\tau}^{n}} = ((\tau^{n})|_0)^{-1}\,\alpha|_0(\tau^{\xi n})|_{(0)\sigma_{\alpha}}\\ \quad \text{and} \quad \alpha|_i = \alpha|_{(0)\sigma_{\tau}^{i}} = ((\tau^{i})|_0)^{-1}\,\alpha|_0(\tau^{\xi i})|_{(0)\sigma_{\alpha}}\ \forall i \in Y - \{0\}\end{array}\right)$$

and these in turn are equivalent to

$$\left(\begin{array}{c} \alpha|_i = \alpha|_0 \tau^{\frac{\xi i - \overline{\xi i}}{n} + \delta(v(\alpha)\xi, \xi i)} = \alpha|_0 \tau^{\mu_i} \\ \text{where } \mu_i = i\left(\frac{\xi - \overline{\xi}}{n}\right) + \sum_{k=0}^{i-1} \delta((v(\alpha)+k)\xi, \xi) \ \forall i \in Y - \{0\} \end{array}\right).$$

Substitute $i = 0$ in

$$\frac{j\xi - \overline{j\xi}}{n} + \delta(i, j\xi) = j\left(\frac{\xi - \overline{\xi}}{n}\right) + \sum_{k=0}^{j-1} \delta(i + k\xi, \xi), \forall i, \xi \in \mathbb{Z}.$$

to get $\sum_{k=0}^{i-1} \delta(k\xi, \xi) = 0$. The rest of the assertion follows directly. □

Corollary 1. *Let $\xi \in U(\mathbb{Z}_n)$ and μ_i be as above. Then $\alpha = (\alpha)^{(1)}(e, \tau^{\mu_1}, ..., \tau^{\mu_{n-1}})$ conjugates τ to τ^{ξ}. In particular, if $\xi \in \mathbb{Z}_n^1$, then*

$$\alpha = (\alpha)^{(1)}(e, \tau^{\frac{\xi-1}{n}}, \tau^{2\frac{\xi-1}{n}}, \cdots, \tau^{(n-1)\frac{\xi-1}{n}})$$

denoted by λ_ξ conjugates α to τ^{ξ}.

Although we have computed above an automorphism which inverts τ, we give another with a simpler description. Define the permutation

$$\varepsilon = (0, n-1)(1, n-2) ... \left(\left[\frac{n-2}{2}\right], \left[\frac{n+1}{2}\right]\right).$$

Then ε inverts $\sigma_\tau = (0, 1, ..., n-1)$ and

$$\iota = \iota^{(1)}\varepsilon$$

inverts τ.

Define

$$\begin{array}{rcl} \Lambda & = & \{\lambda_\xi \mid \xi \in \mathbb{Z}_n^1\}, \\ \Psi & = & \{\lambda_\xi \tau^t \mid \xi \in \mathbb{Z}_n^1, t \in \mathbb{Z}_n\} \end{array}$$

and call Λ the *monic normalizer* of $\overline{\langle\tau\rangle}$.

Proposition 3. *(i) Λ is an abelian group isomorphic to $\mathbb{Z}_n^1$;*
(ii) $\Psi = \Lambda \ltimes \overline{\langle\tau\rangle} \cong \mathbb{Z}_n^1 \ltimes \mathbb{Z}_n$;
(iii) the derived subgroup $\Psi' = \overline{\langle\tau^n\rangle}$.

Proof. (i) Let $\xi, \theta \in \mathbb{Z}_n^1$. Then, as $\lambda_\xi, \lambda_\theta$ and $\lambda_{\xi\theta}$ are inactive, its follows that

$$(\lambda_\xi \lambda_\theta \lambda_{\xi\theta}^{-1})|_i = (\lambda_\xi)|_i (\lambda_\theta)|_i \left((\lambda_{\xi\theta})|_i\right)^{-1}$$

$$= \lambda_\xi \tau^{i\frac{\xi-1}{n}} \lambda_\theta \tau^{i\frac{\theta-1}{n}} \left(\lambda_{\xi\theta} \tau^{i\frac{\xi\theta-1}{n}}\right)^{-1} = \lambda_\xi \lambda_\theta \lambda_\theta^{-1} \tau^{i\frac{\xi-1}{n}} \lambda_\theta \tau^{i\frac{\theta-1}{n}} \tau^{-i\frac{\xi\theta-1}{n}} \lambda_{\xi\theta}^{-1}$$

$$= \lambda_\xi \lambda_\theta \left(\tau^{i\theta\frac{\xi-1}{n}} \tau^{i\frac{\theta-1}{n}} \tau^{-i\frac{\xi\theta-1}{n}}\right) \lambda_{\xi\theta}^{-1} = \lambda_\xi \lambda_\theta \lambda_{\xi\theta}^{-1}, \forall i \in \{0, \cdots, n-1\}.$$

Therefore, $\lambda_\xi \lambda_\theta = \lambda_{\xi\theta}$. In addition, $\lambda_\xi = e$ if and only if $\xi = 1$.

(ii) This factorization is clear.

(iii) Let $\theta = 1 + n\theta', \eta \in \mathbb{Z}_n$. Then

$$\begin{aligned} [\tau^{\eta}, \lambda_{\theta}] &= \tau^{-\eta}\lambda_{\theta^{-1}}\tau^{\eta}\lambda_{\theta} = \\ \tau^{-\eta}\tau^{\eta\theta} &= \tau^{\eta(\theta-1)} = (\tau^{n})^{\eta\theta'} . \end{aligned}$$

□

We prove below the existence of conjugates τ^{α} of τ in $N_{\mathcal{A}_n}\left(\overline{\langle\tau\rangle}\right)$, which lie outside $\overline{\langle\tau\rangle}$. This fact provides us with the first important type of metabelian groups $\overline{\langle\tau\rangle}\,\langle\tau^{\alpha}\rangle$ containing τ.

Proposition 4. *Suppose $\alpha = (\alpha|_0, \alpha|_1, \cdots, \alpha|_{n-1}) \in \mathcal{A}_n$ satisfies $\tau^{\alpha} = \lambda_{\xi}\tau^{\rho}$ for some $\xi \in \mathbb{Z}_n^1$, and $\rho = 1 + \kappa n \in \mathbb{Z}_n^1$. Then*

$$\begin{cases} \alpha|_{i+1} = (\alpha|_0)\,\lambda_{\xi^{i+1}}\tau^{\frac{1}{n}\left[\rho\frac{\xi^{i+1}-1}{\xi-1}-(i+1)\right]} \ (0 \le i \le n-2)\,, \\ \tau^{\alpha|_0} = \lambda_{\xi^n}\tau^{\frac{1}{n}\left[\rho\frac{\xi^{n}-1}{\xi-1}\right]}. \end{cases} .$$

The converse is true for $n \ge 3$ and for $n = 2$ provided $4|\xi - 1$.

Proof. From $\tau^{\alpha} = \lambda_{\xi}\tau^{1+\kappa n}$, we obtain using (4) and (5),

$$\begin{cases} \lambda_{\xi}\tau^{i\frac{\xi-1}{n}+\kappa} = \alpha|_i^{-1}\alpha_{i+1}, \ \text{if } i \in Y - \{n-1\} \\ \lambda_{\xi}\tau^{(n-1)\frac{\xi-1}{n}+\kappa+1} = \alpha|_{n-1}^{-1}\tau\alpha|_0. \end{cases}$$

Therefore,

$$\begin{aligned} &\alpha|_{i+1} = \alpha|_0\lambda_{\xi}\tau^{\kappa}\lambda_{\xi}\tau^{\frac{\xi-1}{n}+\kappa}\cdots\lambda_{\xi}\tau^{i\frac{\xi-1}{n}+\kappa}, \ \text{for } i = 0, 1, \cdots, n-2, \\ &\alpha|_0 = \tau^{-1}\alpha|_{n-1}\lambda_{\xi}\tau^{(n-1)\frac{\xi-1}{n}+\kappa+1}. \end{aligned}$$

The first equations can be expresses as

$$\begin{aligned} \alpha|_{i+1} &= \alpha|_0\lambda_{\xi^{i+1}}\tau^{\kappa\sum_{j=0}^{i}\xi^{j}+\frac{\xi-1}{n}\xi^{i}\sum_{j=1}^{i}j(\xi^{-1})^{j}} \\ &= \alpha|_0\lambda_{\xi^{i+1}}\tau^{\frac{1}{n}\left[(1+\kappa n)\frac{\xi^{i+1}-1}{\xi-1}-(i+1)\right]} \end{aligned}$$

and the last as

$$\begin{aligned} \alpha|_0 &= \tau^{-1}\alpha|_0\lambda_{\xi^n}\tau^{\frac{\xi}{n}\left[(1+\kappa n)\frac{\xi^{n-1}-1}{\xi-1}-(n-1)\right]}\tau^{(n-1)\frac{\xi-1}{n}+\kappa+1} \\ &= \lambda_{\xi^n}\tau^{\frac{1}{n}\left[(1+\kappa n)\frac{\xi^{n}-1}{\xi-1}\right]}. \end{aligned}$$

If $n \ge 3$ then $\tau^{\alpha|_0} = \lambda_{\xi^n}\tau^{\frac{1}{n}\left[(1+\kappa n)\frac{\xi^{n}-1}{\xi-1}\right]}$ satisfies the same conditions as those for α; namely, both $\xi^n, \rho' = \frac{1}{n}\left[(1+\kappa n)\frac{\xi^n-1}{\xi-1}\right]$ are in $\mathbb{Z}_n^1$. If $n = 2$ then $\xi = 1 + 2\xi'$, $\rho' = \frac{1}{2}\left[(1+2\kappa)\frac{\xi^2-1}{\xi-1}\right] = (1+2\kappa)(1+\xi')$ and so, $\rho' \in \mathbb{Z}_2^1$ implies $\xi = 1 + 4\xi''$. □

4. Abelian groups B normalized by τ

Let B be an abelian subgroup of $\mathcal{A}_n$ normalized by τ. For a fixed $\beta \in B$, we define the 'state closure' of $\langle \beta, \tau \rangle$ as the group

$$H = \langle \beta|_i \ \ (i \in Y) \, , \tau \rangle \, .$$

We will be dealing frequently with the following subgroups of H ,

$$\begin{aligned} N &= \left\langle [\beta|_i, \tau^{k_i}] \mid k_i \in \mathbb{Z}, i \in Y \right\rangle \\ M &= N \langle \tau \rangle \, . \end{aligned}$$

When $\sigma_\beta = (\sigma_\tau)^s$ *for some integer* s *we will also be dealing with the subgroups*

$$\begin{aligned} K &= \left\langle N, \ \beta|_i \beta|_{\overline{i+s}} \beta|_{\overline{i+2s}} \cdots \beta|_{\overline{i+(m-1)s}} \mid i \in Y \right\rangle , \\ O &= K \langle \tau \rangle \end{aligned}$$

where $s = \frac{n}{\gcd(n,s)}$.

We show that when n is a power of a prime number p^k, the activity range of β narrows down to a Sylow p-subgroup of Σ_n. This is used to restrict the location of an abelian group B normalized by τ, within $\mathcal{A}_n$

Proposition 5. *Let* $n = p^k$, $\sigma = (0, 1, ..., n-1)$ *and* P *be a Sylow* p*-subgroup* P *of* Σ_n *which contains* σ. *Then*
(i) P *is isomorphic to* $((...(...C_p)wr)C_p)\, wrC_p$, *a wreath product of the cyclic group* C_p *of order* p *iterated* $k-1$ *times; the normalizer of* P *in* Σ_n *is* $N_{\Sigma_n}(P) = P \langle c \rangle$ *where* c *is cyclic of order* $p-1$;
(ii) P *is the unique Sylow* p*-subgroup* P *of* Σ_n *which contains* σ;
(iii) if W *is an abelian subgroup of* Σ_n *normalized by* σ *then* W *is contained in* P;
(iv) the subgroup B *is contained in the layer closure* $L = L\left(N_{\Sigma_p}(P)\right)$.

Proof. (i) The structure of P as an iterated wreath product is well-known. The center of P is $Z = \left\langle z \left(= \sigma^{p^{k-1}}\right) \right\rangle$ and $C_{\Sigma_n}(z) = P$. Therefore, $N_{\Sigma_n}(P) = N_{\Sigma_n}(Z) = P \langle c \rangle$ where c is cyclic of order $p-1$.

(ii) If $\sigma \in P^g$ for some $g \in \Sigma_n$ then $z^g \in C_{\Sigma_n}(\sigma) = \langle \sigma \rangle$ and therefore $\langle z^g \rangle = \langle z \rangle$, $P^g = P$. Thus, P is the unique Sylow p-subgroup of Σ_n to contain σ.

(iii) Let W be an abelian subgroup of Σ_n normalized by σ. Let $V = W < \sigma >$ and V_0 be the stabilizer of 0 in V. Then, since σ is a regular cycle, it follows that $V = V_0 \langle \sigma \rangle$, $V_0 \cap \langle \sigma \rangle = \{e\}$. Suppose that there exists a prime q different from p which divides the order of W and let Q be the unique Sylow q-subgroup of W. Then Q is the unique Sylow q-subgroup of V and $Q \leq V_0$. Therefore, $Q = \{e\}$ and W a p-group. As $\sigma \in W$, we conclude $W \leq P$..

(iv) Since the normal closure of $\langle \sigma_\beta \rangle$ under the action of $\langle \sigma_\tau \rangle$ is an abelian subgroup, it follows that $\sigma_\beta \in P$. Furthermore, as $\left\langle [\beta|_u, \tau^k] \mid k \in \mathbb{Z} \right\rangle$ is an abelian group normalized by τ, it follows that $[\sigma_{\beta|_u}, \sigma] \in P$ and therefore $\sigma^{\sigma_{\beta|_u}} \in P$. Thus, we conclude $\sigma_{\beta|_u} \in N_{\Sigma_n}(P)$ and $\beta \in L$. □

Lemma 5. *Let $\gamma \in \mathcal{A}_n$. Conditions (i), (ii) below are equivalent:*
(i) $[\gamma, \gamma^{\tau^k}] = e$ for all $k \in \mathbb{Z}$;
(ii) $[\tau^k, \gamma, \gamma] = e$ for all $k \in \mathbb{Z}$.
Condition (i) implies
(iii) $\langle [\gamma, \tau^k] \mid k \in \mathbb{Z} \rangle$ is a commutative group.
Condition (iii) implies
$\langle [\gamma|_u, \tau^k] \mid k \in \mathbb{Z} \rangle$ is a commutative group for all indices u.

Proof. First,

$$\begin{aligned} [\gamma, \gamma^{\tau^k}] &= \gamma^{-1}\left(\tau^{-k}\gamma^{-1}\tau^k\right)\gamma\left(\tau^{-k}\gamma\tau^k\right) \\ &= \gamma^{-1}\left(\tau^{-k}\gamma^{-1}\tau^k\gamma\right)\gamma\left(\gamma^{-1}\tau^{-k}\gamma\tau^k\right) \\ &= [\tau^k, \gamma]^\gamma [\gamma, \tau^k] \end{aligned}$$

and so,

$$[\gamma, \gamma^{\tau^k}] = e \Leftrightarrow [\gamma, \tau^k]^\gamma = [\gamma, \tau^k].$$

Furthermore, since

$$[\gamma, \tau^{k_1}]^{\tau^{k_2}} = [\gamma, \tau^{k_2}]^{-1}[\gamma, \tau^{k_1+k_2}] \tag{14}$$

for all integers k_1, k_2, condition (ii) implies

$$\begin{aligned} [\gamma, \tau^{k_1}]^{[\gamma, \tau^{k_2}]} &= [\gamma, \tau^{k_1}]^{\gamma^{-1}\tau^{-k_2}\gamma\tau^{k_2}} = [\gamma, \tau^{k_1}]^{\tau^{-k_2}\gamma\tau^{k_2}} \\ &= \left([\gamma, \tau^{-k_2}]^{-1}[\gamma, \tau^{k_1-k_2}]\right)^{\gamma\tau^{k_2}} = \left([\gamma, \tau^{-k_2}]^{-1}[\gamma, \tau^{k_1-k_2}]\right)^{\tau^{k_2}} \\ &= [\gamma, \tau^{k_1}]. \end{aligned}$$

Finally, we note that by (6) and (7),

$$\begin{aligned} ([\gamma, \tau^{nk}])|_{(i)\sigma_\gamma} &= (\gamma^{-1})|_{(i)\sigma_\gamma}(\tau^{-nk})|_i\,(\gamma|_i)\,(\tau^{nk})|_{(i)\sigma_\gamma} \\ &= \left(\gamma|_i^{-1}\right)\tau^{-k}\,(\gamma|_i)\,\tau^k \\ &= [\gamma|_i, \tau^k]. \end{aligned}$$

Since $[\gamma, \tau^{kn}]$ is inactive for all $k \in \mathbb{Z}$, we obtain $\{[\gamma|_i, \tau^k] \mid k \in \mathbb{Z}\}$ is a commutative set for all i. The rest of the assertion follows by induction on the tree level. □

Obviously, $\langle [\beta, \tau^k] \mid k \in \mathbb{Z} \rangle$ is normalized by τ and if condition (i) holds then it is an abelian normal subgroup of $\langle \beta, \tau \rangle$.

Proposition 6. *Let $l \geq 1$ and suppose $\alpha, \gamma \in \mathrm{Stab}(l)$ satisfy $[\alpha, \gamma^{\tau^x}] = e$ for all $x \in \mathbb{Z}$. Then*

$$\begin{aligned} [\alpha|_u, \gamma|_v{}^{\tau^x}] &= e \ \forall u, v \in \mathcal{M} \\ \textit{having } |u| &= |v| \ \leq l \textit{ and } \forall x \in \mathbb{Z}. \end{aligned}$$

Proof. We start with the case $l = 1$. Write $x = r + kn$ where $r = \overline{x}$.

By (4) and (5),

$$\begin{aligned}\left(\gamma^{\tau^x}\right)|_{(i)\tau^x} &= (\tau^x)|_i^{-1}\gamma|_i(\tau^x)_i,\\ \left(\gamma^{\tau^x}\right)|_i &= \tau^{-k-\delta(i-r,r)}\gamma|_{\overline{i-r}}\tau^{k+\delta(i-r,r)}.\end{aligned}$$

As $[\alpha, \gamma^{\tau^x}] = e$ and $\alpha, \gamma^{\tau^x} \in \mathrm{Stab}(1)$,we have, for all $i, j, r \in Y$ and all $k, x \in \mathbb{Z}$,

$$\begin{aligned}[\alpha|_i, (\gamma^{\tau^x})|_i] &= e,\ [\alpha|_i, \gamma|_{\overline{i-r}}^{\tau^{k+\delta(i-r,r)}}] = e,\\ [\alpha|_i, (\gamma|_j)^{\tau^x}] &= e.\end{aligned}$$

The general case $l \geq 1$ follows by induction. □

We apply the above to $\beta \in B$.

Corollary 2. *Let $\sigma_\beta = e$. Then for all $i, j \in Y$ and for all $x \in \mathbb{Z}$*

$$[\beta|_i, \beta|_j^{\tau^x}] = e.$$

Then we derive further relations in $H = \langle \beta|_i\ \ (i \in Y), \tau\rangle$.

Proposition 7. *Let $\beta \in B$. Then the following relations hold in H for all $v \in \mathbb{Z}$ and for all $i \in Y$:*

(I)

$$\begin{aligned}&\left(\tau^v|_{(i)\sigma_\tau^{-v}}\right)^{-1}\left(\beta|_{(i)\sigma_\tau^{-v}}\right)\left(\tau^v|_{(i)\sigma_\tau^{-v}\sigma_\beta}\right)\left(\beta|_{(i)\sigma_\tau^{-v}\sigma_\beta\sigma_\tau^v}\right)\\ =\ &(\beta|_i)\left(\tau^v|_{(i)\sigma_\beta\sigma_\tau^{-v}}\right)^{-1}\left(\beta|_{(i)\sigma_\beta\sigma_\tau^{-v}}\right)\left(\tau^v|_{(i)\sigma_\beta\sigma_\tau^{-v}\sigma_\beta}\right),\end{aligned}$$

$$[\sigma_\beta, \sigma_\beta^{\sigma_\tau^v}] = e;$$

(II)

$$[\beta|_i, \tau^v]^{\beta|_{(i)\sigma_\beta}} = [\beta|_{(i)\sigma_\beta}, \tau^v];$$

(III)

$$\beta|_{(i)\sigma_\beta}\beta|_{(i)\sigma_\beta^2}\cdots\beta|_{(i)\sigma_\beta^{s_i}}\ \textit{commutes with}\ [\beta|_i, \tau^v]$$

where s_i is the size of the orbit of i under the action of $\langle\sigma_\beta\rangle$.

Proof. (I) Clearly $[\beta, \beta^{\tau^v}] = e$ implies $[\sigma_\beta, \sigma_\beta^{\sigma_\tau^v}] = e$. It also implies

$$\left(\beta|_{(i)\sigma_{\beta^{\tau^v}}}\right)^{-1}\left(\beta^{\tau^v}|_i\right)^{-1}\beta|_i\left(\beta^{\tau^v}|_{(i)\sigma_\beta}\right) = e,$$

$$(\beta^{\tau^v}|_i\left(\beta|_{(i)\sigma_{\beta^{\tau^v}}}\right) = \beta|_i\left(\beta^{\tau^v}|_{(i)\sigma_\beta}\right),$$

$$\begin{aligned}&\left(\tau^v|_{(i)\sigma_{\tau^v}^{-1}}\right)^{-1}\left(\beta|_{(i)\sigma_{\tau^v}^{-1}}\right)\left(\tau^v|_{(i)\sigma_{\tau^v}^{-1}\sigma_\beta}\right)\left(\beta|_{(i)\sigma_{\beta^{\tau^v}}}\right)\\ =\ &(\beta|_i)\left(\tau^v|_{(i)\sigma_\beta\sigma_{\tau^v}^{-1}}\right)^{-1}\left(\beta|_{(i)\sigma_\beta\sigma_{\tau^v}^{-1}}\right)\left((\tau^v)|_{(i)\sigma_\beta\sigma_{\tau^v}^{-1}\sigma_\beta}\right).\end{aligned}$$

(II) On changing v to nv in (I), we obtain:

$$\tau^{-v}\left(\beta|_i\right)\tau^{v}\left(\beta|_{(i)\sigma_\beta}\right)=\left(\beta|_i\right)\tau^{-v}\left(\beta|_{(i)\sigma_\beta}\right)\tau^{v},$$

$$\begin{aligned}&\left(\beta|_{(i)\sigma_\beta}\right)^{-1}\left(\beta|_i^{-1}\tau^{-v}\beta|_i\tau^{v}\right)\left(\beta|_{(i)\sigma_\beta}\right)\\ =\ &\left(\left(\beta|_{(i)\sigma_\beta}\right)^{-1}\beta|_i^{-1}\right)\beta|_i\tau^{-v}\left(\beta|_{(i)\sigma_\beta}\right)\tau^{v}.\end{aligned}$$

(III) From (II), we derive

$$[\beta|_i,\tau^v]^{\left(\beta|_{(i)\sigma_\beta}\beta|_{(i)\sigma_\beta^2}\cdots\beta|_{(i)\sigma_\beta^{s_i}}\right)}=[\beta|_{(i)\sigma_\beta},\tau^v]^{\left(\beta|_{(i)\sigma_\beta^2}\cdots\beta|_{(i)\sigma_\beta^{s_i}}\right)}=\ldots=[\beta|_i,\tau^v].$$

□

5. The case $\beta\in B$ with $\sigma_\beta\in\langle\sigma_\tau\rangle$

This section is devoted to the proof of the second part *(I)* of Theorem B. For this purpose, we introduce the following combination of step functions

$$\Delta_s(i,t)=\delta(i,t-i)-\delta(i-s,t-i)$$

and call it the *Inductor Function.*

Lemma 6. *Let $\beta\in\mathcal{A}_n$ such that $[\beta,\beta^{\tau^x}]=e$ for any $x\in\mathbb{Z}$ and let $\sigma_\beta=\sigma_\tau^s$ for some $s\in Y$. Then,*

$$\begin{aligned}&\tau^{\Delta_s(i,t)}\left(\beta|_{i-s}\right)[\beta|_{i-s},\tau^z]\left(\beta|_t\right)\\ =\ &\left(\beta|_{t-s}\right)\left(\beta|_i\right)[\beta|_i,\tau^z]\tau^{\Delta_s(i+s,t+s)}.\end{aligned}$$

for all $i,t\in\{0,1,\cdots,n-1\}$, $z\in\mathbb{Z}$

Proof. Since $\sigma_\beta=\sigma_\tau^s$, we have $\sigma_{\beta^{\tau^x}}=\sigma_\beta=\sigma_\tau^s$.

From (4), (5), (6) and (7), we obtain

$$\begin{aligned}&\tau^{-\frac{x-\overline{x}}{n}-\delta(j-x,x)}\beta|_{j-x}\tau^{\frac{x-\overline{x}}{n}+\delta(j-x+s,x)}\beta|_{j+s}\\ =\ &\beta|_j\tau^{-\frac{x-\overline{x}}{n}-\delta(j+s-x,x)}\beta|_{j+s-x}\tau^{\frac{x-\overline{x}}{n}+\delta(j+2s-x,x)}\end{aligned}\tag{15}$$

Setting $k=\dfrac{x-\overline{x}}{n}$ and $r=\overline{x}$ and using (15), we have

$$\begin{aligned}&\tau^{-k-\delta(j-r,r)}\beta|_{j-r}\tau^{k+\delta(j+s-r,r)}\beta|_{j+s}\\ &=\beta|_j\tau^{-k-\delta(j+s-r,r)}\beta|_{j+s-r}\tau^{k+\delta(j+2s-r,r)},\end{aligned}\tag{16}$$

for all $r,j\in Y$ and all $k\in\mathbb{Z}$.

Also on setting $t=\overline{j+s}$, $i=\overline{j+s-r}$ and $z=k+\delta(j+s-r,r)=k+\delta(i,t-i)$ and using (16), we obtain

$$\begin{aligned}&\tau^{-z+\delta(i,t-i)-\delta(i-s,t-i)}\beta|_{i-s}\tau^{z}\beta|_t\\ =\ &\beta|_{t-s}\tau^{-z}\beta|_i\tau^{z-\delta(i,t-i)+\delta(i+s,t-i)},\end{aligned}$$

for all $t,i\in\{0,1,\cdots,n-1\}$ and all $z\in\mathbb{Z}$.

Thus, it follows that

$$\begin{aligned} & \tau^{\delta(i,t-i)-\delta(i-s,t-i)}\beta|_{i-s}[\beta|_{i-s},\tau^z]\beta|_t \\ = \quad & \beta|_{t-s}\beta|_i[\beta|_i,\tau^z]\tau^{-\delta(i,t-i)+\delta(i+s,t-i)} \end{aligned}$$

for all $t, i \in \{0, 1, \cdots, n-1\}$ and all $z \in \mathbb{Z}$. $\square$

We develop below some properties of the Δ_s function to be used in the sequel.

Proposition 8. *The inductor function satisfies*

(i) $\Delta_s(i,t) = \delta(i,-s) - \delta(t,-s) = \begin{cases} 0, & \text{if } \overline{t}, \overline{i} \geq \overline{s} \text{ or } \overline{t}, \overline{i} < \overline{s} \\ 1, & \textit{if } \overline{t} < \overline{s} \leq \overline{i} \\ -1, & \textit{if } \overline{i} < \overline{s} \leq \overline{t} \end{cases}$,

(ii) $\Delta_s(i,t) = -\Delta_s(t,i)$,

(iii) $\Delta_s(i+s, t+s) = -\Delta_{-s}(i,t)$,

(iv) $\Delta_s(i,t) = \Delta_s(i,z) + \Delta_s(z,t)$,

(v) $\displaystyle\sum_{k=0}^{\frac{n}{(s,n)}-1} \Delta_s(i+ks, t+ks) = 0$,

(vi) $\displaystyle\sum_{k=0}^{n-1} \Delta_s(k,t) = \begin{cases} n-\overline{s}, & \textit{if } \overline{t} < \overline{s} \\ -\overline{s} & \textit{if } \overline{t} \geq \overline{s} \end{cases}$

for all $i, t, z \in \mathbb{Z}$.

Proof.

(i) Using the definition $\delta(i,j) = \frac{\overline{i}+\overline{j}-\overline{i+j}}{n}$ we have

$$\begin{aligned} \Delta_s(i,t) &= \frac{\overline{i}+\overline{t-i}-\overline{t}}{n} - \frac{\overline{i-s}+\overline{t-i}-\overline{t-s}}{n} \\ &= \frac{\overline{i}+\overline{-s}-\overline{i-s}}{n} - \frac{\overline{t}+\overline{-s}-\overline{t-s}}{n} \\ &= \delta(i,-s) - \delta(t,-s) \\ &= \begin{cases} 0, & \text{if } \overline{t}, \overline{i} \geq \overline{s} \text{ or } \overline{t}, \overline{i} < \overline{s} \\ 1, & \text{if } \overline{t} < \overline{s} \leq \overline{i} \\ -1, & \text{if } \overline{i} < \overline{s} \leq \overline{t} \end{cases}. \end{aligned}$$

(ii) Follows from (i).

(iii) Calculate

$$\begin{aligned} \Delta_s(i+s,t+s) &= \delta(i+s,t-i) - \delta(i,t-i) \\ &= -\left(\delta(i,t-i) - \delta(i+s,t-i)\right) \\ &= -\Delta_{-s}(i,t). \end{aligned}$$

(iv) This part follows from (i).

(v) From the definition of the Polarizer function

$$\sum_{k=0}^{\frac{n}{(n,s)}-1} \delta(i+ks, t-i) = \sum_{k=0}^{\frac{n}{(n,s)}-1} \delta(i+(k-1)s, t-i)$$

(vi) Finally, we have

$$\begin{aligned} \sum_{k=0}^{n-1} \Delta_s(k,t) &= \textstyle\sum_{k=0}^{\overline{s}-1} \Delta_s(k,t) + \sum_{k=\overline{s}}^{n-1} \Delta_s(k,t) \\ &\overset{(i)}{=} \begin{cases} n-\overline{s}, & \text{if } \overline{t} < \overline{s} \\ -\overline{s}, & \text{if } \overline{t} \geq \overline{s} \end{cases} . \end{aligned}$$

□

With the use of the inductor function notation we obtain

Proposition 9. *The following relations are verified in* $H = \langle \beta|_i \ \ (i \in Y), \tau \rangle$, *for all* $x, z \in \mathbb{Z}$ *and all* $i, t \in Y$:

(I) $\tau^{\Delta_s(i,t)} \beta|_{\overline{i-s}} \beta|_t = \beta|_{\overline{t-s}} \beta|_i \tau^{\Delta_s(i+s,t+s)}$;

(II) $[\beta|_{\overline{i-s}}, \tau^z]^{\beta|_t \tau^{-\Delta_s(i+s,t+s)}} = [\beta|_i, \tau^z]$;

(III) $[[\beta|_i, \tau^z], [\beta|_t, \tau^x]] = e$.

Proof. Returning to Lemma 6, we have

$$\begin{aligned} &\tau^{\Delta_s(i,t)} \left(\beta|_{i-s}\right) [\beta|_{i-s}, \tau^z] \left(\beta|_t\right) \\ = \ &\left(\beta|_{t-s}\right) \left(\beta|_i\right) [\beta|_i, \tau^z] \tau^{\Delta_s(i+s,t+s)}. \end{aligned}$$

Consequently,

$$\tau^{\Delta_s(i,t)} \beta|_{\overline{i-s}} \beta|_t = \beta|_{\overline{t-s}} \beta|_i \tau^{\Delta_s(i+s,t+s)} \tag{17}$$

and

$$[\beta|_{\overline{i-s}}, \tau^z]^{\beta|_t \tau^{-\Delta_s(i+s,t+s)}} = [\beta|_i, \tau^z], \tag{18}$$

for all $t, i \in Y$ and all $z \in \mathbb{Z}$.

From (18) and (14), $N = \left\langle [\beta|_i, \tau^{k_i}] \mid k_i \in \mathbb{Z}, i \in Y \right\rangle$ is a normal subgroup of H. Moreover, by applying alternately the above equations, we obtain

$$\begin{aligned} [\beta|_i, \tau^z]^{[\beta|_t, \tau^k]} &= [\beta|_i, \tau^z]^{\beta|_t^{-1} \tau^{-k} \beta|_t \tau^k} \\ &= [\beta|_i, \tau^z]^{\left(\tau^{-\Delta_s(i+s,t+s)} \tau^{\Delta_s(i+s,t+s)} \beta|_t^{-1} \tau^{-k} \beta|_t \tau^k\right)} \\ &\overset{(14)}{=} \left([\beta|_i, \tau^{-\Delta_s(i+s,t+s)}]^{-1} . [\beta|_i, \tau^{z-\Delta_s(i+s,t+s)}]\right)^{\left(\tau^{\Delta_s(i+s,t+s)} \beta|_t^{-1} \tau^{-k} \beta|_t \tau^k\right)} \\ &\overset{(18)}{=} \left([\beta|_{\overline{i-s}}, \tau^{-\Delta_s(i+s,t+s)}]^{-1} . [\beta|_{\overline{i-s}}, \tau^{z-\Delta_s(i+s,t+s)}]\right)^{\tau^{-k} \beta|_t \tau^k} \\ &\overset{(14)}{=} \begin{pmatrix} \left([\beta|_{\overline{i-s}}, \tau^{-k}]^{-1} . [\beta|_{i-s}, \tau^{-k-\Delta_s(i+s,t+s)}]\right)^{-1} \\ \left([\beta|_{\overline{i-s}}, \tau^{-k}]^{-1} . [\beta|_{\overline{i-s}}, \tau^{-k+z-\Delta_s(i+s,t+s)}]\right) \end{pmatrix}^{\beta|_t \tau^k} \end{aligned}$$

$$= \left([\beta|_{\overline{i-s}}, \tau^{-k-\Delta_s(i+s,t+s)}]^{-1}.[\beta|_{\overline{i-s}}, \tau^{-k+z-\Delta_s(i+s,t+s)}]\right)^{\beta|_t\tau^k}$$

$$\overset{(18)}{=} \left([\beta|_i, \tau^{-k-\Delta_s(i+s,t+s)}]^{-1}.[\beta|_i, \tau^{-k+z-\Delta_s(i+s,t+s)}]\right)^{\tau^{k+\Delta_s(i+s,t+s)}}$$

$$\overset{(14)}{=} [\beta|_i, \tau^z].$$

□

Corollary 3. *Let $\beta \in A_n$ such that $[\beta, \beta^{\tau^x}] = e$ for every $x \in \mathbb{Z}$ with $\sigma_\beta = \sigma_\tau^s$ for some $s \in \{0, 1, \cdots, n-1\}$. Then*

$$M = \left\langle [\beta|_i, \tau^{k_i}], \tau \mid k_i \in \mathbb{Z}, 0 \leq i \leq n-1 \right\rangle$$

is a normal metabelian subgroup of H.

Proof. By Proposition 9 $N = \left\langle [\beta|_i, \tau^{k_i}] \mid k_i \in \mathbb{Z}, 0 \leq i \leq n-1 \right\rangle$ is abelian and normal in H. Since $N\tau \in Z(H/N)$, it follows that $M = N\langle\tau\rangle$ is a normal subgroup of H and is clearly metabelian. □

We are ready to prove part *(II) (i) of* Theorem B.

Theorem 1. *Let $\beta \in \mathcal{A}_n$ be such that $[\beta, \beta^{\tau^x}] = e, \forall x \in \mathbb{Z}$ and $\sigma_\beta = \sigma_\tau^s$ for some $s \in Y$ and $H = \langle \beta|_0, \cdots, \beta|_{n-1}, \tau \rangle$. Then,*

(i) *the group $O = \left\langle [\beta|_i, \tau^x], \beta|_j\beta|_{j+s}\cdots\beta|_{j+(m-1)s}, \tau \mid i, j \in Y, x \in \mathbb{Z}_n \right\rangle$ is an abelian normal subgroup of H;*

(ii) *the quotient group H/O is isomorphic to a subgroup of $C_m \wr C_n$. In particular, H is metabelian-by-finite.*

Proof. (i) Recall

$$\begin{array}{rcl} N & = & \left\langle [\beta|_i, \tau^{k_i}] \mid k_i \in \mathbb{Z}, i \in Y \right\rangle, \\ K & = & N\left\langle \beta|_j\beta|_{\overline{j+s}}\cdots\beta|_{j+(m-1)s} \mid j \in Y \right\rangle \end{array}$$

where $m = \frac{n}{\gcd(n,s)}$. Then, by Proposition 9, N is an abelian normal subgroup of H.

By (18), we have

$$\begin{array}{rl} & [\beta|_i, \tau^z]^{\beta|_j\beta|_{\overline{j+s}}\cdots\beta|_{\overline{j+(m-1)s}}} \\ = & [\beta|_{i+s}, \tau^z]^{\tau^{\Delta_t(i+2s,j+s)}\beta|_{\overline{j+s}}\cdots\beta|_{\overline{j+(m-1)s}}} \\ = & [\beta|_{i+2s}, \tau^z]^{\tau^{\Delta_s(i+2s,j+s)+\Delta_s(i+3s,j+2s)}\beta|_{\overline{j+2s}}\cdots\beta|_{\overline{j+(m-1)s}}} \\ = & [\beta|_i, \tau^z]^{\tau^{\sum_{k=0}^{m-1}\Delta_s(i+(k+1)s,j+ks)}} \\ \overset{\text{Prop.8}(v)}{=} & [\beta|_i, \tau^z] \end{array}$$

Thus,

$$[[\beta|_i, \tau^z], (\beta^m)|_j] = e, \forall i, j \in Y, \forall z \in \mathbb{Z} \tag{19}$$

Since $\sigma_\beta = \sigma_\tau^s$, we have by Lemma 2

$$[(\beta^m)|_i, (\beta^m)|_j] = e, \forall i, j \in Y. \tag{20}$$

Moreover,

$$(\beta^m)|_i^{\tau} = (\beta^m)|_i[(\beta^m)|_i, \tau]. \tag{21}$$

Since $[\beta, \beta^{\tau^x}] = e, \forall x \in \mathbb{Z}$, it follows that $[\beta^m, \beta^{\tau^x}] = e, \forall x \in \mathbb{Z}$.
Therefore, by (6) and (7),

$$e = (\beta^m)|_{(i)\sigma_{\beta^{\tau^x}}}^{-1} (\beta^{\tau^x})|_i^{-1} (\beta^m)|_i (\beta^{\tau^x})|_{(i)\sigma_{\beta^m}}, \forall x \in \mathbb{Z}, \forall i \in Y.$$

Now, as $\sigma_\beta = \sigma_\tau^s$ and $\sigma_{\beta^m} = e$, we reach

$$(\beta^m)|_{\overline{i+s}} = (\beta^m)|_i^{(\beta^{\tau^x})|_i}, \forall x \in \mathbb{Z}, \forall i \in Y. \tag{22}$$

By (4) and (5), the following

$$(\beta^{\tau^x})_i = (\tau^x)^{-1}_{(i)\sigma_{\tau^x}^{-1}} \beta|_{(i)\sigma_{\tau^x}^{-1}} (\tau^x)|_{(i)\sigma_{\tau^x}^{-1}\sigma_\beta} = (\tau^x)|_{\overline{i-x}}^{-1} \beta|_{\overline{i-x}} (\tau^x)_{\overline{i-x+s}}$$

holds for all $i \in Y$ and all $x \in \mathbb{Z}$.
From which we derive

$$(\beta^{\tau^x})|_i = \tau^{-\frac{x-\overline{x}}{n} - \delta(i-x,x)} \beta|_{\overline{i-x}} \tau^{\frac{x-\overline{x}}{n} + \delta(i-x+s,x)}, \tag{23}$$

for all $i \in Y$ and all $x \in \mathbb{Z}$.
Therefore, by (22) and (23),

$$(\beta^m)|_{\overline{i+s}} = (\beta^m)|_i^{\tau^{-\frac{x-\overline{x}}{n} - \delta(i-x,x)} \beta|_{\overline{i-x}} \tau^{\frac{x-\overline{x}}{n} + \delta(i-x+s,x)}},$$

for all $i \in Y$ and all $x \in \mathbb{Z}$..
On writing $x = kn + \overline{x} = kn + r, r \in \mathbb{Z}$ in the above equation, we obtain

$$(\beta^m)|_{\overline{i+s}} = (\beta^m)|_i^{\tau^{-k-\delta(i-r,r)} \beta|_{\overline{i-r}} \tau^{k+\delta(i-r+s,r)}}$$

$$\Rightarrow (\beta^m)|_{\overline{i+s}}^{\tau^{-k-\delta(i-r+s,r)}} = (\beta^m)|_i^{\beta|_{\overline{i-r}} \tau^{-k-\delta(i-r,r)} [\tau^{-k-\delta(i-r,r)}, \beta|_{\overline{i-r}}]}$$

$$\Rightarrow (\beta^m)|_{\overline{i+s}}^{\tau^{-k-\delta(i-r+s,r)} [\beta|_{\overline{i-r}}, \tau^{-k-\delta(i-r,r)}] \tau^{k+\delta(i-r,r)}} = (\beta^m)|_i^{\beta|_{\overline{i-r}}}$$

for all $i, r \in Y$ and all $k \in \mathbb{Z}$.
By (19), (21) and using the fact that N is abelian and normal in H, we find

$$(\beta^m)|_{\overline{i+s}}^{\tau^{\delta(i-r,r)-\delta(i-r+s,r)}} = (\beta^m)|_i^{\beta|_{\overline{i-r}}}$$

$$\Rightarrow (\beta^m)|_{\overline{i+s}}^{\tau^{\delta(i-r,i-r+s)}} = (\beta^m)|_i^{\beta|_{\overline{i-r}}}$$

for all $i, r \in Y$.
On setting $j = \overline{i-r}$, we get

$$(\beta^m)|_{\overline{i+s}}^{\tau^{\delta(j,j+s)}} = (\beta^m)|_i^{\beta|_j} \tag{24}$$

for all $i, j \in Y$.

Further, by using equations (19),(20) (21), (24) and

$$(\beta^m)|_i = \beta|_i \beta|_{\overline{i+s}} \cdots \beta|_{\overline{i+(m-1)s}}, \tag{25}$$

we conclude that also K is an abelian normal subgroup of H.

Now, $O = K\langle\tau\rangle$ is metabelian. Moreover it is normal in H, because

$$\tau^{\beta|_i} = \tau\tau^{-1}\tau^{\beta|_i} = \tau[\tau, \beta|_i] \in O$$

for all $i \in Y$.

(ii) Consider the following Fibonacci type group

$$X = \left\langle b_0, \cdots, b_{n-1} \mid b_i b_{\overline{j+s}} = b_j b_{\overline{i+s}}, b_i b_{\overline{i+s}} \cdots b_{\overline{i+(m-1)s}} = e, \forall i, j \in Y \right\rangle.$$

Equations (17) and (18) show that $\dfrac{H}{M}$ is a homomorphic image of X. We will prove that X is isomorphic to a subgroup of

the wreath product $C_m \wr C_n$.

As a matter of fact the group $C_m \wr C_n$ has the presentation

$$\left\langle u, a \mid u^m = e, a^n = e, u^{a^i} u^{a^j} = u^{a^j} u^{a^i} \right\rangle.$$

On defining $b = a^s u^{-1}$, we have

$$\begin{aligned} u^m = e \quad & (a^{-s}b)^m = e \\ & \Rightarrow (\underbrace{a^{-s}b \cdots a^{-s}b}_{m \text{ terms}})^{a^{-s+i}} = e \\ & \Rightarrow b^{a^i} b^{a^{i+s}} \cdots b^{a^{i+(m-1)s}} = e. \end{aligned}$$

Also, the commutation relation

$$u^{a^i} u^{a^j} = u^{a^j} u^{a^i}$$

implies

$$\begin{aligned} & (b^{-1}a^s)^{a^i}(b^{-1}a^s)^{a^j} = (b^{-1}a^s)^{a^j}(b^{-1}a^s)^{a^i} \\ \Rightarrow \quad & (a^{-s}b)^{a^j}(a^{-s}b)^{a^i} = (a^{-s}b)^{a^i}(a^{-s}b)^{a^j} \\ \Rightarrow \quad & b^{a^j} a^{-s} b^{a^i} = b^{a^i} a^{-s} b^{a^j} \\ \Rightarrow \quad & b^{a^j} b^{a^{i+s}} = b^{a^i} b^{a^{j+s}}. \end{aligned}$$

Thus, by using Tietze transformations we conclude that $C_m \wr C_n$ has the presentation

$$\left\langle a, b \mid a^n = e, b^{a^j} b^{a^{i+s}} = b^{a^i} b^{a^{j+s}}, b^{a^i} b^{a^{i+s}} \cdots b^{a^{i+(m-1)s}} = e, \forall i, j \in Y \right\rangle.$$

Then, on introducing $b_i = b^{a^i}, i = 0, \cdots, n-1$, the above presentation is expressed as

$$\left\langle a, b_0, \cdots, b_{n-1} \mid a^n = e, b_i = b_0^{a^i},\ b_j b_{\overline{i+s}} = b_i b_{\overline{j+s}},\ b_i b_{\overline{i+s}} \cdots b_{\overline{i+(m-1)s}} = e, \right.$$
$$\left. \forall i, j \in Y \right\rangle .$$

□

The next results leads to a proof of Theorem C.

Lemma 7. *Let $\sigma = (0, 1, ..., n-1) \in \Sigma_n$ and let L be the layer closure of $\langle\sigma\rangle$ in $\mathcal{A}_n$. Suppose $\beta = (\beta|_0, \beta|_1, \cdots, \beta|_{n-1})\sigma_\beta \in L$ satisfies $[\beta, \beta^{\tau^x}] = e$ for all $x \in \mathbb{Z}$. Write $\sigma_\beta = \sigma^s$ and $\sigma_{\beta|_i} = \sigma^{m_i}$ for all $i \in Y$. Then for all $i, j \in Y$, the following congruence holds*

$$\Delta_s(i,t) + m_{\overline{i-s}} + m_t \equiv m_{\overline{t-s}} + m_i + \Delta_s(i+s, t+s) \bmod n, \tag{26}$$

Proof. Since $\sigma_{\beta|_i} = \sigma^{m_i}$, we conclude by (17),

$$\sigma^{\Delta_s(i,t)+m_{\overline{i-s}}+m_t} = \sigma^{m_{\overline{t-s}}+m_i+\Delta_s(i+s,t+s)}$$

and therefore, $\Delta_s(i,t) + m_{\overline{i-s}} + m_t \equiv m_{\overline{t-s}} + m_i + \Delta_s(i+s, t+s) \bmod n$. □

Lemma 8. *Maintain the notation of the previous lemma and let n be an odd integer. Then,*

$$\sigma_{(\beta^n)|_0} = \sigma_{(\beta|_0\beta|_1\cdots\beta|_{n-1})} = \sigma.$$

Proof. From

$$\Delta_1(i,t) + m_{\overline{i-1}} + m_t \equiv m_{\overline{t-1}} + m_i + \Delta_1(i+1, t+1) \bmod n$$

we conclude

$$\begin{aligned} &\sum_{i=0}^{n-2} \sum_{t=i+1}^{n-1} \left(\Delta_1(i,t) + m_{\overline{i-1}} + m_t\right) \\ \equiv\ &\sum_{i=0}^{n-2} \sum_{t=i+1}^{n-1} \left(m_{\overline{t-1}} + m_i + \Delta_1(i+1, t+1)\right) \bmod n. \end{aligned}$$

Now,

$$\begin{aligned} \sum_{i=0}^{n-2} \sum_{t=i+1}^{n-1} \Delta_1(i,t) &\overset{\text{Prop.8(i)}}{=} \sum_{t=1}^{n-1} \Delta_1(0,t) \overset{\text{Prop.8(ii)}}{=} \sum_{t=0}^{n-1} \Delta_1(0,t) \\ &\overset{\text{Prop.8(ii)}}{=} \sum_{t=0}^{n-1} -\Delta_1(t,0) \overset{\text{Prop.8(vi)}}{=} -(n-1), \end{aligned}$$

$$\begin{aligned} \sum_{i=0}^{n-2} \sum_{t=i+1}^{n-1} \Delta_1(i+1,t+1) &\overset{\text{Prop.8(i)}}{=} \sum_{i=0}^{n-2} \Delta_1(i+1,0) \overset{\text{Prop.8(ii)}}{=} \sum_{i=0}^{n-1} \Delta_1(i,0) \\ &\overset{\text{Prop.8(vi)}}{=} (n-1), \end{aligned}$$

$$\sum_{i=0}^{n-2}\sum_{t=i+1}^{n-1}\left(m_{\overline{i-1}}+m_t\right)=2(n-1)m_{n-1}+(n-2)\sum_{k=0}^{n-2}m_k$$

and

$$\sum_{i=0}^{n-2}\sum_{t=i+1}^{n-1}(m_{\overline{t-1}}+m_i)=n\sum_{k=0}^{n-1}m_k.$$

Since n is odd, we have

$$\sum_{k=0}^{n-1}m_k\equiv 1 \bmod n$$

and therefore, $\sigma_{\beta|_0\cdots\beta|_{n-1}}=\sigma^{(m_0+\ldots m_{n-1})}=\sigma.$ □

We prove Theorem C below.

Theorem 2. *Let n be an odd number, $\sigma=(0,\cdots,n-1)\in\Sigma_n$ and let L be the layer closure of $\langle\sigma\rangle$ in A_n. Let s an integer relatively prime to n and $\beta=(\beta|_0,\beta|_1,\cdots,\beta|_{n-1})\sigma^s\in L$ be such that $[\beta,\beta^{\tau^x}]=e$ for all $x\in Z$. Then β is a conjugate of τ in L.*

Proof. We start with the case $s=1$. The element

$$\alpha(1)=(e,\beta|_0^{-1},(\beta|_0\beta|_1)^{-1},\cdots,(\beta|_0\cdots\beta|_{n-2})^{-1})\in \mathrm{Stab}_G(1)$$

conjugates β to

$$\beta^{\alpha(1)}=(e,\cdots,e,\beta|_0\cdots\beta|_{n-1})\sigma.$$

By Lemma8 we find $\sigma_{\beta|_0\beta|_1\cdots\beta|_{n-1}}=\sigma$. Moreover by Proposition 6,

$$[(\beta^n)|_0,(\beta^n)|_0^{\tau^x}]=[\beta|_0\beta|_1\cdots\beta|_{n-1},(\beta|_0\beta|_1\cdots\beta|_{n-1})^{\tau^x}]=e,$$

for all integers x. Therefore $\beta|_0\beta|_1\cdots\beta|_{n-1}$ satisfies the hypothesis of the theorem. The process can be repeated until we obtain a sequence $(\alpha(k))_{k\in\mathbb{N}}$ such that $\beta^{\alpha(1)\alpha(2)\cdots\alpha(k)\cdots}=\tau$, where $\alpha(k)\in\mathrm{Stab}_G(k)$ satisfies $\alpha(k)|_u=\alpha(k)|_v$ for all $u,v\in\mathcal{M}$ with $|u|=|v|=k-1$.

Now, suppose more generally s is such $\gcd(s,n)=1$ and let k be a minimum positive integer for which $sk\equiv 1 \bmod(n)$. Then β^k satisfies the hypothesis of the first part and so, there exists $\alpha\in G$ such that $(\beta^k)^\alpha=\tau$. Since k is invertible in $\mathbb{Z}_n$, there exists an automorphism γ of the tree such that $\tau^\gamma=\tau^{k^{-1}}$. Thus, $\beta^{\alpha\gamma^{-1}}=\tau$. □

6. Solvable groups for $n=p$, a prime number.

We will prove in this section the case $n=p$ of Theorem A.

Let B be an abelian subgroup of $Aut(T_p)$ normalized by τ and let $\beta\in B$. By Lemma 5, $\sigma_\beta\in\langle\sigma_\tau\rangle$ and therefore in effect we have two cases, $\sigma_\beta=e,\sigma_\tau$.

Proposition 10. *Suppose $\sigma_\beta=\sigma_\tau$. Then, $\sigma_{\beta|_i}\in\langle\sigma_\tau\rangle$ for all $i\in Y$.*

Proof. By theorem 1, O is a normal subgroup of H and $\frac{H}{O}$ is isomorphic to a subgroup of $C_p \wr C_p$.

By Lemma 5, O is a subgroup of $\langle \sigma_\tau \rangle$ modulo $Stab_p(1)$.

Therefore, H is a p-group modulo $Stab_p(1)$ and by Lemma 5, we have $\sigma_{\beta|_i} \in \langle \sigma_\tau \rangle$. $\square$

Theorem 3. *Let p be a prime number and $\beta \in \mathrm{Aut}(T_p)$ such that $\sigma_\beta = \sigma_\tau^s$ for some integer s relatively prime to p. Suppose $[\beta, \beta^{\tau^x}] = e$ for all $x \in \mathbb{Z}$. Then β is conjugate to τ in $\mathrm{Aut}(T_p)$.*

Proof. Suppose $s = 1$. Recall that

$$\alpha(1) = (e, \beta|_0^{-1}, (\beta|_0\beta|_1)^{-1}, \cdots, (\beta|_0 \cdots \beta|_{p-2})^{-1}) \in \mathrm{Stab}_G(1)$$

conjugates β to its normal form

$$\beta^{\alpha(1)} = (e, \cdots, e, \beta|_0 \cdots \beta|_{p-1})\sigma.$$

By Lemma 8 we have $\sigma_{\beta|_0\beta|_1\cdots\beta|_{p-1}} = \sigma_\tau$. Moreover by Proposition 6,

$$[\beta^p|_0, (\beta^p|_0)^{\tau^x}] = [\beta|_0\beta|_1 \cdots \beta|_{p-1}, (\beta|_0\beta|_1 \cdots \beta|_{p-1})^{\tau^x}] = e,$$

for all integers x. Therefore $\beta|_0\beta|_1 \cdots \beta|_{n-1}$ satisfies the condition of the theorem.. This process can be repeated to produce a sequence $(\alpha(k))_{k\in\mathbb{N}}$ such that $\beta^{\alpha(1)\alpha(2)\cdots\alpha(k)\cdots} = \tau$, where $\alpha(k) \in \mathrm{Stab}(k)$ satisfies $\alpha(k)|_u = \alpha(k)|_v$ for all $u, v \in \mathcal{M}$ where $|u| = |v| = k - 1$.

Now, to the general case, s such $\gcd(p, s) = 1$. Let k be the minimum positive integer which is the inverse of s modulo p. Then, $\sigma|_{\beta^k} = \sigma_\tau$ and β^k satisfies the hypotheses. Thus there exists $\alpha \in \mathcal{A}_p$ such that $\left(\beta^k\right)^\alpha = \tau$. Let k^{-1} be the inverse of k in $U(\mathbb{Z}_n)$; then $\beta^\alpha = \tau^{k^{-1}}$. There exists $\gamma \in N_{\mathcal{A}_p}\overline{<\tau>}$ which conjugates τ to $\tau^{k^{-1}}$ and so, $(\beta^\alpha)^{\gamma^{-1}} = \tau$. $\square$

Lemma 9. *Let p be a prime number and $\beta \in \mathrm{Aut}(T_p)$ such that $[\beta, \beta^{\tau^x}] = e$ for all $x \in \mathbb{Z}$. Then, there exists a tree level m and a conjugate μ of τ such that $\beta \in \times_{p^m}\overline{\langle \mu \rangle}$ and there exists an index u of length m such that $\beta|_u = \mu$.*

Proof. Let m be the minimum tree level such that $\sigma_{\beta|_u} \neq e$ for some $|u| = m$. Therefore, $\sigma_{\beta|_u} = \sigma_\tau^s$ for some integer s such that $\gcd(p, s) = 1$ and so, $\mu = \beta|_u$ is conjugate to τ in $\mathrm{Aut}(T_p)$. Since $\beta \in \mathrm{Stab}(m)$, by Proposition 6 $[\mu, \beta|_v] = e$ for all indices v such that $|v| = m$. Therefore, $\beta|_v \in \overline{\langle \mu \rangle}$ for all v such that $|v| = m$. $\square$

Theorem 4. *Let p be a prime number, $\sigma = (0, 1, \cdots, p-1) \in \Sigma_p$, $F = N_{\Sigma_p}(\langle \sigma \rangle)$, $\Gamma_0 = N_{\mathcal{A}}(\overline{<\tau>}$. Let G be a finitely generated solvable subgroup of Aut(T_p) which contains the p-adic adding machine τ. Then, there exists an integer $t \geq 1$ such that G is conjugate to a subgroup of*

$$\times_p (\cdots (\times_p (\times_p \Gamma_0 \rtimes F) \rtimes) \cdots) \rtimes F.$$

Proof. We may suppose G has derived length $d \geq 2$. Let B be the $(d-1)$-th term of the derived series of G. By Theorem 9, there exists a level t such that B is a subgroup of $V = \times_{p^t} \overline{\langle \mu \rangle}$ where $\mu = \tau^{\alpha}$ for some $\alpha \in Aut\,(T_n)$.

We will show that G is a subgroup of

$$\dot{J} = \times_p \left(\cdots \left(\times_p \left(\times_p \left(\Gamma_0 \right)^{\alpha} \rtimes \Sigma_p \right) \rtimes \Sigma_p \right) \cdots \right) \rtimes \Sigma_p,$$

where $\times_p$ appears t times.

Let $\gamma \in G \backslash \dot{J}$. Then there exists an index w of length t such that $\gamma|_w \notin (\Gamma_0)^{\alpha}$. Since τ is transitive on all levels of the tree , by Theorem 9, there exists $\beta \in B$ such that $\beta|_w = \mu^{\eta}$ for some $\eta \in U(\mathbb{Z}_p)$.

Write $v = w^{\gamma}$. Then,

$$\left(\beta^{\gamma} \right)|_v \overset{(9)}{=} \left(\beta|_{v^{\gamma^{-1}}} \right)^{\gamma|_{v^{\gamma^{-1}}}} = \left(\beta|_w \right)^{\gamma|_w} \notin \overline{\langle \mu \rangle},$$

and this implies $\beta^{\gamma} \notin B \leq \overline{\langle \mu \rangle}$ and $\gamma \notin G$. Hence, G is a subgroup of $\dot{J}$.

Now, since G is a solvable group containing τ, there exist G_i $(0 \leq i \leq t)$ solvable subgroups of Σ_p containing $\sigma = (0, 1, \cdots, p-1)$ such that G is a subgroup of

$$R_t\,(\alpha) = \times_p \left(\cdots \left(\times_p \left(\times_p \left(\Gamma_0 \right)^{\alpha} \rtimes G_1 \right) \rtimes G_2 \right) \cdots \right) \rtimes G_t.$$

Since for all i, we have $G_i \leq F$ we may substitute the $G_i's$ by F. Finally, $R_t\,(\alpha)$ is a conjugate of $R_t\,(1)$ by the diagonal automorphism $\alpha^{(t)}$. $\square$

7. Two cases for n even

7.1. The case $\sigma_\beta = (\sigma_\tau)^{\frac{n}{2}}$.

Theorem 5. *Let n be an even number, $\beta \in \mathcal{A}_n$ such that $\sigma_\beta = \sigma_\tau^{\frac{n}{2}}$ and $[\beta, \beta^{\tau^x}] = e$ for all $x \in \mathbb{Z}$. Then $H = \langle \beta|_i \ (0 \leq i \leq n-1), \tau \rangle$ is a metabelian subgroup of $\mathcal{A}_n$.*

Proof. Define the subgroup

$$R = \left\langle [\beta|_t, \tau^k],\ \beta|_i \beta|_{i+\frac{n}{2}},\ \beta|_j^2 \tau^{-\Delta(j, j+\frac{n}{2})} \mid k \in \mathbb{Z} \text{ and } i, j, t \in Y \right\rangle.$$

Denote $\Delta_{\frac{n}{2}}(i,j)$ by $\Delta(i,j)$.

We will prove that N is an abelian normal subgroup of H.

(I) R is normal in H :

– $\left\langle [\beta|_i, \tau^k] \right\rangle^H \leq R$:

$$[\beta|_{i+\frac{n}{2}}, \tau^k]^{\beta|_j} \overset{(18)}{=} [\beta|_i, \tau^k]^{\tau^{\Delta(j,i)}};$$

– $\left\langle \beta|_i \beta_{i+\frac{n}{2}} \right\rangle^H \leq R$:

$$\begin{aligned} \left(\beta|_i \beta|_{i+\frac{n}{2}} \right)^{\tau^k} &= \left(\beta|_i \beta|_{i+\frac{n}{2}} \right) . [\beta|_i \beta|_{i+\frac{n}{2}}, \tau^k] \\ &= \left(\beta|_i \beta|_{i+\frac{n}{2}} \right) [\beta|_i, \tau^k]^{\beta|_{i+\frac{n}{2}}} [\beta|_{i+\frac{n}{2}}, \tau^k] \end{aligned}$$

$$\stackrel{(18)}{=} \left(\beta|_i\beta|_{i+\frac{n}{2}}\right) [\beta|_{i+\frac{n}{2}}, \tau^k]^{\tau^{\Delta(i+\frac{n}{2},i+\frac{n}{2})}} [\beta|_{i+\frac{n}{2}}, \tau^k] \stackrel{\text{Prop.8}}{=} \beta|_i\beta|_{i+\frac{n}{2}}[\beta|_{i+\frac{n}{2}}, \tau^k]^2$$

(27)
$$\begin{aligned}
(\beta|_i\beta|_{i+\frac{n}{2}})^{\beta|_j} &= \left(\beta|_j^{-1}\beta|_i\beta|_{i+\frac{n}{2}}\beta|_j\right) \tau^{\Delta(j+\frac{n}{2},i+\frac{n}{2})}\tau^{-\Delta(j+\frac{n}{2},i+\frac{n}{2})} \\
&\stackrel{(17)}{=} \left(\beta|_j^{-1}\beta|_i\right) \tau^{\Delta(j,i)} \left(\beta|_{j+\frac{n}{2}}\beta|_i\right) \tau^{-\Delta(j+\frac{n}{2},i+\frac{n}{2})} \\
&= \left(\beta|_j^{-1}\beta|_i\beta|_{j+\frac{n}{2}}\right) \tau^{\Delta(j,i)}.[\tau^{\Delta(j,i)}, \beta|_{j+\frac{n}{2}}].\beta|_i\tau^{-\Delta(j+\frac{n}{2},i+\frac{n}{2})} \\
&\stackrel{(17)}{=} \left(\beta|_j^{-1}\right) \tau^{\Delta(j+\frac{n}{2},i+\frac{n}{2})} \left(\beta|_j\beta|_{i+\frac{n}{2}}\right). \\
&\quad [\tau^{\Delta(j,i)}, \beta|_{j+\frac{n}{2}}].\beta|_i\tau^{-\Delta(j+\frac{n}{2},i+\frac{n}{2})} \\
&= \tau^{\Delta(j+\frac{n}{2},i+\frac{n}{2})}.[\tau^{\Delta(j+\frac{n}{2},i+\frac{n}{2})}, \beta|_j]. \\
&\quad \beta|_{i+\frac{n}{2}}[\tau^{\Delta(j,i)}, \beta|_{j+\frac{n}{2}}]\beta|_i\tau^{-\Delta(j+\frac{n}{2},i+\frac{n}{2})} \\
&\stackrel{Prop.8}{=} \tau^{-\Delta(j,i)}[\tau^{-\Delta(j,i)}, \beta|_j].\beta|_{i+\frac{n}{2}}. \\
&\quad [\tau^{\Delta(j,i)}, \beta|_{j+\frac{n}{2}}]\beta|_i\tau^{\Delta(j,i)} \\
&\stackrel{(18)}{=} \tau^{-\Delta(j,i)}\beta|_{i+\frac{n}{2}}.[\tau^{-\Delta(j,i)}, \beta|_{j+\frac{n}{2}}]^{\tau^{\Delta(j,i)}}.[\tau^{\Delta(j,i)}, \beta|_{j+\frac{n}{2}}].\beta|_i\tau^{\Delta(j,i)} \\
&\stackrel{(14)}{=} \left(\beta|_{i+\frac{n}{2}}\beta|_i\right)^{\tau^{\Delta(j,i)}}.
\end{aligned}$$

– $\left\langle \beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}\right\rangle^H \leq R$:

$$\begin{aligned}
(\beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})})^{\tau^k} &= \beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}.[\beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}, \tau^k] \\
= \quad & \beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}.[\beta|_j^2, \tau^k]^{\tau^{-\Delta(j,j+\frac{n}{2})}} \\
= \quad & \beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}\left([\beta|_j, \tau^k]^{\beta|_j}.[\beta|_j, \tau^k]\right)^{\tau^{-\Delta(j,j+\frac{n}{2})}} \\
\stackrel{(18)}{=} \quad & \beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}\left([\beta|_{j+\frac{n}{2}}, \tau^k]^{\tau^{\Delta(j,j+\frac{n}{2})}}.[\beta|_j, \tau^k]\right)^{\tau^{-\Delta(j,j+\frac{n}{2})}} \\
= \quad & \beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}[\beta|_{j+\frac{n}{2}}, \tau^k][\beta|_j, \tau^k]^{\tau^{-\Delta(j,j+\frac{n}{2})}}.
\end{aligned}$$

By Proposition 8 and 9, we can show

(28)
$$\left(\beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}\right)^{\beta|_i} = \left(\beta|_{j+\frac{n}{2}}^2\tau^{-\Delta(j+\frac{n}{2},j)}[\tau^{-\Delta(j+\frac{n}{2},j)}, \beta|_{j+\frac{n}{2}}]\right)^{\tau^{\Delta(i,j)}}.$$

(II) The subgroup R is abelian:

(29)
$$[\beta|_i, \tau^k]^{\beta|_j\tau^t} \stackrel{Prop.9}{=} [\beta|_i, \tau^k]^{\tau^t\beta|_j};$$

(30)
$$[\beta|_i, \tau^k]^{\beta|_j\beta|_{j+\frac{n}{2}}} \stackrel{(18)}{=} [\beta|_{i+\frac{n}{2}}, \tau^k]^{\tau^{\Delta(j,i+\frac{n}{2})}\beta|_{j+\frac{n}{2}}} \stackrel{(29)}{=} [\beta|_{i+\frac{n}{2}}, \tau^k]^{\beta|_{j+\frac{n}{2}}\tau^{\Delta(j,i+\frac{n}{2})}}$$

$$\overset{(18)}{=} [\beta|_i, \tau^k]^{\tau^{\Delta(j+\frac{n}{2},i)+\Delta(j,i+\frac{n}{2})}} \overset{\text{Prop.8}}{=} [\beta|_i, \tau^k]$$

$$\begin{aligned} & [\beta|_i, \tau^k]^{\beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}} \overset{(18)}{=} [\beta|_{i+\frac{n}{2}}, \tau^k]^{\tau^{\Delta(j,i+\frac{n}{2})}\beta|_j\tau^{-\Delta(j,j+\frac{n}{2})}} \\ \overset{(29)}{=} \; & [\beta|_{i+\frac{n}{2}}, \tau^k]^{\beta|_j\tau^{\Delta(j,i+\frac{n}{2})-\Delta(j,j+\frac{n}{2})}} \overset{(18)}{=} [\beta|_i, \tau^k]^{\tau^{\Delta(j,i)+\Delta(j,i+\frac{n}{2})-\Delta(j,j+\frac{n}{2})}} \\ \overset{\text{Prop.8}}{=} \; & [\beta|_i, \tau^k] \end{aligned} \tag{31}$$

$$\begin{aligned} \left(\beta|_i\beta|_{i+\frac{n}{2}}\right)^{\beta|_j\beta|_{j+\frac{n}{2}}} & \overset{(27)}{=} \left(\beta|_{i+\frac{n}{2}}\beta|_i\right)^{\tau^{\Delta(j,i)}\beta|_{j+\frac{n}{2}}} \\ & = \left(\beta|_{i+\frac{n}{2}}\beta|_i\right)^{\left(\beta|_{j+\frac{n}{2}}\tau^{\Delta(j,i)}[\tau^{\Delta(j,i)},\beta|_{j+\frac{n}{2}}]\right)} \\ & \overset{(27)}{=} \left(\beta|_i\beta|_{i+\frac{n}{2}}\right)^{\left(\tau^{\Delta(j+\frac{n}{2},i+\frac{n}{2})+\Delta(j,i)}.[\tau^{\Delta(j,i)},\beta|_{j+\frac{n}{2}}]\right)} \\ & \overset{\text{Prop.8}}{=} \left(\beta|_i\beta|_{i+\frac{n}{2}}\right)^{[\tau^{\Delta(j,i)},\beta|_{j+\frac{n}{2}}]} \\ & \overset{(30)}{=} \beta|_i\beta|_{i+\frac{n}{2}} \end{aligned}$$

$$\begin{aligned} (\beta|_i\beta|_{i+\frac{n}{2}})^{\beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}} & \overset{(27)}{=} (\beta|_{i+\frac{n}{2}}\beta|_i)^{\tau^{\Delta(j,i)}\beta|_j\tau^{-\Delta(j,j+\frac{n}{2})}} \\ & = (\beta|_{i+\frac{n}{2}}\beta|_i)^{\beta|_j\tau^{\Delta(j,i)}[\tau^{\Delta(j,i)},\beta|_j]\tau^{-\Delta(j,j+\frac{n}{2})}} \\ & = (\beta|_i\beta|_{i+\frac{n}{2}})^{\tau^{\Delta(j,i+\frac{n}{2})+\Delta(j,i)}[\tau^{\Delta(j,i)},\beta|_j]\tau^{-\Delta(j,j+\frac{n}{2})}} \\ & \overset{\text{Prop.8}}{=} (\beta|_i\beta|_{i+\frac{n}{2}})^{[\tau^{\Delta(j,i)},\beta|_j]^{\tau^{\Delta(j+\frac{n}{2},j)}}} \\ & \overset{\text{Prop.9}}{=} \beta|_i\beta|_{i+\frac{n}{2}} \end{aligned}$$

Let

$$\alpha = \beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}[\tau^{-\Delta(j,j+\frac{n}{2})}, \beta|_j]. \tag{32}$$

Then,

$$\begin{aligned} & \left(\beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}\right)^{\beta|_i^2\tau^{-\Delta(i,i+\frac{n}{2})}} \\ \overset{(28)}{=} \; & \left(\beta|_{j+\frac{n}{2}}^2\tau^{-\Delta(j+\frac{n}{2},j)}.[\tau^{-\Delta(j+\frac{n}{2},j)}, \beta|_{j+\frac{n}{2}}]\right)^{\tau^{\Delta(i,j)}\beta|_i\tau^{-\Delta(i,i+\frac{n}{2})}} \\ = \; & \left(\beta|_{j+\frac{n}{2}}^2\tau^{-\Delta(j+\frac{n}{2},j)}.[\tau^{-\Delta(j+\frac{n}{2},j)}, \beta|_{j+\frac{n}{2}}]\right)^{\left(\beta|_i\tau^{\Delta(i,j)}.[\tau^{\Delta(i,j)},\beta|_i].\tau^{-\Delta(i,i+\frac{n}{2})}\right)} \\ = \; & \left(\left(\beta|_{j+\frac{n}{2}}^2\tau^{-\Delta(j+\frac{n}{2},j)}\right)^{\beta|_i}.[\tau^{-\Delta(j+\frac{n}{2},j)}, \beta|_{j+\frac{n}{2}}]^{\beta|_i}\right)^{\left(\tau^{\Delta(i,j)}.[\tau^{\Delta(i,j)},\beta|_i].\tau^{-\Delta(i,i+\frac{n}{2})}\right)} \\ \overset{(18)}{=} \; & \left(\left(\beta|_{j+\frac{n}{2}}^2\tau^{-\Delta(j+\frac{n}{2},j)}\right)^{\beta|_i}.[\tau^{-\Delta(j+\frac{n}{2},j)}, \beta|_j]^{\tau^{\Delta(i,j)}}\right)^{\left(\tau^{\Delta(i,j)}.[\tau^{\Delta(i,j)},\beta|_i].\tau^{-\Delta(i,i+\frac{n}{2})}\right)} \\ \overset{(28)}{=} \; & \left(\alpha^{\tau^{\Delta(i,j+\frac{n}{2})}}.[\tau^{-\Delta(j+\frac{n}{2},j)}, \beta|_j]^{\tau^{\Delta(i,j)}}\right)^{\left(\tau^{\Delta(i,j)}.[\tau^{\Delta(i,j)},\beta|_i].\tau^{-\Delta(i,i+\frac{n}{2})}\right)} \\ = \; & \left(\alpha.[\tau^{-\Delta(j+\frac{n}{2},j)}, \beta|_j]^{\tau^{\Delta(i,j)-\Delta(i,j+\frac{n}{2})}}\right)^{\left(\tau^{\Delta(i,j+\frac{n}{2})+\Delta(i,j)}.[\tau^{\Delta(i,j)},\beta|_i].\tau^{-\Delta(i,i+\frac{n}{2})}\right)} \end{aligned}$$

$$\overset{\text{Prop.8}}{=} \left(\alpha.[\tau^{-\Delta(j+\frac{n}{2},j)}, \beta|_j]^{\tau^{\Delta(j+\frac{n}{2},j)}}\right)^{\left(\tau^{\Delta(i,i+\frac{n}{2})}[\tau^{\Delta(i,j)},\beta|_i]\tau^{-\Delta(i,i+\frac{n}{2})}\right)}$$

$$\overset{(32)}{=} \left(\beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}[\tau^{-\Delta(j,j+\frac{n}{2})}, \beta|_j][\tau^{\Delta(j+\frac{n}{2},j)}, \beta|_j]^{-1}\right)^{[\tau^{\Delta(i,j)},\beta|_i]^{\tau^{-\Delta(i,i+\frac{n}{2})}}}$$

$$\overset{\text{Prop.8}}{=} \left(\beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}\right)^{[\tau^{\Delta(i,j)},\beta|_i]^{\tau^{-\Delta(i,i+\frac{n}{2})}}}$$

$$\overset{\text{Prop.9 e (31)}}{=} \beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}.$$

Moreover, since

$$\begin{aligned}
R\left(\beta|_i\right) R\left(\beta|_j\right) &= R\left(\beta|_i\right)\left(\beta|_j\right) \overset{\text{Prop.5}}{=} R\tau^{\Delta(j,i+\frac{n}{2})}\beta|_{j+\frac{n}{2}}\beta|_{i+\frac{n}{2}}\tau^{\Delta(j,i+\frac{n}{2})} \\
&= R\beta|_{j+\frac{n}{2}}\beta|_{i+\frac{n}{2}}\tau^{2\Delta(j,i+\frac{n}{2})} = R\beta|_j^{-1}\beta|_i^{-1}\tau^{2\Delta(j,i+\frac{n}{2})} \\
&= R\beta|_j^{-1}\beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}\beta|_i^{-1}\beta|_i^2\tau^{-\Delta(i,i+\frac{n}{2})}\tau^{2\Delta(j,i+\frac{n}{2})} \\
&= R\beta|_j\beta|_i\tau^{-\Delta(j,j+\frac{n}{2})-\Delta(i,i+\frac{n}{2})+2\Delta(j,i+\frac{n}{2})} \\
&\overset{\text{Prop.8}}{=} R\beta|_j\beta|_i = R\beta|_jN\beta|_i
\end{aligned}$$

and

$$R\beta|_i = R\beta|_{i+\frac{n}{2}}^{-1}, \quad R\beta|_i^2 = R\tau^{\Delta(i,i+\frac{n}{2})}, \forall i,j \in Y,$$

we conclude $\dfrac{H}{R}$ is a homomorphic image of

$$\mathbb{Z} \times \underbrace{C_2 \times \cdots \times C_2}_{\frac{n}{2}\ \text{terms}}.$$

□

7.2. **The case σ_β transposition.** We prove in this section part *(II) (ii)* of Theorem B.

Theorem 6. *Let n be an even number and B an abelian subgroup of $\mathcal{A}_n$ normalized by τ. Suppose $\beta = (\beta|_0, \beta|_1, \cdots, \beta|_{n-1})\sigma_\beta \in B$ where σ_β is a transposition. Then $H = \langle \beta|_i \ (0 \leq i \leq n-1), \tau\rangle$ is a metabelian group.*

We prove progressively that

$$\begin{aligned}
N &= \left\langle [\beta|_i, \tau^k] \mid k \in \mathbb{Z}, i \in Y \right\rangle, \\
U &= \left\langle N, \quad \beta|_j \mid j \neq 0, \frac{n}{2} \right\rangle, \\
V &= \left\langle U, \quad \beta|_{\frac{n}{2}}\beta|_0, \quad \tau\left(\beta|_0\right)^2 \right\rangle
\end{aligned}$$

are normal abelian subgroups of H, from which it follows that $\frac{H}{V}$ is cyclic and therefore H metabelian.

Lemma 10. *The degree of the tree n is even and σ_β is $\langle\sigma_\tau\rangle$-conjugate to the transposition $\left(0, \frac{n}{2}\right)$.*

Proof. On conjugating by an appropriate power of σ_τ, we may assume $\sigma_\beta = (0, j)$. The conjugate of σ_β by σ_τ^i is the transposition $(i, j+i)$. In particular, $(j, 2j)$ is a conjugate which is supposed to commute with $(0, j)$. Therefore, $\{0, j\} = \{j, 2j\}$, $2j = 0$ modulo(n), $n = 2n'$ and $j = n'$. $\square$

We go back to part (I) of the Proposition 7,

$$\begin{aligned} &\left(\tau^v|_{(i)\sigma_\tau^{-v}}\right)^{-1}\left(\beta|_{(i)\sigma_\tau^{-v}}\right)\left(\tau^v|_{(i)\sigma_\tau^{-v}\sigma_\beta}\right)\left(\beta|_{(i)\sigma_\tau^{-v}\sigma_\beta\sigma_\tau^v}\right) \\ = \ &(\beta|_i)\left(\tau^v|_{(i)\sigma_\beta\sigma_\tau^{-v}}\right)^{-1}\left(\beta|_{(i)\sigma_\beta\sigma_\tau^{-v}}\right)\left(\tau^v|_{(i)\sigma_\beta\sigma_\tau^{-v}\sigma_\beta}\right)\end{aligned}$$

and set in it $j = (i)\,\sigma_\tau^{-v}$, $v = kn + r$, $r = \overline{v}$ to obtain

$$(\tau^v)|_j^{-1}\beta|_j(\tau^v)|_{(j)\sigma_\beta}\beta|_{(j)\sigma_\beta\sigma_\tau^v} \tag{33}$$

$$= \ \beta|_{(j)\sigma_\tau^v}(\tau^v)|^{-1}_{(j)\sigma_\tau^v\sigma_\beta\sigma_\tau^{-v}}\beta|_{(j)\sigma_\tau^v\sigma_\beta\sigma_\tau^{-v}}(\tau^v)_{(j)\sigma_\tau^v\sigma_\beta\sigma_\tau^{-v}\sigma_\beta}. \tag{34}$$

Proposition 11. *The following cases hold for different pairs (j, r).*

- *For $j = 0$ there are 3 subcases*
  - *If $r = 0$, then*

$$[\beta|_0, \tau^k]^{\beta|_{\frac{n}{2}}} = [\beta|_{\frac{n}{2}}, \tau^k], \ \forall k \in \mathbb{Z}; \tag{35}$$

  - *If $r = \frac{n}{2}$, then*

$$\beta|_0\tau\beta|_0 = \beta|_{\frac{n}{2}}\tau^{-1}\beta|_{\frac{n}{2}}, \tag{36}$$

and

$$[\beta|_0, \tau^k]^{\tau\beta|_0} = [\beta|_{\frac{n}{2}}, \tau^k], \forall k \in \mathbb{Z}. \tag{37}$$

  - *If $r \neq 0$ and $r \neq \frac{n}{2}$, then*

$$\tau^{\delta(\frac{n}{2},r)}\beta|_0\beta|_{\frac{n}{2}+r} = \beta|_r\tau^{\delta(\frac{n}{2},r)}\beta|_0, \forall r \in Y - \{0, \frac{n}{2}\} \tag{38}$$

and

$$[\beta|_0, \tau^k]^{\beta|_r} = [\beta|_0, \tau^k], \forall k \in \mathbb{Z}. \tag{39}$$

- *For $j = \frac{n}{2}$ there are 3 subcases*
  - *If $r = 0$, then*

$$[\beta|_{\frac{n}{2}}, \tau^k]^{\beta|_0} = [\beta|_0, \tau^k], \ \forall k \in \mathbb{Z}; \tag{40}$$

  - *If $r = \frac{n}{2}$, then*

$$\tau^{-1}\beta|^2_{\frac{n}{2}} = \beta|_0^2\tau, \tag{41}$$

and

$$[\beta|_{\frac{n}{2}}, \tau^k]^{\beta|_{\frac{n}{2}}\tau^{-1}} = [\beta|_0, \tau^k], \forall k \in \mathbb{Z}; \tag{42}$$

– *If* $r \neq 0$ *and* $r \neq \frac{n}{2}$, *then*

$$\tau^{-\delta(\frac{n}{2},r)}\beta|_{\frac{n}{2}}\beta|_r = \beta|_{\frac{n}{2}+r}\tau^{-\delta(\frac{n}{2},r)}\beta|_{\frac{n}{2}}, \forall r \in Y - \{0, \frac{n}{2}\} \tag{43}$$

and

$$[\beta|_{\frac{n}{2}}, \tau^k]^{\beta|_r} = [\beta|_{\frac{n}{2}}, \tau^k], \forall k \in \mathbb{Z}, \forall r \in Y - \{0, \frac{n}{2}\}. \tag{44}$$

- *For* $j \neq 0$ *and* $j \neq \frac{n}{2}$, *there are* 5 *subcases:*
  – *If* $j \neq n - r$ *and* $j \neq \frac{n}{2} - r$, *then*

$$\beta|_j\beta_t = \beta|_t\beta|_j, \forall j, t \in Y - \{0, \frac{n}{2}\} \tag{45}$$

and

$$[\beta|_j, \tau^k]^{\beta|_t} = [\beta|_j, \tau^k], \forall j, t \in Y - \{0, \frac{n}{2}\} \tag{46}$$

– *If* $j = n - r$ *and* $0 < r < \frac{n}{2}$, *then*

$$\tau^{-1}\beta|_{j+\frac{n}{2}}\tau\beta|_0 = \beta|_0\beta|_j, \forall j \in \{1, 2, \cdots, \frac{n}{2} - 1\} \tag{47}$$

and

$$[\beta|_{j+\frac{n}{2}}, \tau^k]^{\tau\beta|_0} = [\beta|_j, \tau^k], \forall j \in \{1, 2, \cdots, \frac{n}{2} - 1\} \tag{48}$$

– *If* $j = n - r$ *and* $\frac{n}{2} < r \leq n - 1$, *then*

$$\beta|_j\beta|_0 = \beta|_0\beta|_{\frac{n}{2}+j}, \forall j \in \{1, \cdots, \frac{n}{2} - 1\} \tag{49}$$

and

$$[\beta|_j, \tau^k]^{\beta|_0} = [\beta|_{\frac{n}{2}+j}, \tau^k], \forall k \in \mathbb{Z}, \forall j \in \{1, \cdots, \frac{n}{2} - 1\} \tag{50}$$

– *If* $j = \frac{n}{2} - r$ *and* $0 < r < \frac{n}{2}$, *then*

$$\beta|_j\beta|_{\frac{n}{2}} = \beta|_{\frac{n}{2}}\tau^{-1}\beta|_{j+\frac{n}{2}}\tau, \forall j \in \{1, \cdots, \frac{n}{2} - 1\} \tag{51}$$

and

$$[\beta|_j, \tau^k]^{\beta|_{\frac{n}{2}}\tau^{-1}} = [\beta|_{\frac{n}{2}+j}, \tau^k], \forall k \in \mathbb{Z}, \forall j \in \{1, \cdots, \frac{n}{2} - 1\} \tag{52}$$

– *If* $j = \frac{n}{2} - r$ *and* $\frac{n}{2} < r \leq n - 1$, *then*

$$\beta|_{\frac{n}{2}}\beta|_j = \beta|_{\frac{n}{2}+j}\beta|_{\frac{n}{2}}, \forall j \in \{1, \cdots, \frac{n}{2} - 1\} \tag{53}$$

and

$$[\beta|_j, \tau^k] = [\beta|_{\frac{n}{2}+j}, \tau^k]^{\beta|_{\frac{n}{2}}}, \forall k \in \mathbb{Z}, \forall j \in \{1, \cdots, \frac{n}{2} - 1\}. \tag{54}$$

Proof. We will prove just the last case. As $j \notin \{0, \frac{n}{2}, n-r, \frac{n}{2}-r\}$, we have

$$\begin{aligned} (j)\,\sigma_\tau^v &= (j)\,\sigma_\beta\sigma_\tau^v = j+r, \\ (j)\,\sigma_\beta &= (j)\,\sigma_\tau^v\sigma_\beta\sigma_\tau^{-v} = (j)\,\sigma_\tau^v\sigma_\beta\sigma_\tau^{-v}\sigma_\beta = j. \end{aligned}$$

Therefore,

$$\begin{aligned} &\big((\tau^v)|_j^{-1}\beta|_j(\tau^v)|_j\beta|_{j+r} = \beta|_{j+r}(\tau^v)|_j^{-1}\beta|_j(\tau^v)_j, \forall v \in \mathbb{Z}\big) \\ \Leftrightarrow\ &\big(\tau^{-k-\delta(j,r)}\beta|_j\tau^{k+\delta(j,r)}\beta|_{j+r} = \beta|_{j+r}\tau^{-k-\delta(j,r)}\beta|_j\tau^{k+\delta(j,r)}, \forall k \in \mathbb{Z}\big) \\ \Leftrightarrow\ &\big(\beta|_j[\beta|_j,\tau^{k+\delta(j,r)}]\beta|_{j+r} = \beta|_{j+r}\beta|_j[\beta|_j,\tau^{k+\delta(j,r)}], \forall k \in \mathbb{Z}\big), \end{aligned}$$

$$\beta|_j\beta_t = \beta|_t\beta|_j, \forall j,t \in Y - \{0, \frac{n}{2}\} \tag{55}$$

and

$$[\beta|_j,\tau^k]^{\beta|_t} = [\beta|_j,\tau^k], \forall j,t \in Y - \{0, \frac{n}{2}\}. \tag{56}$$

□

Lemma 11. *The group $N = \big\langle [\beta|_i,\ \ \tau^k] \mid k \in \mathbb{Z}, i \in Y \big\rangle$ is an abelian normal subgroup of H.*

Proof. Define

$$N_i = \big\langle [\beta|_i,\tau^k] \mid k \in \mathbb{Z} \big\rangle$$

for each $i \in Y$. Then, $N = \langle N_i \mid i \in Y \rangle$, each N_i is an abelian subgroup normalized by τ and

$$[\beta|_i,\tau^k]^{\beta|_j^{-1}} = [\beta|_i,\tau^k], \forall k \in \mathbb{Z}, \forall i,j \in Y, j \neq 0, \frac{n}{2} \tag{57}$$

We have $[N_i, N_j] = 1, \forall i,j \in Y, j \neq 0, \frac{n}{2}$, because

$$\begin{aligned} [\beta|_i,\tau^k]^{[\beta|_j,\tau^t]} &= [\beta|_i,\tau^k]^{\beta|_j^{-1}\tau^{-t}\beta|_j\tau^t} \overset{(57)}{=} [\beta|_i,\tau^k]^{\tau^{-t}\beta|_j\tau^t} \\ &\overset{(14)}{=} \big([\beta|_i,\tau^{-t}]^{-1}[\beta|_i,\tau^{k-t}]\big)^{\beta|_j\tau^t} \\ &\overset{(57)}{=} \big([\beta|_i,\tau^{-t}]^{-1}[\beta|_i,\tau^{k-t}]\big)^{\tau^t} \\ \overset{(14)}{=} [\beta|_i,\tau^k]^{\tau^{-t}\tau^t} &= [\beta|_i,\tau^k], \forall k,t \in \mathbb{Z}, \end{aligned}$$

$\forall i,j \in Y, j \neq 0, \frac{n}{2}$.

Furthermore, $[N_0, N_{\frac{n}{2}}] = 1$, because

$$\begin{aligned} [\beta|_{\frac{n}{2}},\tau^k]^{[\beta|_0,\tau^t]} &= [\beta|_{\frac{n}{2}},\tau^k]^{\beta|_0^{-1}\tau^{-t}\beta|_0\tau^t} \overset{(37)}{=} [\beta|_0,\tau^k]^{\tau\tau^{-t}\beta|_0\tau^t} \\ &\overset{(14)}{=} \big([\beta|_0,\tau^{-t}]^{-1}[\beta|_0,\tau^{k-t}]\big)^{\tau\beta|_0\tau^t} \\ &\overset{(37)}{=} \big([\beta|_{\frac{n}{2}},\tau^{-t}]^{-1}[\beta|_{\frac{n}{2}},\tau^{k-t}]\big)^{\tau^t} \end{aligned}$$

$$\overset{(14)}{=} [\beta|_{\frac{n}{2}}, \tau^k]^{\tau^{-t}\tau^t} = [\beta|_{\frac{n}{2}}, \tau^k], \forall k, t \in \mathbb{Z}.$$

Therefore N is abelian.

Now, equation (57) implies

$$N_i = N_i^{\beta|_j} = N_i^{\beta|_j^{-1}}, \forall i, j \in Y, j \neq 0, \frac{n}{2}; \tag{58}$$

equations (14), (35) imply

$$\left\{ N_{\frac{n}{2}} = N_0^{\beta|_0},\ N_0 = N_{\frac{n}{2}}^{\beta|_0^{-1}} \right. ; \tag{59}$$

equation (40) implies

$$\left\{ N_0 = N_{\frac{n}{2}}^{\beta|_0},\ N_{\frac{n}{2}} = N_0^{\beta|_0^{-1}} \right. ; \tag{60}$$

equations (14), (42) imply

$$\left\{ N_0 = N_{\frac{n}{2}}^{\beta|_{\frac{n}{2}}},\ N_{\frac{n}{2}} = N_0^{\beta|_{\frac{n}{2}}^{-1}} \right. ; \tag{61}$$

equations (14), (48) imply

$$\left\{ N_j = N_{j+\frac{n}{2}}^{\beta|_0}, N_{j+\frac{n}{2}} = N_j^{\beta|_0^{-1}} \right. , \forall j \in \{1, \cdots, \frac{n}{2} - 1\}; \tag{62}$$

equations (14) and (50) imply

$$\left\{ N_{j+\frac{n}{2}} = N_j^{\beta|_0},\ N_j = N_{j+\frac{n}{2}}^{\beta|_0^{-1}} \right. , \forall j \in \{1, \cdots, \frac{n}{2} - 1\}; \tag{63}$$

equations (14) (52) imply

$$\left\{ N_{j+\frac{n}{2}} = N_j^{\beta|_{\frac{n}{2}}},\ N_j = N_{j+\frac{n}{2}}^{\beta|_{\frac{n}{2}}^{-1}} \right. , \forall j \in \{1, \cdots, \frac{n}{2} - 1\}; \tag{64}$$

equations (14), (54) imply

$$\left\{ N_j = N_{j+\frac{n}{2}}^{\beta|_{\frac{n}{2}}},\ N_{j+\frac{n}{2}} = N_j^{\beta|_{\frac{n}{2}}^{-1}} \right. , \forall j \in \{1, \cdots, \frac{n}{2} - 1\}. \tag{65}$$

Thus (57)-(65) prove

$$\begin{aligned} N &= \langle N_i \mid i \in Y \rangle \\ &= \left\langle [\beta|_i, \tau^k] \mid \forall i, k \in \mathbb{Z} \right\rangle \end{aligned}$$

is an abelian normal subgroup of H. □

Lemma 12. *The group $U = \left\langle N,\quad \beta|_j \mid j \neq 0, \frac{n}{2} \right\rangle$ is a normal abelian subgroup of H.*

Proof. Lemma 11 and equations (39), (44), (45) and (46) show that U is abelian.

The fact that N is normal in H, together with the following assertions prove that U is normal in H.

Let $J = \left\langle \beta_0, \beta_{\frac{n}{2}}, \tau \right\rangle$. Then, for $j \in Y - \{0, \frac{n}{2}\}$, we have

(I) $\langle \beta|_j \rangle^J \leq U$:

$$\beta|_j^{\tau^t} = \beta|_j [\beta|_j, \tau^t];$$

$$\beta|_j^{\beta|_0} \overset{(49)}{=} \beta|_{j+\frac{n}{2}};$$

$$\beta|_j^{\beta|_0^{-1}} \overset{(47)}{=} \tau^{-1} \beta|_{j+\frac{n}{2}} \tau = \beta|_{j+\frac{n}{2}} [\beta|_{j+\frac{n}{2}}, \tau];$$

$$\beta|_j^{\beta|_{\frac{n}{2}}} \overset{(51)}{=} \tau^{-1} \beta|_{j+\frac{n}{2}} \tau = \beta|_{j+\frac{n}{2}} [\beta|_{j+\frac{n}{2}}, \tau];$$

$$\beta|_j^{\beta|_{\frac{n}{2}}^{-1}} \overset{(53)}{=} \beta|_{j+\frac{n}{2}};$$

(II) $\left\langle \beta|_{j+\frac{n}{2}} \right\rangle^J \leq U$:

$$\beta|_{j+\frac{n}{2}}^{\tau^t} = \beta|_{j+\frac{n}{2}} [\beta|_{j+\frac{n}{2}}, \tau^t];$$

$$\begin{aligned} \beta|_{j+\frac{n}{2}}^{\beta|_0} &\overset{(47)}{=} \beta|_0^{-1} \tau \beta|_0 \beta|_j \beta|_0^{-1} \tau^{-1} \beta|_0 \\ &= \left([\beta|_0, \tau]^{-1}\right)^{\tau^{-1}} \beta|_j^{\tau^{-1}} [\beta|_0, \tau]^{\tau^{-1}} \in U; \end{aligned}$$

$$\beta|_{j+\frac{n}{2}}^{\beta|_0^{-1}} \overset{(49)}{=} \beta|_j \in U;$$

$$\beta|_{j+\frac{n}{2}}^{\beta|_{\frac{n}{2}}} \overset{(53)}{=} \beta|_j \in U;$$

$$\begin{aligned} \beta|_{j+\frac{n}{2}}^{\beta|_{\frac{n}{2}}^{-1}} &\overset{(51)}{=} \beta|_{\frac{n}{2}} \tau \beta|_{\frac{n}{2}}^{-1} \beta|_j \beta|_{\frac{n}{2}} \tau^{-1} \beta|_{\frac{n}{2}}^{-1} \\ &= [\beta|_{\frac{n}{2}}, \tau]^{\beta|_{\frac{n}{2}}^{-1} \tau^{-1}} \beta|_j^{\tau^{-1}} \left([\beta|_{\frac{n}{2}}, \tau]^{-1}\right)^{\beta|_{\frac{n}{2}}^{-1} \tau^{-1}}. \end{aligned}$$

Hence, U is a normal abelian subgroup of H. □

Lemma 13. *$V = \left\langle U, \ \beta|_{\frac{n}{2}} \beta|_0, \ \tau \beta|_0^2 \right\rangle$ is a normal abelian subgroup of H.*

Proof. Lemma 12 together with the following assertions prove that V is a normal abelian subgroup of H.

Given $j \in Y - \{0, \frac{n}{2}\}, k \in \mathbb{Z}$, and $J = \left\langle \beta|_0, \beta_{\frac{n}{2}}, \tau, \right\rangle$, we prove

(I) $\beta|_{\frac{n}{2}} \beta|_0 \in C_H(U)$:

$$(\beta|_j)^{\beta|_{\frac{n}{2}} \beta|_0} \overset{(51)}{=} (\beta|_{j+\frac{n}{2}})^{\tau \beta|_0} \overset{(47)}{=} \beta|_j;$$

$$(\beta|_{j+\frac{n}{2}})^{\beta|_{\frac{n}{2}} \beta|_0} \overset{(53)}{=} (\beta|_j)^{\beta|_0} \overset{(49)}{=} \beta|_{j+\frac{n}{2}};$$

$$[\beta|_j, \tau^k]^{\beta|_{\frac{n}{2}} \beta|_0} = [\beta|_j, \tau^k]^{\beta|_{\frac{n}{2}} \tau^{-1} \tau \beta|_0} \overset{(52)}{=} [\beta|_{j+\frac{n}{2}}, \tau^k]^{\tau \beta|_0}$$

$$\overset{(48)}{=} [\beta|_j, \tau^k];$$

$$[\beta|_{j+\frac{n}{2}}, \tau^k]^{\beta|_{\frac{n}{2}}\beta|_0} \overset{(54)}{=} [\beta|_j, \tau^k]^{\beta|_0} \overset{(50)}{=} [\beta|_{j+\frac{n}{2}}, \tau^k];$$

$$[\beta|_0, \tau^k]^{\beta|_{\frac{n}{2}}\beta|_0} \overset{(35)}{=} [\beta|_{\frac{n}{2}}, \tau^k]^{\beta|_0} \overset{(40)}{=} [\beta|_0, \tau^k];$$

$$\begin{aligned}
[\beta|_{\frac{n}{2}}, \tau^k]^{\beta|_{\frac{n}{2}}\beta|_0} &= [\beta|_{\frac{n}{2}}, \tau^k]^{\beta|_{\frac{n}{2}}\tau^{-1}\tau\beta|_0} \\
&\overset{(42)}{=} [\beta|_0, \tau^k]^{\tau\beta|_0} \overset{(37)}{=} [\beta|_{\frac{n}{2}}, \tau^k];
\end{aligned}$$

(II) $\tau\beta|_0^2 \in C_H(U)$:

$$\beta|_j^{\tau\beta|_0^2} = (\beta|_j[\beta|_j, \tau])^{\beta|_0^2} = (\beta|_j^{\beta|_0}[\beta|_j, \tau]^{\beta|_0})^{\beta|_0}$$

$$\overset{(49),(50)}{=} (\beta|_{j+\frac{n}{2}}[\beta|_{j+\frac{n}{2}}, \tau])^{\beta|_0} = \beta|_{j+\frac{n}{2}}^{\tau\beta|_0} \overset{(47)}{=} \beta|_j;$$

$$(\beta|_{j+\frac{n}{2}})^{\tau\beta|_0^2} \overset{(47)}{=} \beta|_j^{\beta|_0} \overset{(49)}{=} \beta|_{j+\frac{n}{2}};$$

$$[\beta|_0, \tau^k]^{\tau\beta|_0^2} \overset{(37)}{=} [\beta|_{\frac{n}{2}}, \tau^k]^{\beta|_0} \overset{(40)}{=} [\beta|_0, \tau^k];$$

$$\begin{aligned}
[\beta|_{\frac{n}{2}}, \tau^k]^{\tau\beta|_0^2} &\overset{(14)}{=} ([\beta|_{\frac{n}{2}}, \tau]^{-1}[\beta|_{\frac{n}{2}}, \tau^{k+1}])^{\beta|_0^2} \\
&\overset{(40)}{=} ([\beta|_0, \tau]^{-1}[\beta|_0, \tau^{k+1}])^{\beta|_0} \\
&\overset{(14)}{=} [\beta|_0, \tau^k]^{\tau\beta|_0} \overset{(37)}{=} [\beta|_{\frac{n}{2}}, \tau^k];
\end{aligned}$$

$$\begin{aligned}
[\beta|_j, \tau^k]^{\tau\beta|_0^2} &\overset{(14)}{=} ([\beta|_j, \tau]^{-1}[\beta|_j, \tau^{k+1}])^{\beta|_0^2} \\
&\overset{(50)}{=} ([\beta|_{j+\frac{n}{2}}, \tau]^{-1}[\beta|_{j+\frac{n}{2}}, \tau^{k+1}])^{\beta|_0} \\
&\overset{(14)}{=} [\beta|_{j+\frac{n}{2}}, \tau^k]^{\tau\beta|_0} \overset{(48)}{=} [\beta|_j, \tau^k];
\end{aligned}$$

$$[\beta|_{j+\frac{n}{2}}, \tau^k]^{\tau\beta|_0^2} \overset{(48)}{=} [\beta|_j, \tau^k]^{\beta|_0} \overset{(50)}{=} [\beta|_{j+\frac{n}{2}}, \tau^k];$$

(III) $\tau\beta|_0^2 \in C_H(\beta|_{\frac{n}{2}}\beta|_0)$:

$$\begin{aligned}
(\beta|_{\frac{n}{2}}\beta|_0)^{\tau\beta|_0^2} &= \beta|_0^{-2}\tau^{-1}\beta|_{\frac{n}{2}}\beta|_0\tau\beta|_0^2 \\
&\overset{(36)}{=} \beta|_0^{-2}\tau^{-1}\beta|_{\frac{n}{2}}\beta|_{\frac{n}{2}}\tau^{-1}\beta|_{\frac{n}{2}}\beta|_0
\end{aligned}$$

$$\begin{aligned}
&= \beta|_0^{-2}\tau^{-1}\beta|_{\frac{n}{2}}^2\tau^{-1}\beta|_{\frac{n}{2}}\beta|_0 = (\tau\beta|_0^2)^{-1}\beta|_{\frac{n}{2}}^2\tau^{-1}\beta|_{\frac{n}{2}}\beta|_0 \\
&\overset{(41)}{=} \beta|_{\frac{n}{2}}\beta|_0;
\end{aligned}$$

(IV) $\left\langle \beta|_{\frac{n}{2}}, \beta|_0 \right\rangle^J \leq V$:

$$(\beta|_{\frac{n}{2}}\beta|_0)^{\tau^k} = \beta|_{\frac{n}{2}}\beta|_0[\beta|_{\frac{n}{2}}\beta|_0, \tau^k] = \beta|_{\frac{n}{2}}\beta|_0[\beta|_{\frac{n}{2}}, \tau^k]^{\beta|_0}[\beta|_0, \tau^k];$$

$$(\beta|_{\frac{n}{2}}\beta|_0)^{\beta|_0} = \beta|_0^{-1}\beta|_{\frac{n}{2}}\beta|_0^2 = \beta|_0^{-1}\beta|_{\frac{n}{2}}\tau^{-1}\tau\beta|_0^2 = \beta|_0^{-1}\beta|_{\frac{n}{2}}^{-1}\beta|_{\frac{n}{2}}^2\tau^{-1}\tau\beta|_0^2$$

$$= (\beta|_{\frac{n}{2}}\beta|_0)^{-1}(\tau\beta|_0^2)^2;$$

$$\beta|_{\frac{n}{2}}\beta|_0 \overset{(t)}{=} (\tau\beta|_0^2)^2((\beta|_{\frac{n}{2}}\beta|_0)^{-1})^{\beta|_0};$$

$$(\beta|_{\frac{n}{2}}\beta|_0)^{\beta|_0^{-1}} \overset{(u)}{=} ((\tau\beta|_0^2)^2)^{\beta|_0^{-1}}(\beta|_{\frac{n}{2}}\beta|_0)^{-1};$$

$$(\beta|_{\frac{n}{2}}\beta|_0)^{\beta|_{\frac{n}{2}}^{-1}} = \beta|_{\frac{n}{2}}^2\beta|_0\beta|_{\frac{n}{2}}^{-1} = \beta|_{\frac{n}{2}}^2\tau^{-1}\tau\beta|_0\beta|_0\beta|_0^{-1}\beta|_{\frac{n}{2}}^{-1}$$

$$\overset{(41)}{=} (\tau\beta|_0^2)^2\beta|_0^{-1}\beta|_{\frac{n}{2}}^{-1} = (\tau\beta|_0^2)^2(\beta|_{\frac{n}{2}}\beta|_0)^{-1};$$

$$(\beta|_{\frac{n}{2}}\beta|_0)^{\beta|_{\frac{n}{2}}} \overset{(x)}{=} (\beta|_{\frac{n}{2}}\beta|_0)^{-1}((\tau\beta|_0^2)^2)^{\beta|_{\frac{n}{2}}}$$

.

(V) $\langle\tau\beta|_0^2\rangle^J \leq V$:

$$(\tau\beta|_0^2)^{\tau^k} = \tau(\beta|_0^2)^{\tau^k} = \tau\beta|_0^2[\beta|_0^2, \tau^k] = \tau\beta|_0^2[\beta|_0, \tau^k]^{\beta|_0}[\beta|_0, \tau^k];$$

$$(\tau\beta|_0^2)^{\beta|_0} = \beta|_0^{-1}\tau\beta|_0^2\beta|_0 = \tau\tau^{-1}\beta|_0^{-1}\tau\beta|_0\beta|_0^2 = \tau[\tau, \beta|_0]\beta|_0^2$$

$$= \tau[\tau, \beta|_0]\tau^{-1}\tau\beta|_0^2 = ([\beta|_0, \tau]^{-1})^{\tau^{-1}}\tau\beta|_0^2;$$

$$(\tau\beta|_0^2)^{\beta|_0^{-1}} = \beta|_0\tau\beta|_0 = \tau\beta|_0[\beta|_0, \tau]\beta|_0 = \tau\beta|_0^2[\beta|_0, \tau]^{\beta|_0};$$

$$(\tau\beta|_0^2)^{\beta|_{\frac{n}{2}}^{-1}} \overset{(p)}{=} \left((\tau\beta|_0^2)^{\beta|_0^{-1}}([\beta|_0, \tau]^{-1})^{\beta|_0}\right)^{\beta|_{\frac{n}{2}}^{-1}}$$

$$= (\tau\beta|_0^2)^{\beta|_0^{-1}\beta|_{\frac{n}{2}}^{-1}}([\beta|_0, \tau]^{-1})^{\beta|_0\beta|_{\frac{n}{2}}^{-1}}$$

$$= (\tau\beta|_0^2)^{(\beta|_{\frac{n}{2}}\beta|_0)^{-1}}([\beta|_0, \tau]^{-1})^{\beta|_0\beta|_{\frac{n}{2}}^{-1}} \overset{(g)}{=} \tau\beta|_0^2([\beta|_0, \tau]^{-1})^{\beta|_0\beta|_{\frac{n}{2}}^{-1}};$$

$$(\tau\beta|_0^2)^{\beta|_{\frac{n}{2}}} \overset{(q)}{=} \tau\beta|_0^2[\beta|_0, \tau]^{\beta|_0}.$$

□

8. Solvable groups for $n = 4$.

Let B be an abelian subgroup of $\mathcal{A}_4 = Aut(T_4)$ normalized by τ and let $\beta \in B$. Then, by Proposition 5 , $\sigma_\beta \in D = \langle (0,1,2,3), (0,2) \rangle$, the unique Sylow 2-subgroup of Σ_4 which contains $\sigma = \sigma_\tau = (0,1,2,3)$.

The normalizer of $\overline{\langle \tau \rangle}$ here is $\Gamma_0 = N_{\mathcal{A}_4}\left(\overline{\langle \tau \rangle}\right) = \langle \Lambda, \iota \rangle$ where Λ is the monic normalizer and where $\iota = \iota^{(1)}(0,3)(1,2)$ inverts τ.

Given a group W, the subgroup generated by the square of its elements is denoted by W^2.

Lemma 14. *Let $L = L(D)$ be the layer closure of D above. If $\gamma \in L^2$ then $\gamma\tau$ is conjugate to τ.*

Proof. If $\alpha \in L$ then $\sigma_{\alpha^2} \in \langle \sigma^2 \rangle$ and the product in any order of the states $(\alpha^2)|_i$ $(0 \leq i \leq 3)$ belongs to $S = L^2$.

Let $\gamma \in S$. Then $\gamma\tau$ is transitive on the $1st$ level of the tree and $(\gamma\tau)^4$ is inactive with conjugate 1st level states, where the first state is

$$(\gamma|_0)(\gamma|_1)(\gamma|_2)(\gamma|_3)\,\tau \text{ if } \sigma_\gamma = e,$$

and

$$(\gamma|_0)(\gamma|_3)(\gamma|_2)(\gamma|_1)\,\tau \text{ if } \sigma_\gamma = \sigma^2;$$

in both cases the element is contained in $S^2\tau$. Therefore, $\gamma\tau$ is transitive on the 2nd level of the tree. Now use induction to prove that $\gamma\tau$ is transitive on all levels of the tree. □

8.1. Cases $\sigma_\beta \in \{(0,3)(1,2), (0,1)(2,3)\}$.

We will show that these cases cannot occur. We note that σ_τ conjugates $(0,1)(2,3)$ to $(0,3)(1,2)$. Since the argument for β applies to β^τ, it is sufficient to consider the first case.

Suppose $\sigma_\beta = (0,1)(2,3)$. Then,

$$\beta^\tau = \left(\tau^{-1}(\beta|_3), \beta|_0, \beta|_1, \beta|_2\tau\right)(\sigma_\beta)^{\sigma_\tau}.$$

On substituting $\alpha = \beta^\tau$ in $\theta = [\beta, \alpha]$ and in (7)

$$\theta|_{(i)\sigma_{\alpha\beta}} = \left(\beta|_{(i)\sigma_\alpha}\right)^{-1}(\alpha|_i)^{-1}(\beta|_i)\left(\alpha|_{(i)\sigma_\beta}\right), \forall i \in Y. \tag{66}$$

we get $\theta = e$ and

$$e = \left(\beta|_{(i)\sigma_{\beta^\tau}}\right)^{-1}(\beta^\tau|_i)^{-1}(\beta|_i)\left(\beta^\tau|_{(i)\sigma_\beta}\right), \forall i \in Y \tag{67}$$

and so for the index $i = 0$, we obtain

$$\begin{aligned} e &= (\beta|_3)^{-1}\left(\tau^{-1}(\beta|_3)\right)^{-1}(\beta|_0)(\beta|_0), \\ e &= (\beta|_3)^{-2}\,\tau\,(\beta|_0)^2 \end{aligned}$$

which is impossible.

8.2. **Cases** $\sigma_\beta \in \{(0,2),(1,3)\}$.

Lemma 15. *Let* $\alpha, \gamma \in Aut(T_4)$ *be such that*

$$\begin{array}{rcl} \sigma_\alpha, \sigma_\gamma & \in & \langle (0,1,2,3),(0,2) \rangle, \\ \tau^{-1}\alpha^2 & = & \gamma^2 \tau, \\ [\alpha, \tau^k]^\gamma & = & [\gamma, \tau^k] \end{array}$$

for all $k \in \mathbb{Z}$. *Then,*

$$\sigma_\alpha, \sigma_\gamma \in \langle \sigma \rangle, \qquad \sigma_\alpha \sigma_\gamma = \sigma^{\pm 1}.$$

Proof. From the second and third equations above, we have $\sigma^{-1}\sigma_\alpha^2 = \sigma_\gamma^2 \sigma$ and $[\sigma_\alpha, \sigma^k]^{\sigma_\gamma} = [\sigma_\gamma, \sigma^k]$.

(i) Suppose $\sigma_\gamma^2 = e$. Then $\sigma_\alpha^2 = \sigma^2$ and therefore, $\sigma_\alpha = \sigma^{\pm 1}$, $[\sigma_\alpha, \sigma^k]^{\sigma_\gamma} = [\sigma_\gamma, \sigma^k] = e$ for all k; thus, $\sigma_\gamma \in \langle \sigma \rangle$ and $\sigma_\gamma \in \langle \sigma^2 \rangle$, $\sigma_\alpha \sigma_\gamma = \sigma^{\pm 1}$ follows.

(ii) Suppose $o(\sigma_\gamma) = 4$. Then, $\sigma_\gamma = \sigma^{\pm 1}$ and $\sigma_\alpha^2 = e$. Since $[\sigma_\alpha, \sigma^k]^{\sigma_\gamma} = e$ for all k, we obtain $\sigma_\alpha \in \langle \sigma \rangle$, $\sigma_\alpha^2 = e$ and $\sigma_\alpha \in \langle \sigma^2 \rangle$ .Therefore, $\sigma_\alpha \sigma_\gamma = \sigma^{\pm 1}$. □

(1) Suppose $\sigma_\beta = (0,2)$. Then by the analysis in Section 7.2, we conclude

$$V = \left\langle [\beta|_i, \tau^k], \beta|_1, \beta|_3, \beta|_2\beta|_0, \tau\beta|_0^2 \mid i \in Y \right\rangle$$

is an abelian normal subgroup of H.

By Lemma 14 , $\tau\beta|_0^2 = \mu$ is a conjugate of τ. As V is abelian, there exist $\xi, t_1, t_2 \in \mathbb{Z}_4$ such that

$$\mu = \tau\beta|_0^2, \beta|_2\beta|_0 = \mu^\xi, \beta|_1 = \mu^{t_1}, \beta|_3 = \mu^{t_2}.$$

Therefore,

$$\beta|_2 = \mu^\xi \beta|_0^{-1}, \tau = \mu\beta|_0^{-2}.$$

On substituting $\gamma = \beta_0$ and $\alpha = \beta_2$ in Lemma 15, we obtain $\sigma_{\alpha\gamma} = \sigma_{\beta|_2\beta|_0} = \sigma^{\pm 1}$. Thus, from $\beta|_2\beta|_0 = \mu^\xi$, we reach $\xi \in U(Z_4)$.

By (41), we have

$$\beta|_2^2 \tau^{-1} = \tau\beta|_0^2.$$

It follows then that

$$\begin{array}{rcl} \mu^\xi \beta|_0^{-1} \mu^\xi \beta|_0^{-1} \beta|_0^2 \mu^{-1} & = & \mu, \\ \left(\mu^\xi\right)^{\beta|_0} & = & \mu^{2-\xi}. \end{array}$$

Therefore,

$$\mu^{\beta|_0} = \mu^{\frac{2-\xi}{\xi}} \tag{68}$$

where $\frac{2-\xi}{\xi} \in \mathbb{Z}_4^1$.

By Equation (49) we have

$$\beta|_1^{\beta|_0} = \beta|_3.$$

It follows that

$$\left(\mu^{t_1}\right)^{\beta|_0} = \mu^{t_2},\ \mu^{t_1\frac{2-\xi}{\xi}} = \mu^{t_2},\ t_2 = t_1\frac{2-\xi}{\xi}.$$

We have reached the form of β,

$$\beta = (\beta|_0, \mu^{t_1}, \mu^{\xi}\beta|_0^{-1}, \mu^{t_1\frac{2-\xi}{\xi}})(0,2)$$

where $\mu = \tau^{\alpha}$ for some $\alpha \in Aut(T_4)$.

Now, since

$$\beta|_0 = \left(\lambda_{\frac{2-\xi}{\xi}}\tau^m\right)^{\alpha}$$

for some $m \in \mathbb{Z}_4$, we have

$$\begin{aligned} \mu^{t_1} &= (\tau^{t_1})^{\alpha}, \\ \mu^{\xi}\beta|_0^{-1} &= \left(\tau^{\xi}\left(\lambda_{\frac{2-\xi}{\xi}}\tau^m\right)^{-1}\right)^{\alpha} \\ &= \left(\lambda_{\frac{\xi}{2-\xi}}\tau^{(\xi-m)\frac{\xi}{2-\xi}}\right)^{\alpha}. \end{aligned}$$

Thus

$$\beta = (\lambda_{\frac{2-\xi}{\xi}}\tau^m, \tau^{t_1}, \lambda_{\frac{\xi}{2-\xi}}\tau^{(\xi-m)\frac{\xi}{2-\xi}}, \tau^{t_1\frac{2-\xi}{\xi}})^{\alpha^{(1)}}(0,2)$$

and

$$\begin{aligned} \tau &= \mu\beta|_0^{-2} \\ &= \left(\tau\left(\lambda_{\frac{2-\xi}{\xi}}\tau^m\right)^{-2}\right)^{\alpha} \\ &= \left(\lambda_{(\frac{\xi}{2-\xi})^2}\tau^{\left(1-\frac{2m}{\xi}\right)\left(\frac{\xi}{2-\xi}\right)^2}\right)^{\alpha} \end{aligned}$$

We note that in case $\xi = 1$ and β has the form

$$\beta = (\tau^m, \tau^{t_1}, \tau^{1-m}, \tau^{t_1})^{\alpha^{(1)}}(0,2)$$

where $\tau = (\tau^{1-2m})^{\alpha}$; therefore,

$$\beta = (\tau^{\frac{m}{1-2m}}, \tau^{\frac{t_1}{1-2m}}, \tau^{\frac{1-m}{1-2m}}, \tau^{\frac{t_1}{1-2m}})(0,2).$$

(2) Suppose $\sigma_\beta = (1,3)$. Then, $\gamma = \beta^{\tau}$ satisfies $[\gamma, \gamma^{\tau^k}] = e$. Therefore, the previous case applies and

$$\gamma = (\lambda_{\frac{2-\xi}{\xi}}\tau^m, \tau^{t_1}, \lambda_{\frac{\xi}{2-\xi}}\tau^{(\xi-m)\frac{\xi}{2-\xi}}, \tau^{t_1\frac{2-\xi}{\xi}})^{\alpha^{(1)}}(0,2),$$

where

$$\tau = \left(\lambda_{(\frac{\xi}{2-\xi})^2}\tau^{\left(1-\frac{2m}{\xi}\right)\left(\frac{\xi}{2-\xi}\right)^2}\right)^{\alpha} = (e,e,e,\left(\lambda_{(\frac{\xi}{2-\xi})^2}\tau^{\left(1-\frac{2m}{\xi}\right)\left(\frac{\xi}{2-\xi}\right)^2}\right)^{\alpha})\sigma_\tau.$$

Hence, β has the form

$$\beta = \gamma^{\tau^{-1}} = (\tau^{t_1}, \lambda_{\frac{2-\xi}{\xi}}\tau^{1+m-\xi}, \tau^{t_1\frac{2-\xi}{\xi}}, \lambda_{\frac{\xi}{2-\xi}}\tau^{(1-m)\frac{\xi}{2-\xi}})^{\alpha^{(1)}}(1,3).$$

8.3. **The case** $\sigma_\beta = (\sigma_\tau)^2 = (0,2)(1,3)$**.** We know that

$$V = \left\langle N, \beta|_i \beta|_{i+2}, \beta|_j^2 \tau^{-\Delta(j,j+2)} \mid i,j,t \in Y \text{ and } k \in \mathbb{Z} \right\rangle$$

is an abelian normal subgroup of H and

$$\tau^{\Delta(i,j)} \beta|_{i+2} \beta|_j \tau^{\Delta(i,j)} = \beta|_{j+2} \beta|_i, \tag{69}$$

by analysis of the case 7.1.

From Lemmas 12 and 13, we have

$$\tau\beta|_0^2 = \mu,\ \beta|_2\beta|_0 = \mu^{\xi_0},\ \beta|_3\beta|_1 = \mu^{\xi_1},\ \tau\beta|_1^2 = \mu^{\xi_2}$$

where $\mu = \tau^\alpha$ and $\xi_0, \xi_1, \xi_2 \in U(\mathbb{Z}_4)$. Therefore,

$$\tau = \mu\beta|_0^{-2} \tag{70}$$

$$\beta|_2 = \mu^{\xi_0}\beta|_0^{-1} \tag{71}$$

$$\beta|_3 = \mu^{\xi_1}\beta|_1^{-1} \tag{72}$$

$$\tau = \mu^{\xi_2}\beta|_1^{-2}. \tag{73}$$

Now, we let i,j take their values from Y in (69). Note that (i,j) and (j,i) produce equivalent equations and the case where $i=j$ is a tautology. Thus we have to treat the cases $(i,j) = (0,1), (0,2), (1,3), (2,3), (0,3), (1,2)$. Indeed, the last two cases turn out to be superfluous.

(i) Substitute $i=0, j=2$ in (69), to obtain

$$\beta|_2^2\tau^{-1} = \tau\beta|_0^2 \tag{74}$$

Use (70) and (71) in (74) to get

$$\mu^{\xi_0}\beta|_0^{-1}\mu^{\xi_0}\beta|_0^{-1}\beta|_0^2\mu^{-1} = \mu$$

and so,

$$(\mu^{\xi_0})^{\beta|_0} = \mu^{2-\xi_0}.$$

Therefore,

$$\mu^{\beta|_0} = \mu^{\frac{2-\xi_0}{\xi_0}} \tag{75}$$

Since $\frac{2-\xi_0}{\xi_0} \in \mathbb{Z}_4^1$, we find

$$\beta|_0 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}\right)^\alpha. \tag{76}$$

From (71),

$$\beta|_2 = \mu^{\xi_0}\beta|_0^{-1} = \left(\tau^{\xi_0}\tau^{-m_0}\lambda_{\frac{\xi_0}{2-\xi_0}}\right)^\alpha = \left(\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}\right)^\alpha. \tag{77}$$

(ii) Substitute $i = 1, j = 3$ in (69) to get

$$\beta|_3^2\tau^{-1} = \tau\beta|_1^2. \tag{78}$$

On using (72) and (73) in (78), we obtain

$$\mu^{\xi_1}\beta|_1^{-1}\mu^{\xi_1}\beta|_1^{-1}\beta|_1^2\mu^{-\xi_2} = \mu^{\xi_2}$$

and so,

$$(\mu^{\xi_1})^{\beta|_1} = \mu^{2\xi_2-\xi_1}.$$

Therefore,

$$\mu^{\beta|_1} = \mu^{\frac{2\xi_2-\xi_1}{\xi_1}}. \tag{79}$$

Since $\frac{2\xi_2-\xi_1}{\xi_1} \in \mathbb{Z}_4^1$, we have

$$\beta|_1 = \left(\lambda_{\frac{2\xi_2-\xi_1}{\xi_1}}\tau^{m_1}\right)^\alpha. \tag{80}$$

By (72), we find

$$\beta|_3 = \mu^{\xi_1}\beta|_1^{-1} = \left(\tau^{\xi_1}\tau^{-m_1}\lambda_{\frac{\xi_1}{2\xi_2-\xi_1}}\right)^\alpha = \left(\lambda_{\frac{\xi_1}{2\xi_2-\xi_1}}\tau^{(\xi_1-m_1)\frac{\xi_1}{2\xi_2-\xi_1}}\right)^\alpha. \tag{81}$$

(iii) Substitute $i = 0, j = 1$ in (69) to get

$$\beta|_2\beta|_1 = \beta|_3\beta|_0. \tag{82}$$

Use (76), (77), (80) and (81) in (82), to obtain

$$\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}\lambda_{\frac{2\xi_2-\xi_1}{\xi_1}}\tau^{m_1} = \lambda_{\frac{\xi_1}{2\xi_2-\xi_1}}\tau^{(\xi_1-m_1)\frac{\xi_1}{2\xi_2-\xi_1}}\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}$$

and so,

$$\lambda_{\frac{\xi_0}{2-\xi_0}\frac{2\xi_2-\xi_1}{\xi_1}}\tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}\frac{2\xi_2-\xi_1}{\xi_1}+m_1} = \lambda_{\frac{\xi_1}{2\xi_2-\xi_1}\frac{2-\xi_0}{\xi_0}}\tau^{(\xi_1-m_1)\frac{\xi_1}{2\xi_2-\xi_1}\frac{2-\xi_0}{\xi_0}+m_0}.$$

Therefore,

$$\left(\frac{\xi_1}{2\xi_2-\xi_1}\right)^2 = \left(\frac{\xi_0}{2-\xi_0}\right)^2 \tag{83}$$

and

$$(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}\frac{2\xi_2-\xi_1}{\xi_1} + m_1 = (\xi_1-m_1)\frac{\xi_1}{2\xi_2-\xi_1}\frac{2-\xi_0}{\xi_0} + m_0. \tag{84}$$

(iv) Substitute $i = 2, j = 3$ in (69) to get

$$\beta|_0\beta|_3 = \beta|_1\beta|_2. \tag{85}$$

Use (76), (77), (80) and (81) in (85), to obtain

$$\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}\lambda_{\frac{\xi_1}{2\xi_2-\xi_1}}\tau^{(\xi_1-m_1)\frac{\xi_1}{2\xi_2-\xi_1}} = \lambda_{\frac{2\xi_2-\xi_1}{\xi_1}}\tau^{m_1}\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}$$

and so,

$$\lambda_{\frac{\xi_0}{2-\xi_0}\frac{\xi_1}{2\xi_2-\xi_1}}\tau^{m_0\frac{\xi_1}{2\xi_2-\xi_1}+(\xi_1-m_1)\frac{\xi_1}{2\xi_2-\xi_1}} = \lambda_{\frac{2\xi_2-\xi_1}{\xi_1}\frac{\xi_0}{2-\xi_0}}\tau^{m_1\frac{\xi_0}{2-\xi_0}+(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}.$$

Therefore,

$$\left(\frac{\xi_1}{2\xi_2-\xi_1}\right)^2 = \left(\frac{\xi_0}{2-\xi_0}\right)^2$$

and

$$m_0\frac{\xi_1}{2\xi_2-\xi_1} + (\xi_1-m_1)\frac{\xi_1}{2\xi_2-\xi_1} = m_1\frac{\xi_0}{2-\xi_0} + (\xi_0-m_0)\frac{\xi_0}{2-\xi_0}. \tag{86}$$

We have from (83)

$$\frac{\xi_0}{2-\xi_0} = \pm\frac{\xi_1}{2\xi_2-\xi_1}. \tag{87}$$

(a) If

$$\frac{\xi_0}{2-\xi_0} = \frac{\xi_1}{2\xi_2-\xi_1},$$

then

$$2\xi_2\xi_0 - \xi_1\xi_0 = 2\xi_1 - \xi_1\xi_0,$$

and so,

$$\xi_2 = \frac{\xi_1}{\xi_0}. \tag{88}$$

From (84), we get

$$m_1 = \frac{\xi_1-\xi_0}{2} + m_0. \tag{89}$$

(b) If

$$\frac{\xi_0}{2-\xi_0} = -\frac{\xi_1}{2\xi_2-\xi_1}$$

then by (84) and (86),

$$m_0 - \xi_0 + m_1 = m_1 - \xi_1 + m_0$$

$$m_0 + \xi_1 - m_1 = -m_1 - \xi_0 + m_0,$$

which implies $\xi_1 = \xi_0 = 0$,which is impossible.

Now by (88) and (89), we have

$$\beta|_1 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{\frac{\xi_1-\xi_0}{2}+m_0}\right)^\alpha \tag{90}$$

and

$$\beta|_3=\left(\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{\left(\frac{\xi_1+\xi_0}{2}-m_0\right)\frac{\xi_0}{2-\xi_0}}\right)^{\alpha}. \tag{91}$$

Therefore,

$$\beta=(\beta|_0,\beta|_1,\beta|_2,\beta|_3)(0,2)(1,3)$$

where $\beta|_0,\beta|_1,\beta|_2$ and $\beta|_3$ are described in (76),(90), (77) and (91), respectively, and

$$\begin{aligned}\tau &= \mu\beta|_0^{-2}\\ &= \left(\tau\left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}\right)^{-2}\right)^{\alpha}\\ &= \left(\lambda_{(\frac{\xi_0}{2-\xi_0})^2}\tau^{\left(1-\frac{2m_0}{\xi_0}\right)\left(\frac{\xi_0}{2-\xi_0}\right)^2}\right)^{\alpha}.\end{aligned}$$

(v) The cases $(i,j)=(1,2),(0,3)$ in (69) do not add any more information about β.

Summarizing, we have found

$$\beta|_0=\left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}\right)^{\alpha},\ \beta|_1=\left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{\frac{\xi_1-\xi_0}{2}+m_0}\right)^{\alpha}, \tag{92}$$

$$\beta|_2=\left(\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}\right)^{\alpha},\beta|_3=\left(\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{\left(\frac{\xi_1+\xi_0}{2}-m_0\right)\frac{\xi_0}{2-\xi_0}}\right)^{\alpha}, \tag{93}$$

$$\tau=\left(\lambda_{(\frac{\xi_0}{2-\xi_0})^2}\tau^{\left(1-\frac{2m_0}{\xi_0}\right)\left(\frac{\xi_0}{2-\xi_0}\right)^2}\right)^{\alpha}. \tag{94}$$

In the particular case where $\xi_0=1$, β has the form

$$\beta=(\tau^{\frac{m_0}{1-2m_0}},\tau^{\frac{\frac{\xi_1-1}{2}+m_0}{1-2m_0}},\tau^{\frac{1-m_0}{1-2m_0}},\tau^{\frac{\frac{\xi_1+1}{2}-m_0}{1-2m_0}})(0,2)(1,3)$$

where $\tau=(\tau^{1-2m_0})^{\alpha}$.

8.4. **Cases $\sigma_\beta\in\{e,\sigma_\tau,\sigma_\tau^{-1}\}$.** (1) Suppose $\sigma_\beta=e$ and let β stabilize the kth level of the tree. Then by Proposition 6, we have

$$[\beta|_u,\beta|_v^{\tau^{\xi}}]=e,\ \text{for all } u,v\in\mathcal{M} \text{ with } |u|=|v|=k.$$

Therefore, $\dot N=\langle\beta|_w\mid |w|=k,w\in\mathcal{M}\rangle$ is abelian and so is its normal closure $\dot M$ under $\left\langle\dot N,\tau\right\rangle$. Also, active elements in $\dot M$ are characterized in 8.1, 8.2, 8.3 and 8.4. In particular, there exists $\kappa\in\dot M$ such that $\sigma_\kappa=(0,2)(1,3)$ and $\beta\in\times_{p^k}C(\kappa)$.

(2) Suppose $\sigma_\beta = \sigma_\tau = (0,1,2,3)$. Then, clearly the element

$$\beta^2 = (\beta|_0\beta|_1,\ \beta|_1\beta|_2,\ \beta|_2\beta|_3,\ \beta|_3\beta|_0)(0,2)(1,3)$$

satisfies $[\beta^2, (\beta^2)^{\tau^k}] = e$ for all $k \in \mathbb{Z}_4$. Therefore, by the previous analysis, we have

$$\beta|_0\beta|_1 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}\right)^\alpha, \tag{95}$$

$$\beta|_1\beta|_2 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{\frac{\xi_1-\xi_0}{2}+m_0}\right)^\alpha, \tag{96}$$

$$\beta|_2\beta|_3 = \left(\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}\right)^\alpha, \tag{97}$$

$$\beta|_3\beta|_0 = \left(\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{\left(\frac{\xi_1+\xi_0}{2}-m_0\right)\frac{\xi_0}{2-\xi_0}}\right)^\alpha, \tag{98}$$

$$\tau = \left(\lambda_{(\frac{\xi_0}{2-\xi_0})^2}\tau^{\left(1-\frac{2m_0}{\xi_0}\right)\left(\frac{\xi_0}{2-\xi_0}\right)^2}\right)^\alpha. \tag{99}$$

Therefore,

$$\beta|_0\beta|_1\beta|_2\beta|_3 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}\right)^\alpha = \left(\tau^{\frac{\xi_0^2}{2-\xi_0}}\right)^\alpha,$$

$$\beta|_1\beta|_2\beta|_3\beta|_0 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{\frac{\xi_1-\xi_0}{2}+m_0}\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{\left(\frac{\xi_1+\xi_0}{2}-m_0\right)\frac{\xi_0}{2-\xi_0}}\right)^\alpha = \left(\tau^{\frac{\xi_1\xi_0}{2-\xi_0}}\right)^\alpha.$$

It follows that

$$\left(\tau^{\frac{\xi_0^2}{2-\xi_0}}\right)^{\alpha\beta|_0} = \left(\tau^{\frac{\xi_1\xi_0}{2-\xi_0}}\right)^\alpha$$

and

$$(\tau^\alpha)^{\beta|_0} = \left(\tau^{\frac{\xi_1}{\xi_0}}\right)^\alpha \tag{100}$$

Substitute $\eta = \frac{\xi_1}{\xi_0}$ in (100) to get

$$\beta|_0 = (\psi_\eta\tau^{m_1})^\alpha, \tag{101}$$

where

(102) $$\psi_\eta = \begin{cases} \lambda_\eta, & \text{if } \eta \in \mathbb{Z}_4^1 \\ \theta\lambda_{-\eta}, & \text{if } -\eta \in \mathbb{Z}_4^1 \end{cases},$$

$$\theta = \theta^{(1)}(e, \tau^{-1}, \tau^{-1}, \tau^{-1})(1,3)$$

(an invertor of τ). Note that

$$\psi_\eta\lambda_\xi = \psi_\eta\psi_\xi = \psi_{\eta\xi} = \psi_{\xi\eta} = \psi_\xi\psi_\eta = \lambda_\xi\psi_\eta$$

for all $\xi \in \mathbb{Z}_4^1$.

By (95) and (101),

(103) $$\beta|_1 = \left(\tau^{-m_1}\psi_{\eta^{-1}}\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}\right)^\alpha = \left(\psi_{\frac{2-\xi_0}{\eta\xi_0}}\tau^{-m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right)+m_0}\right)^\alpha.$$

Also, by (96) and (101),

(104) $$\begin{aligned} \beta|_2 &= \left(\tau^{m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right)-m_0}\psi_{\frac{\eta\xi_0}{2-\xi_0}}\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{\frac{\eta\xi_0-\xi_0}{2}+m_0}\right)^\alpha \\ &= \left(\psi_\eta\tau^{\left[m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right)-m_0\right]\eta+\frac{\eta\xi_0-\xi_0}{2}+m_0}\right)^\alpha. \end{aligned}$$

Furthermore, by (98) and (101),

(105) $$\begin{aligned} \beta|_3 &= \left(\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{\left(\frac{\eta\xi_0+\xi_0}{2}-m_0\right)\frac{\xi_0}{2-\xi_0}}\tau^{-m_1}\psi_{\eta^{-1}}\right)^\alpha \\ &= \left(\psi_{\frac{\xi_0}{\eta(2-\xi_0)}}\tau^{\left[\left(\frac{\eta\xi_0+\xi_0}{2}-m_0\right)\frac{\xi_0}{2-\xi_0}-m_1\right]\eta^{-1}}\right)^\alpha. \end{aligned}$$

Setting $i = 1$ and $t = 2$ in (17), we obtain

(106) $$\beta|_0\beta|_2 = \beta|_1^2.$$

Use (101), (103), (104) and (105) in (106), to get

(107) $$\begin{aligned} &\psi_\eta\tau^{m_1}\psi_\eta\tau^{\left[m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right)-m_0\right]\eta+\frac{\eta\xi_0-\xi_0}{2}+m_0} \\ = \;&\psi_{\frac{2-\xi_0}{\eta\xi_0}}\tau^{-m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right)+m_0}\psi_{\frac{2-\xi_0}{\eta\xi_0}}\tau^{-m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right)+m_0} \end{aligned}$$

which is the same as

(108) $$\begin{aligned} &\psi_{\eta^2}\tau^{m_1\eta+\left[m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right)-m_0\right]\eta+\frac{\eta\xi_0-\xi_0}{2}+m_0} \\ = \;&\psi_{\left(\frac{2-\xi_0}{\eta\xi_0}\right)^2}\tau^{\left[-m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right)+m_0\right]\left(\frac{2-\xi_0}{\eta\xi_0}\right)-m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right)+m_0}. \end{aligned}$$

Therefore,

$$\eta^2 = \left(\frac{2-\xi_0}{\eta\xi_0}\right)^2 \tag{109}$$

and

$$m_1\eta + \left[m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right) - m_0\right]\eta + \frac{\eta\xi_0 - \xi_0}{2} + m_0$$
$$= \left[-m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right) + m_0\right]\left(\frac{2-\xi_0}{\eta\xi_0}\right) - m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right) + m_0$$

(a) Suppose

$$\eta = -\frac{2-\xi_0}{\eta\xi_0} \tag{110}$$

(or what is the same

$$\left(\eta^2 - 1\right)\xi_0 = -2). \tag{111}$$

Then on substituting this in the above equation, we get

$$(\eta - 1)\,\xi_0 = 0$$

contradicting the previous equation.

(b) Suppose

$$\eta = \frac{2-\xi_0}{\eta\xi_0}. \tag{112}$$

Then,

$$\xi_0 = \frac{2}{\eta^2+1} \tag{113}$$

and this leads to

$$m_0 = 2m_1 + \frac{\eta - 1}{2\eta(\eta^2+1)}. \tag{114}$$

On substituting (113) and (114) in(103), (104), (105) and (99), we find

$$\beta|_1 = \left(\psi_\eta \tau^{m_1(2-\eta) + \frac{\eta-1}{2\eta(\eta^2+1)}}\right)^\alpha \tag{115}$$

$$\beta|_2 = \left(\psi_\eta \tau^{m_1(\eta^2-2\eta+2) + \frac{\eta^2-1}{2\eta(\eta^2+1)}}\right)^\alpha, \tag{116}$$

$$\beta|_3 = \left(\psi_{\eta^{-3}} \tau^{\frac{2\eta^2+\eta+1}{2\eta^4(\eta^2+1)} - m_1\left(\frac{\eta^2+2}{\eta^3}\right)}\right)^\alpha, \tag{117}$$

$$\tau = \left(\psi_{\eta^{-4}}\tau^{\frac{\eta+1}{2\eta^5}-2m_1\left(\frac{\eta^2+1}{\eta^4}\right)}\right)^{\alpha}. \tag{118}$$

Substitute $i = 0$, $t = 1$ in (17), to get

$$\beta|_3\beta|_1 = \tau\beta|_0^2. \tag{119}$$

Using (101), (115), (116), (117) and (118) in (119), we obtain

$$\begin{aligned} & \psi_{\eta^{-3}}\tau^{\frac{2\eta^2+\eta+1}{2\eta^4(\eta^2+1)}-m_1\left(\frac{\eta^2+2}{\eta^3}\right)}\psi_\eta\tau^{m_1(2-\eta)+\frac{\eta-1}{2\eta(\eta^2+1)}} \\ = \;& \psi_{\eta^{-4}}\tau^{\frac{\eta+1}{2\eta^5}-2m_1\left(\frac{\eta^2+1}{\eta^4}\right)}\psi_\eta\tau^{m_1}\psi_\eta\tau^{m_1}. \end{aligned}$$

Thus,

$$\begin{aligned} & \psi_{\eta^{-2}}\tau^{\frac{2\eta^2+\eta+1}{2\eta^3(\eta^2+1)}-m_1\left(\frac{\eta^2+2}{\eta^2}\right)+m_1(2-\eta)+\frac{\eta-1}{2\eta(\eta^2+1)}} \\ = \;& \psi_{\eta^{-2}}\tau^{\frac{\eta+1}{2\eta^3}-2m_1\left(\frac{\eta^2+1}{\eta^2}\right)+m_1\eta+m_1}, \end{aligned}$$

which implies

$$(\eta-1)m_1 = 0 \tag{120}$$

and thus,

$$m_1 = 0 \text{ or } \eta = 1.$$

- If $m_1 = 0$ we get

$$\begin{aligned} \beta &= (\psi_\eta, \psi_\eta\tau^{\frac{\eta-1}{2\eta(\eta^2+1)}}, \psi_\eta\tau^{\frac{\eta^2-1}{2\eta(\eta^2+1)}}, \psi_{\eta^{-3}}\tau^{\frac{2\eta^2+\eta+1}{2\eta^4(\eta^2+1)}})^{\alpha^{(1)}}\sigma_\tau \\ &= \tau^\gamma, \end{aligned} \tag{121}$$

where

$$\gamma = \left(\lambda_{\frac{2}{\eta^2(\eta^2+1)}}\right)^{(1)}(e, \psi_\eta, \psi_{\eta^2}\tau^{\frac{\eta-1}{2\eta(\eta^2+1)}}, \psi_{\eta^3}\tau^{\frac{2\eta^2-n-1}{2\eta(\eta^2+1)}})\alpha^{(1)} \tag{122}$$

and

$$\tau = \left(\psi_{\eta^{-4}}\tau^{\frac{\eta+1}{2\eta^5}}\right)^{\alpha}. \tag{123}$$

- If $\eta = 1$ we get

$$\beta = (\tau^{m_1}, \tau^{m_1}, \tau^{m_1}, \tau^{1-3m_1})^{\alpha^{(1)}}(0, 1, 2, 3) \tag{124}$$

and

$$\tau = \left(\tau^{1-4m_1}\right)^{\alpha}, \tag{125}$$

which produce

$$\begin{aligned} \beta &= (\tau^{\frac{m_1}{1-4m_1}}, \tau^{\frac{m_1}{1-4m_1}}, \tau^{\frac{m_1}{1-4m_1}}, \tau^{\frac{1-3m_1}{1-4m_1}})(0,1,2,3) \\ &= (\tau^{\frac{m_1}{1-4m_1}}, \tau^{\frac{m_1}{1-4m_1}}, \tau^{\frac{m_1}{1-4m_1}}, \tau^{\frac{m_1}{1-4m_1}})\tau \\ &= \tau^{\frac{4m_1}{1-4m_1}}\tau = \tau^{\frac{1}{1-4m_1}} = \tau^{\lambda_{\frac{1}{1-4m_1}}} \end{aligned} . \tag{126}$$

(3) Suppose $\sigma_\beta = \sigma_\tau^{-1} = (0,3,2,1)$. Then, β^{-1} satisfies the previous case. Therefore, as θ inverts τ, we have

$$\beta = \left(\beta^{-1}\right)^{-1} = (\tau^{\gamma})^{-1} = (\tau)^{\theta\gamma} \tag{127}$$

or

$$\beta = \tau^{\theta\lambda_{\frac{1}{1-4m_1}}}, \tag{128}$$

where $m_1 \in \mathbb{Z}_4$,

$$\gamma = \left(\lambda_{\frac{2}{\eta^2(\eta^2+1)}}\right)^{(1)} (e, \psi_\eta, \psi_{\eta^2}\tau^{\frac{\eta-1}{2\eta(\eta^2+1)}}, \psi_{\eta^3}\tau^{\frac{2\eta^2-n-1}{2\eta(\eta^2+1)}})\alpha^{(1)}, \tag{129}$$

$\eta \in U(\mathbb{Z}_4)$ and

$$\tau = \left(\psi_{\eta^{-4}}\tau^{\frac{\eta+1}{2\eta^5}}\right)^{\alpha}. \tag{130}$$

8.5. **Final Step.** We finish the proof of the second part of Theorem A. In order to treat the remaining case where the activity of β is a 4-cycle, we use the fact that $\beta^2 \in B$, which we have already described. Next, from the description of the centralizer of β^2, we are able to pin down the form of β.

Proposition 12. *Let $\beta = (\beta|_0, \beta|_1, \beta|_2, \beta|_3)(0,2)(1,3)$ be such that $(\beta|_0)(\beta|_2) = \tau^{\theta_1}$ and $(\beta|_1)(\beta|_3) = \tau^{\theta_2}$, for some $\theta_1, \theta_2 \in Aut(T_4)$. Then, β is conjugate to τ^2.*

Proof. Let $\alpha = (e, e, \beta|_0^{-1}, \beta|_3^{-1})$. Then,

$$\beta^{\alpha} = (e, e, \ \beta|_0\beta|_2 \ , \beta|_1\beta|_3)(0,2)(1,3). \tag{131}$$

Therefore, substituting $\beta|_0\beta|_2 = \tau^{\theta_1}$ and $\beta|_1\beta|_3 = \tau^{\theta_2}$ in the above equation, we have

$$\beta^{\alpha} = (e, e, \tau^{\theta_1}, \tau^{\theta_2})(0,2)(1,3).$$

Conjugating β^{α} by $\gamma = (\theta_1^{-1}, \theta_2^{-1}, \theta_1^{-1}, \theta_2^{-1})$ we produce

$$\beta^{\alpha\gamma} = \tau^2.$$

□

We show below that active elements of B produce within B elements conjugate to τ^2.

Proposition 13. *Let $\beta \in B$ with nontrivial σ_β. Then*

(i) *If $\sigma_\beta = \sigma_\tau^2$, then β is a conjugate of τ^2.*
(ii) *If $\sigma_\beta \in \{(0,2),(1,3)\}$, then $\beta\beta^\tau$ is a conjugate τ^2.*
(iii) *If $\sigma_\beta \in \{\sigma_\tau, \sigma_\tau^{-1}\}$, then β^2 is a conjugate of τ^2.*

Proof. It is enough to prove (i), since (ii), (iii) are just special cases.

If $\sigma_\beta = \sigma_\tau^2$, then

$$\beta|_0 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}} \tau^{m_0}\right)^\alpha, \ \beta|_1 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}} \tau^{\frac{\xi_1-\xi_0}{2}+m_0}\right)^\alpha, \tag{132}$$

$$\beta|_2 = \left(\lambda_{\frac{\xi_0}{2-\xi_0}} \tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}\right)^\alpha, \beta|_3 = \left(\lambda_{\frac{\xi_0}{2-\xi_0}} \tau^{\left(\frac{\xi_1+\xi_0}{2}-m_0\right)\frac{\xi_0}{2-\xi_0}}\right)^\alpha, \tag{133}$$

$$\tau = \left(\lambda_{(\frac{\xi_0}{2-\xi_0})^2} \tau^{\left(1-\frac{2m_0}{\xi_0}\right)\left(\frac{\xi_0}{2-\xi_0}\right)^2}\right)^\alpha, \tag{134}$$

where $\xi_0, \xi_1 \in U(\mathbb{Z}_4),\ m_0 \in \mathbb{Z}_4$.

Therefore,

$$\beta|_0\beta|_2 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}} \tau^{m_0} \lambda_{\frac{\xi_0}{2-\xi_0}} \tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}\right)^\alpha = \left(\tau^{\frac{\xi_0^2}{2-\xi_0}}\right)^\alpha = (\tau)^{{}^{\psi}\frac{\xi_0^2}{2-\xi_0}{}^{\alpha}}$$

$$\beta|_1\beta|_3 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}} \tau^{\frac{\xi_1-\xi_0}{2}+m_0} \lambda_{\frac{\xi_0}{2-\xi_0}} \tau^{\left(\frac{\xi_1+\xi_0}{2}-m_0\right)\frac{\xi_0}{2-\xi_0}}\right)^\alpha = \left(\tau^{\frac{\xi_1\xi_0}{2-\xi_0}}\right)^\alpha = \tau^{{}^{\psi}\frac{\xi_1\xi_0}{2-\xi_0}{}^{\alpha}}$$

It follows from Proposition 12, that β is a conjugate of τ^2. $\square$

Corollary 4. *Suppose $\beta \in B$ is an active element. Then, B is conjugate to a subgroup of the centralizer $C(\tau^2)$.*

Proposition 14. *Let $\gamma \in C(\tau^2)$. Then,*

$$\gamma = (\tau^{m_0}, \tau^{m_1}, \tau^{m_0+\delta((0)\sigma_\gamma,\, 2)}, \tau^{m_1+\delta((1)\sigma_\gamma,\, 2)})\sigma_\gamma, \tag{135}$$

where $m_0, m_1 \in \mathbb{Z}_4, \sigma_\gamma \in C_{\Sigma_4}(\sigma^2)$.

Proof. Write $\gamma = (\gamma|_0, \gamma|_1, \gamma|_2, \gamma|_3)\sigma_\gamma$. Then $\tau^2\gamma = \gamma\tau^2$ translates to

$$\begin{aligned} & (e,e,\tau,\tau)(0,2)(1,3)(\gamma|_0,\gamma|_1,\gamma|_2,\gamma|_3)\sigma_\gamma \\ = \ & (\gamma|_0,\gamma|_1,\gamma|_2,\gamma|_3)\sigma_\gamma(e,e,\tau,\tau)(0,2)(1,3), \end{aligned}$$

and this in turn translates to

$$\begin{aligned} & (\gamma|_2,\gamma|_3,\tau\gamma|_0,\tau\gamma|_1)(0,2)(1,3)\sigma_\gamma \\ = \ & \begin{array}{l}(\gamma|_0,\gamma|_1,\gamma|_2,\gamma|_3)\cdot \\ \sigma_\gamma(\tau^{\delta(0,2)},\tau^{\delta(1,2)},\tau^{\delta(2,2)},\tau^{\delta(3,2)})(0,2)(1,3)\end{array} \\ = \ & \begin{array}{l}(\gamma|_0,\gamma|_1,\gamma|_2,\gamma|_3) \\ (\tau^{\delta((0)\sigma_\gamma,2)},\tau^{\delta((1)\sigma_\gamma,2)},\tau^{\delta((2)\sigma_\gamma,2)},\tau^{\delta((3)\sigma_\gamma,2)})\sigma_\gamma(0,2)(1,3)\end{array} \\ = \ & (\gamma|_0\tau^{\delta((0)\sigma_\gamma,2)},\gamma|_1\tau^{\delta((1)\sigma_\gamma,2)},\gamma|_2\tau^{\delta((2)\sigma_\gamma,2)},\gamma|_3\tau^{\delta((3)\sigma_\gamma,2)})\sigma_\gamma(0,2)(1,3) \end{aligned}$$

Thus, we have

$$\begin{cases} \gamma|_2 = \gamma|_0 \tau^{\delta((0)\sigma_\gamma,2)}, \\ \gamma|_3 = \gamma|_1 \tau^{\delta((1)\sigma_\gamma,2)}, \\ \tau\gamma|_0 = \gamma|_2 \tau^{\delta((2)\sigma_\gamma,2)}, \\ \tau\gamma|_1 = \gamma|_3 \tau^{\delta((3)\sigma_\gamma,2)}. \end{cases}$$

Hence,

$$\begin{cases} \gamma|_2 = \gamma|_0 \tau^{\delta((0)\sigma_\gamma,2)},\ \gamma|_3 = \gamma|_1 \tau^{\delta((1)\sigma_\gamma,2)}, \\ \tau^{\gamma|_0} = \tau^{\delta((0)\sigma_\gamma,2)+\delta((2)\sigma_\gamma,2)} = \tau,\ \tau^{\gamma|_1} = \tau^{\delta((1)\sigma_\gamma,2)+\delta((3)\sigma_\gamma,2)} = \tau \end{cases}.$$

Therefore, there exist $m_0, m_1 \in \mathbb{Z}_4$ such that

$$\begin{cases} \gamma|_0 = \tau^{m_0},\ \gamma|_1 = \tau^{m_1}, \\ \gamma|_2 = \tau^{m_0+\delta((0)\sigma_\gamma,2)},\ \gamma|_3 = \tau^{m_1+\delta((1)\sigma_\gamma,2)} \end{cases}.$$

Hence, γ has the form

$$\gamma = (\tau^{m_0}, \tau^{m_1}, \tau^{m_0+\delta((0)\sigma_\gamma,2)}, \tau^{m_1+\delta((1)\sigma_\gamma,2)})\sigma_\gamma, \tag{136}$$

where $\sigma_\gamma \in C_{\Sigma_4}(\sigma^2)$. □

Corollary 5. *The centralizer of τ^2 in $\mathcal{A}_4$ is*

$$C(\tau^2) = \langle (e,e,\tau,e)(0,2),\ \tau,\ (\tau^{m_0}, \tau^{m_1}, \tau^{m_0}, \tau^{m_1}) \mid m_0, m_1 \in \mathbb{Z}_4 \rangle .$$

Corollary 6. *Let $\gamma \in C(\tau^2)$ be such that $\sigma_\gamma \in \langle (0,2)(1,3) \rangle$. Then*

$$\gamma \in \left\langle (\tau^{m_0}, \tau^{m_1}, \tau^{m_0}, \tau^{m_1}),\ \tau^2 \mid m_0, m_1 \in \mathbb{Z}_4 \right\rangle .$$

Proposition 15. *Let $\dot{H} = \langle (\tau^{m_0}, \tau^{m_1}, \tau^{m_0}, \tau^{m_1}), \tau^2 \mid m_0, m_1 \in \mathbb{Z}_4 \rangle$. Then the normalizer $N_{\mathcal{A}_4}(\dot{H})$ is the group*

$$\left\langle C(\tau^2), (\psi_{2m_0+1}, \psi_{2m_1+1}, \psi_{2m_0+1}\tau^{m_0}, \psi_{2m_1+1}\tau^{m_1}) \mid m_0, m_1 \in \mathbb{Z}_4 \right\rangle ,$$

where, for each $\eta \in U(\mathbb{Z}_4)$, ψ_η is defined by (102) and

$$\tau^{\psi_\eta} = \tau^\eta .$$

Proof. Note that $\dot{H}$ is an abelian group. Let $\alpha \in N_{\mathcal{A}_4}(\dot{H})$. Then,

$$(\tau^2)^\alpha = (\tau^{m_0}, \tau^{m_1}, \tau^{m_0+1}, \tau^{m_1+1})(0,2)(1,3),$$

where $m_0, m_1 \in \mathbb{Z}_4$.

Suppose α is inactive. Then,

$$\begin{aligned} & (\tau^{m_0}, \tau^{m_1}, \tau^{m_0+1}, \tau^{m_1+1})(0,2)(1,3) \\ = \ & (\alpha|_0^{-1}, \alpha|_1^{-1}, \alpha|_2^{-1}, \alpha|_3^{-1})(e,e,\tau,\tau)(0,2)(1,3)(\alpha|_0, \alpha|_1, \alpha|_2, \alpha|_3) \\ = \ & (\alpha|_0^{-1}, \alpha|_1^{-1}, \alpha|_2^{-1}, \alpha|_3^{-1})(e,e,\tau,\tau)(\alpha|_2, \alpha|_3, \alpha|_0, \alpha|_1)(0,2)(1,3) \\ = \ & (\alpha|_0^{-1}\alpha|_2, \alpha|_1^{-1}\alpha|_3, \alpha|_2^{-1}\tau\alpha|_0, \alpha|_3^{-1}\tau\alpha|_1)(0,2)(1,3) \end{aligned}$$

which produces

$$\begin{cases} \alpha|_0^{-1}\alpha|_2 = \tau^{m_0},\ \alpha|_1^{-1}\alpha|_3 = \tau^{m_1}, \\ \alpha|_2^{-1}\tau\alpha|_0 = \tau^{m_0+1},\ \alpha|_3^{-1}\tau\alpha|_1 = \tau^{m_1+1} \end{cases}.$$

Therefore,

$$\begin{cases} \alpha|_2 = \alpha|_0 \tau^{m_0},\ \alpha|_3 = \alpha|_1 \tau^{m_1}, \\ \alpha|_0^{-1} \tau \alpha|_0 = \tau^{2m_0+1},\ \alpha|_1^{-1} \tau \alpha|_1 = \tau^{2m_1+1}. \end{cases}$$

Thus,

$$\alpha = (\alpha|_0, \alpha|_1, \alpha|_2, \alpha|_3) = (\psi_{2m_0+1}, \psi_{2m_1+1}, \psi_{2m_0+1}\tau^{m_0}, \psi_{2m_1+1}\tau^{m_1})$$

satisfies

$$(\tau^2)^\alpha = (\tau^{m_0}, \tau^{m_1}, \tau^{m_0+1}, \tau^{m_1+1})(0,2)(1,3).$$

□

Theorem 7. *Let G be a finitely generated solvable subgroup of $Aut(T_4)$ which contains τ. Then, G is a subgroup of*

$$\times_4 \left(\cdots \left(\times_4 \left(\times_4 N_{\mathcal{A}_4}(H)^\alpha \rtimes S_4 \right) \rtimes S_4 \right) \cdots \right) \rtimes S_4 \tag{137}$$

for some $\alpha \in \mathcal{A}_4$.

Proof. As in the case $n = p$, we assume G has derived length $d \geq 2$ and let B be the $(d-1)$th term of the derived series of G. Then, B is an abelian group normalized by τ. On analyzing the case 8.4 and the final step, there exists a level t such that B is a subgroup of $\dot{V} = \times_{4^k} C(\mu^2)$,where $\mu = \tau^\alpha$ for some $\alpha \in \mathcal{A}_4$ and where $\sigma_{\mu^2} = (0,2)(1,3)$. There also exists $\beta \in B$ such that $\beta|_u = \mu^2$ for some index $u \in \mathcal{M}$.

Moreover, if T is the normalizer of $C(\tau^2)$, then clearly, T^α is the normalizer of $C(\mu^2)$.

We will show now that G is a subgroup of

$$\dot{J} = \times_4 \left(\cdots \left(\times_4 \left(\times_4 N_{\mathcal{A}_4}(H)^\alpha \rtimes S_4 \right) \rtimes S_4 \right) \cdots \right) \rtimes S_4$$

where the cartesian product $\times_4$appears t times..

Let $\gamma \notin \dot{J}$. Since $\gamma \notin \dot{J}$, there exists $w \in \mathcal{M}$ having $|w| = t$ and $\gamma|_w \notin T^\alpha$. Since τ is transitive on all levels of the tree, by Corollary 6 we can conjugate β by an appropriate power of τ to get $\theta \in B$ such that

$$\theta|_w = \mu^2 \text{ or } \theta|_w = \left(\mu^2\right)^\tau = \left((\tau^{m_0}, \tau^{m_1}, \tau^{m_0+1}, \tau^{m_1+1})(0,2)(1,3)\right)^\alpha,$$

where $m_0, m_1 \in \mathbb{Z}_4$. Thus, for $v = w^\gamma$ we have

$$(\theta^\gamma)|_v \overset{(9)}{=} \theta|_{v^{\gamma^{-1}}}^{\gamma_{v^{\gamma^{-1}}}} = \theta|_w^{\gamma|_w} \notin C(\mu^2)$$

which implies $\theta^\gamma \notin B \leq \dot{V}$ and $\gamma \notin G$. Hence, G is a subgroup of $\dot{J}$. □

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