The Six-Point Circle Theorem *

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Abstract

Given ΔABC and angles $\alpha, \beta, \gamma \in (0, \pi)$ with $\alpha + \beta + \gamma = \pi$, we study the properties of the triangle DEF which satisfies: (i) $D \in BC$, $E \in AC$, $F \in AB$, (ii) $\measuredangle D = \alpha$, $\measuredangle E = \beta$, $\measuredangle F = \gamma$, (iii) ΔDEF has the minimal area in the class of triangles satisfying (i) and (ii). In particular, we show that minimizer ΔDEF, exists, is unique and is a pedal triangle, corresponding to a certain pedal point P. Permuting the roles played by the angles α, β, γ in (ii), yields a total of six such area-minimizing triangles, which are pedal relative to six pedal points, say, $P_1, ..., P_6$. The main result of the paper is the fact that there exists a circle which contains all six points.

1 Introduction

Consider the following question in planar geometry: *given three, nonconcurrent lines, L_1, L_2, L_3, along with three angles α, β, $\gamma \in (0, \pi)$ satisfying $\alpha + \beta + \gamma = \pi$, find three points X_1, X_2, X_3 such that*

$$X_1 \in L_1,\ X_2 \in L_2,\ X_3 \in L_3, \tag{1.1}$$

and

$$\measuredangle X_2 X_1 X_3 = \alpha, \quad \measuredangle X_1 X_2 X_3 = \beta, \quad \measuredangle X_1 X_3 X_2 = \gamma. \tag{1.2}$$

Somewhat informally, with A, B, C playing the roles of the points of mutual intersection of the lines L_1, L_2, L_3, this asks whether given a fixed triangle ABC, one can inscribe a triangle $X_1X_2X_3$ of an arbitrarily prescribed shape. For a triangle, "shape" simply signifies the measures of its angles (in the current scenario α, β, γ).

Although not necessarily obvious, this question turns out to have an affirmative answer. In fact, the collection $\mathcal{I}_{\alpha\beta\gamma}$ of all triangles $X_1X_2X_3$ satisfying (1.1) and (1.2) has infinite cardinality. Since all triangles in $\mathcal{I}_{\alpha\beta\gamma}$ have the *same shape*, what distinguishes them is their *size*. It is not too difficult to show that $\mathcal{I}_{\alpha\beta\gamma}$ contains triangles of arbitrarily large areas, hence, the most distinguished triangle in $\mathcal{I}_{\alpha\beta\gamma}$ is the one of the *smallest* area, called, for the purposes of this introduction, $\Delta_{\alpha\beta\gamma}$. Part of our paper is devoted to studying this minimizer, which enjoys many interesting properties. Among other things we show that this triangle of minimal area is unique, and is also a *pedal triangle*. Recall that a pedal triangle is a triangle whose vertices are the projections of a point, called the pedal point of the pedal triangle, onto the sides of another triangle. Let $P_{\alpha\beta\gamma}$ be the pedal point of $\Delta_{\alpha\beta\gamma}$.

Of course, similar considerations apply to the situation when the ordered triplet (α, β, γ) in (1.2) is replaced by any of its permutations

$$(\alpha, \beta, \gamma),\ (\alpha, \gamma, \beta),\ (\beta, \alpha, \gamma),\ (\beta, \gamma, \alpha),\ (\gamma, \alpha, \beta),\ (\gamma, \beta, \alpha). \tag{1.3}$$

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As before, these permutations lead to considering the collections $\mathcal{I}_{\alpha\beta\gamma}$, $\mathcal{I}_{\alpha\gamma\beta}$, ..., defined as above, which in turn have their own minimizers,

$$\Delta_{\alpha\beta\gamma},\ \Delta_{\alpha\gamma\beta},\ \Delta_{\beta\alpha\gamma},\ \Delta_{\beta\gamma\alpha},\ \Delta_{\gamma\alpha\beta},\ \Delta_{\gamma\beta\alpha}. \tag{1.4}$$

Once again, they are all pedal triangles with respect to six points, say,

$$P_{\alpha\beta\gamma},\ P_{\alpha\gamma\beta},\ P_{\beta\alpha\gamma},\ P_{\beta\gamma\alpha},\ P_{\gamma\alpha\beta},\ P_{\gamma\beta\alpha}. \tag{1.5}$$

Remarkably, even though these six points are obtained via constructions that start from starkly different sets of premises, they turn out to be concyclic. That is,

Theorem 1.1 (THE SIX POINT CIRCLE THEOREM)**.** *There exists a circle that contains all six points in (1.5).*

This is the main result of the paper (cf. Theorem 3.2 for a formal statement). The starting point in its proof is to consider, besides the area-minimizing triangle inscribed in ΔABC and having prescribed angles, the *area-maximizing triangle circumscribed to* ΔABC, having the same angles. This dual aspect of the problem under consideration is very useful in understanding the nature of the extremal triangles of a prescribed shape. A concrete step in this regard is proving a structure theorem, detailing how these three triangles are related to one another. For instance, we show that *the area of the original triangle is the geometric mean of the area-minimizer and the area-maximizer triangles*; cf. Theorem 2.4. Since the conclusion in Theorem 1.1 involves a circle, it is not unnatural that another important tool in its proof is the *geometrical inversion*.

In §2, the properties of the area-minimizing and area-maximizing triangles having prescribed angles and being, respectively, inscribed and circumscribed in a given triangle are examined. Finally, §3 is devoted to presenting the proof of The Six Point Circle Theorem. The presentation in this paper is largely self-contained. For more basic concepts and definitions related to the geometry of the triangle the interested reader is referred to, e.g. [2], and the references therein.

2 Extremal Triangles of a Prescribed Shape

Throughout, we let ΔABC denote the triangle with vertices A, B, C and denote by $|\Delta ABC|$ the area of $\Delta\, ABC$.

Given a triangle $A_1A_2A_3$ along with a triangle $B_1B_2B_3$ inscribed in it, we first recall a result from [3] that gives a procedure for obtaining a triangle, $C_1C_2C_3$, that is inscribed in $\Delta B_1B_2B_3$ and is homotopic to $\Delta A_1A_2A_3$. By definition, two triangles are homotopic if their corresponding sides are parallel. The concurrence point of the lines passing through the corresponding vertices of two homotopic triangles is the homotopy center. The justification of the following useful result is left to the interested reader (a proof is given in [3]).

Proposition 2.1. *Let $\Delta A_1A_2A_3$ be arbitrary and assume that $B_1 \in A_2A_3$, $B_3 \in A_2A_1$, $B_2 \in A_1A_3$ (see Figure 1 below). Take $C_1 \in B_2B_3$, $C_2 \in B_1B_3$, $C_3 \in B_1B_2$ such that*

$$\frac{A_1B_3}{B_3A_2} = \frac{B_2C_3}{C_3B_1}, \quad \frac{A_2B_1}{B_1A_3} = \frac{B_3C_1}{C_1B_2}, \quad \frac{A_3B_2}{B_2A_1} = \frac{B_1C_2}{C_2B_3}, \tag{2.6}$$

Then $\Delta A_1A_2A_3$ and $\Delta C_1C_2C_3$ are homotopic and, in addition, $|\Delta B_1B_2B_3|$ is the geometric mean of $|\Delta A_1A_2A_3|$ and $|\Delta C_1C_2C_3|$, i.e.

$$|\Delta B_1B_2B_3|^2 = |\Delta A_1A_2A_3| \cdot |\Delta C_1C_2C_3|. \tag{2.7}$$

Conversely, if $\Delta A_1A_2A_3$ and $B_1 \in A_2A_3$, $B_3 \in A_2A_1$, $B_2 \in A_1A_3$ are given and $C_1 \in B_2B_3$, $C_2 \in B_1B_3$, $C_3 \in B_1B_2$ are such that $\Delta A_1A_2A_3$ and $\Delta C_1C_2C_3$ are homotopic, then (2.6) and (2.7) hold.

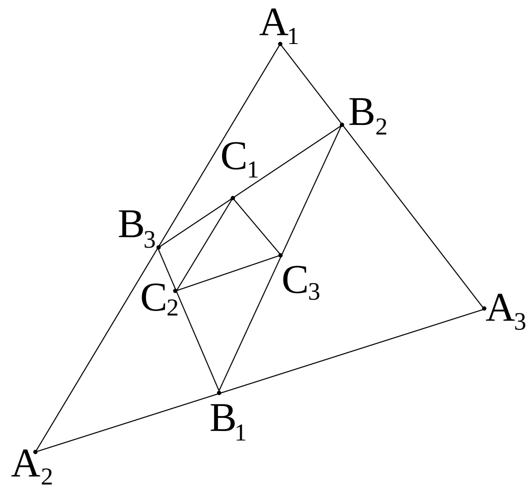

Figure 1

Definition 2.2. *(i) Call a triangle XYZ* `inscribed` *in a given triangle ABC if each of the lines AB, BC and AC contain precisely one of the vertices X, Y, Z. Call ΔXYZ* `circumscribed` *to ΔABC if ΔABC is inscribed in ΔXYZ.*

(ii) Given two fixed triangles ABC, called of `reference`*, and MNP, called* `fundamental`*, let $\mathcal{I}$ be the set of all triangles which are similar to the fundamental one and are inscribed in the triangle of reference, and let $\mathcal{C}$ be the set of all triangles similar to the fundamental one and circumscribed to the reference one.*

(iii) Given $\Delta\, ABC$, triangle of reference, and a fundamental triangle with angles α, β, γ, denote by $\mathcal{I}_{\alpha\beta\gamma}$ the set of all triangles XYZ with X, Y, Z belonging, respectively, to the lines BC, AC, AB, and for which $\measuredangle X$, $\measuredangle Y$, $\measuredangle Z$ coincide with α, β, γ (in this order).

Likewise, let $\mathcal{C}_{\alpha\beta\gamma}$ denote the set of all triangles XYZ for which A, B, C belonging, respectively, to the lines YZ, XZ, XY, and for which $\measuredangle X$, $\measuredangle Y$, $\measuredangle Z$ coincide with α, β, γ (in this order).

Clearly,

$$\begin{aligned} \mathcal{I} &= \mathcal{I}_{\alpha\beta\gamma} \bigcup \mathcal{I}_{\alpha\gamma\beta} \bigcup \mathcal{I}_{\beta\alpha\gamma} \bigcup \mathcal{I}_{\beta\gamma\alpha} \bigcup \mathcal{I}_{\gamma\alpha\beta} \bigcup \mathcal{I}_{\gamma\beta\alpha}, \\ \mathcal{C} &= \mathcal{C}_{\alpha\beta\gamma} \bigcup \mathcal{C}_{\alpha\gamma\beta} \bigcup \mathcal{C}_{\beta\alpha\gamma} \bigcup \mathcal{C}_{\beta\gamma\alpha} \bigcup \mathcal{C}_{\gamma\alpha\beta} \bigcup \mathcal{C}_{\gamma\beta\alpha}. \end{aligned} \tag{2.8}$$

Our next result shows the existence of area-minimizers and area-maximizers in the above classes. The issue of uniqueness for these extremal triangles will be dealt with in Theorem 2.4 below.

Theorem 2.3. *There exists a triangle $\Delta\, M_1N_1P_1 \in \mathcal{I}_{\alpha\beta\gamma}$ which has the smallest area amongst all the triangles in the class $\mathcal{I}_{\alpha\beta\gamma}$. Also, there exists a triangle $\Delta\, M_2N_2P_2 \in \mathcal{C}_{\alpha\beta\gamma}$ which has the largest area amongst all the triangles in the class $\mathcal{C}_{\alpha\beta\gamma}$. In addition, any two such triangles satisfy*

$$|\Delta M_1N_1P_1| \cdot |\Delta M_2N_2P_2| = |\Delta ABC|^2. \tag{2.9}$$

In fact, similar results are valid for each of the classes appearing in (2.8), including $\mathcal{I}$ and $\mathcal{C}$.

Proof. It suffices to only deal with the class $\mathcal{I}_{\alpha\beta\gamma}$ since a similar reasoning applies to all the classes in (2.8). The first issue is to show that there exists at least one area-minimizer in $\mathcal{I}_{\alpha\beta\gamma}$. We briefly sketch the argument for this, which is based on results from Calculus and Analytical Geometry. To get started, fix a unit vector $\vec{w} = w_1\vec{i} + w_2\vec{j}$ and consider the class of all triangles XYZ satisfying

$$\measuredangle X = \alpha, \quad \measuredangle Y = \beta, \quad \measuredangle Z = \gamma, \quad X \in BC, \quad Z \in AB, \quad XZ \parallel \vec{w}. \tag{2.10}$$

It is then clear that the coordinates of Y and Z depend linearly on those of X, with coefficients varying continuously with w_1, w_2. Since X moves on a line (i.e., BC), it follows that the geometrical locus of the vertex Y is itself a line. In addition, the characteristics of this line, called L, depend continuously on w_1, w_2. Now, the position of X which, under the constraints (2.10), yields a triangle XYZ in the class $\mathcal{I}_{\alpha\beta\gamma}$, is the one corresponding to the case when $Y = L \cap AC$. Linear algebra considerations then show that the coordinates of X in this critical case are continuously dependent on w_1, w_2. Using the formula which expresses the area of a triangle in terms of the coordinates of its vertices, we then deduce that $|\Delta XYZ|$ -corresponding the case when $\Delta XYZ \in \mathcal{I}_{\alpha\beta\gamma}$- is a continuous function in w_1, w_2. Since w_1 and w_2 vary in a compact set, it follows that the assignment $(w_1, w_2) \mapsto |\Delta XYZ|$ is bounded and it attains its extrema. In particular, there exists an area-minimizing triangle in $\mathcal{I}_{\alpha\beta\gamma}$.

To prove that there exists (at least) one area-maximizing triangle in $\mathcal{C}_{\alpha\beta\gamma}$, we proceed as follows. Starting with an area-minimizing triangle $\Delta M_1N_1P_1$ (whose existence has just been established), construct a a new triangle, $\Delta M_2N_2P_2$ by taking the parallels through A, B and C to the sides of $\Delta\, M_1N_1P_1$. As a result, $\Delta M_2N_2P_2$ is homotopic to $\Delta\, M_1N_1P_1$ and circumscribed to $\Delta\, ABC$. Hence, $\Delta\, M_2N_2P_2 \in \mathcal{C}_{\alpha\beta\gamma}$ and, thanks to Proposition 2.1,

$$|\Delta M_2N_2P_2| \cdot |\Delta M_1N_1P_1| = |\Delta ABC|^2. \tag{2.11}$$

We claim that $\Delta M_2N_2P_2$ is area-maximizing in the class $\mathcal{C}_{\alpha\beta\gamma}$. Indeed, if $\Delta M''N''P'' \in \mathcal{C}_{\alpha\beta\gamma}$ is arbitrary, let $M' \in BA$, $N' \in BC$, $P' \in AC$ (cf. Figure 2) be such that

$$\frac{AM'}{M'B} = \frac{P''C}{CN''}, \quad \frac{BN'}{N'C} = \frac{M''A}{AP''}, \quad \frac{CP'}{P'A} = \frac{N''B}{BM''}. \tag{2.12}$$

Proposition 2.1 gives that $\Delta\, M'N'P'$ and $\Delta\, M''N''P''$ are homotopic (thus $\Delta\, M'N'P' \in \mathcal{I}_{\alpha\beta\gamma}$) and

$$|\Delta M''N''P''| \cdot |\Delta M'N'P'| = |\Delta ABC|^2. \tag{2.13}$$

Since by definition $|\Delta M'N'P'| \geq |\Delta M_1N_1P_1|$, (2.11), (2.13) imply that $|\Delta M''N''P''| \leq |\Delta M_2N_2P_2|$. This justifies our claim that $\Delta M_2N_2P_2$ is area-maximizing in the class $\mathcal{C}_{\alpha\beta\gamma}$.

Going further, we note that proving (2.9) uses a similar circle of ideas. Concretely, let $M' \in BA$, $N' \in BC$, $P' \in AC$ (cf. Figure 2) be such that

$$\frac{AM'}{M'B} = \frac{P_2C}{CN_2}, \quad \frac{BN'}{N'C} = \frac{M_2A}{AP_2}, \quad \frac{CP'}{P'A} = \frac{N_2B}{BM_2}. \tag{2.14}$$

Proposition 2.1 shows that $\Delta\, M'N'P'$ and $\Delta\, M_2N_2P_2$ are homotopic and

$$|\Delta M_2N_2P_2| \cdot |\Delta M'N'P'| = |\Delta ABC|^2. \tag{2.15}$$

The former property also entails $\Delta\, M'N'P' \in \mathcal{I}_{\alpha\beta\gamma}$ and, hence,

$$|\Delta M_1N_1P_1| \leq |\Delta M'N'P'|. \tag{2.16}$$

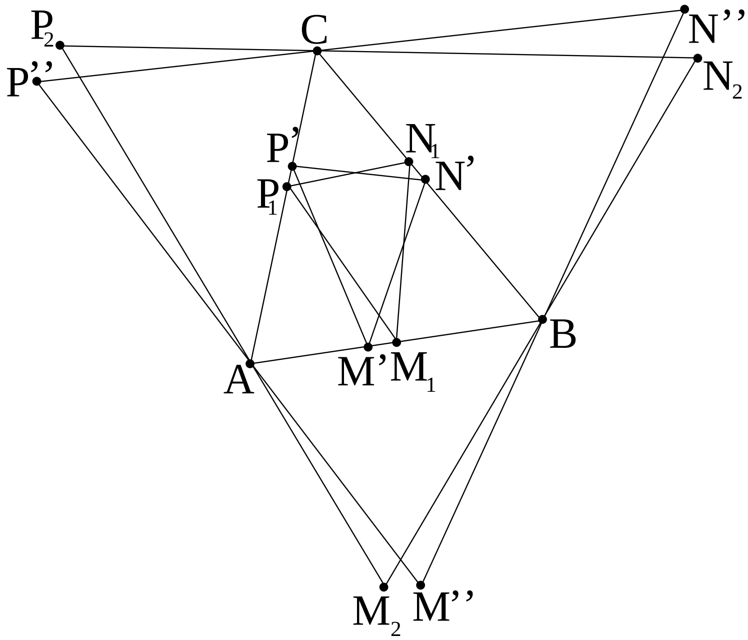

Figure 2

The parallels through A, B and C to the sides of the triangle $\Delta\, M_1N_1P_1$ determine the triangle $M''N''P''$ which is homotopic to $\Delta\, M_1N_1P_1$ and circumscribed to $\Delta\, ABC$. As a result, $\Delta\, M''N''P'' \in \mathcal{C}_{\alpha\beta\gamma}$ which forces

$$|\Delta M''N''P''| \leq |\Delta M_2N_2P_2|. \tag{2.17}$$

In this context, Proposition 2.1 also gives that

$$|\Delta M''N''P''| \cdot |\Delta M_1N_1P_1| = |\Delta ABC|^2. \tag{2.18}$$

By combining (2.15)-(2.18) we obtain

$$\begin{aligned} |\Delta ABC|^2 &= |\Delta M_1N_1P_1||\Delta M''N''P''| \\ &\leq |\Delta M_2N_2P_2||\Delta M'N'P'| = |\Delta ABC|^2, \end{aligned} \tag{2.19}$$

so that, we actually have $|\Delta M''N''P''| = |\Delta M_2N_2P_2|$, $|\Delta M_1N_1P_1| = |\Delta M'N'P'|$. Therefore, $|\Delta ABC|^2 = |\Delta M_1N_1P_1||\Delta M''N''P''| = |\Delta M_1N_1P_1||\Delta M_2N_2P_2|$, finishing the proof of (2.9). □

Remark 1. *It is implicit in the proof of the above theorem that if $\Delta M_1N_1P_1$ is an area-minimizer in the class $\mathcal{I}_{\alpha\beta\gamma}$, then the triangle $M_2N_2P_2$ that is in $\mathcal{C}_{\alpha\beta\gamma}$ and is homotopic to $\Delta M_1N_1P_1$ (constructed as in Proposition 2.1) is an area-maximizer in the class $\mathcal{C}_{\alpha\beta\gamma}$. Conversely, if $\Delta M_2N_2P_2$ is an area-maximizer in the class $\mathcal{C}_{\alpha\beta\gamma}$, then the triangle $M_1N_1P_1$ that is in $\mathcal{I}_{\alpha\beta\gamma}$ and is homotopic to $\Delta M_2N_2P_2$ (again, constructed as in Proposition 2.1) is an area-minimizer in the class $\mathcal{I}_{\alpha\beta\gamma}$.*

The main result in this section describes the structure of the configuration consisting of a given triangle ABC along with the area-minimizing triangle in $\mathcal{I}_{\alpha\beta\gamma}$ and the area-maximizing triangle in $\mathcal{C}_{\alpha\beta\gamma}$. Recall that an antipedal triangle of a point P with respect to ΔABC is the triangle whose vertices are the intersection points of the perpendiculars to PA, PB, and PC, passing through A, B, and C, respectively. Also, the isogonal of a point P in ΔABC is the concurrence point of the reflections of the lines PA, PB, and PC across the angle bisectors of A, B, and C.

Theorem 2.4. *Retain notation and conventions introduced earlier. Given ΔABC, reference triangle, and ΔMNP fundamental triangle with angles $\alpha, \beta, \gamma \in (0, \pi)$, let $\mathcal{I}_{\alpha\beta\gamma}$ and $\mathcal{C}_{\alpha\beta\gamma}$ be as before.*

Then there exist a unique $\Delta\, EFD \in \mathcal{I}_{\alpha\beta\gamma}$ which is area-minimizing in this class, along with a unique $\Delta\, QRS \in \mathcal{C}_{\alpha\beta\gamma}$ which is area-maximizing in this class. In addition, these triangles enjoy the following properties:

(i) The triangles EFD and QRS are homotopic;

(ii) The triangles EFD and QRS, are the pedal and the antipedal triangles of some points K and L, respectively, with respect to $\Delta\, ABC$;

(iii) Let T be the concurrence point of the projections of Q, R, S on the sides of the triangle ABC. Then the pairs of points (L, K) and (T, L) are isogonal in the triangles ABC and QRS, respectively;

(iv) If O is the homotopy center of $\Delta\, EFD$ and $\Delta\, QRS$, then O, K, T lie on the same line.

Before proceeding with the proof of this theorem, a comment is in order.

Remark 2. *As a corollary of the above theorem, there exists a unique area-minimizing triangle in each of the classes $\mathcal{I}$, $\mathcal{I}_{\alpha\beta\gamma}$, ..., listed in the first line of (2.8). Moreover, there exists a unique area-maximizing triangle in each of the classes $\mathcal{C}$, $\mathcal{C}_{\alpha\beta\gamma}$, ..., listed in the second line of (2.8). In each case, similar properties to $(i) - (iv)$ above hold.*

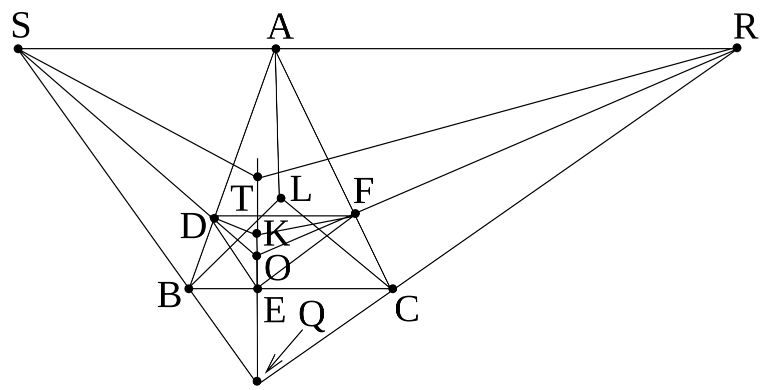

Figure 3

Proof of Theorem 2.4. By design, the triangles QRS and MNP are similar (cf. Figure 3), so that $\measuredangle Q = \measuredangle M$, $\measuredangle R = \measuredangle N$, $\measuredangle S = \measuredangle P$. If we do not impose the condition that the triangle QRS has maximal area in the class $\mathcal{C}_{\alpha\beta\gamma}$, then the triangle QRS is not completely determined. There are, however, certain restrictions on the location of its vertices, due solely to the membership to $\mathcal{C}_{\alpha\beta\gamma}$. Concretely, the point S must lie on the circle passing through A and B and such that the arc $\overset{\frown}{AB} = 2\measuredangle S$. Let us denote by O_1 the center of this circle (see Figure 4). Likewise, the point Q lies on a circle (whose center we denote by O_2) passing through B and C and such that $\overset{\frown}{BC} = 2\measuredangle Q$. Finally, the point R lies on a circle (centered at some point, denoted by O_3) passing through A and C and for which $\overset{\frown}{AC} = 2\measuredangle R$.

Since $\measuredangle M + \measuredangle N + \measuredangle P = \measuredangle Q + \measuredangle R + \measuredangle S = \pi$, it is not difficult to see that these three circles, have a common point, which we denote by L. Observe next that a particular position of the point S on the circle (O_1) determines also the positions of the points Q and R on the circles (O_2) and (O_3), respectively. The idea is now to determine the position of the point S on the circle (O_1) for which the triangle QRS has maximal area.

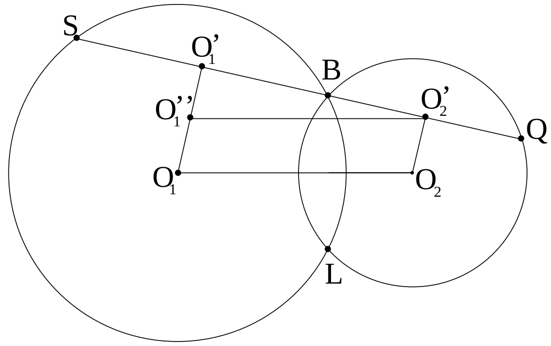

Figure 4

Since the angles of ΔQRS are fixed, maximizing $|\Delta QRS|$ amounts to maximizing the length of one of its sides, say SQ. Let O_1' and O_2' be, respectively, the projections of O_1 and O_2 onto the line SQ, and let $O_1'' \in O_1'O_1$ be such that $O_1''O_2' \parallel O_1O_2$. Then we can write

$$O_1O_2 = O_1''O_2' \geq O_1'O_2' = BO_1' + BO_2' = \frac{BS}{2} + \frac{BQ}{2} = \frac{SQ}{2}. \tag{2.20}$$

This implies $2O_1O_2 \geq SQ$, an inequality from which we see that the maximum of $|\Delta QRS|$ is attained when $SQ \parallel O_1O_2$.

Since the location of the points O_1, O_2, O_3 depends only on ΔABC and angles α, β, γ, the result just established shows that there exists a *unique* area-maximizing triangle in $\mathcal{C}_{\alpha\beta\gamma}$. Given an area-minimizing triangle ΔDEF in the class $\mathcal{I}_{\alpha\beta\gamma}$, we can associate with it a homotopic triangle ΔQRS in $\mathcal{C}_{\alpha\beta\gamma}$, which can be constructed following the recipe given in Proposition 2.1. Using the Remark 1, the latter triangle is area-maximizing in $\mathcal{C}_{\alpha\beta\gamma}$ and, hence, must coincide with the unique area-maximizing triangle described in the first part of the current proof. This shows that ΔDEF is itself uniquely determined by the property that it is area-minimizing in $\mathcal{I}_{\alpha\beta\gamma}$. This concludes the proof of the first claim made in the statement of Theorem 2.4. The above considerations also show that ΔDEF and ΔQRS are homotopic, thus proving item (i).

As regards (ii), let us recall from the first part of the proof that the triangle QRS has a maximal area when $S = LO_1 \cap (O_1)$, $Q = LO_2 \cap (O_2)$, $R = LO_3 \cap (O_3)$. In particular, this implies that the triangles QRS and $O_1O_2O_3$ are homotopic with homotopy center L and dilation factor $\frac{1}{2}$. Next, since $O_1O_2 \parallel SQ$ and $O_1O_2 \perp BL$, it follows that $BL \perp SQ$. Similarly $LC \perp RQ$ and $LA \perp SR$. Altogether, these observations imply that the triangle SRQ is the antipedal of L with respect to the triangle ABC. Now, using that $DF \parallel SR$, $DE \parallel SQ$, and $FE \parallel RQ$ we obtain that $AL \perp DF$, $BL \perp DE$, and $CL \perp EF$. This means that the projections of the vertices of ΔABC on the sides of ΔDEF are concurrent, therefore establishing that the triangles ABC and DEF are orthologic. As is well-know, this gives that the perpendiculars from D, E, F onto AB, BC, AC respectively, are also concurrent in a point, say K. Hence, ΔDEF is the pedal triangle of K with respect to the triangle ABC. This finishes the proof of (ii).

Consider next (iii). Since the triangles ABC and SQR are orthologic, there exists a point T such that $ST \perp AB$, $QT \perp BC$, and $RT \perp AC$. In the quadrilateral $ADKF$ we have

$$\measuredangle KAF = \measuredangle FDK \quad \text{and} \quad \measuredangle FDK + \measuredangle FDA = \measuredangle FDA + \measuredangle LAD = \frac{\pi}{2}. \tag{2.21}$$

Therefore,

$$\measuredangle FDK = \measuredangle LAD. \tag{2.22}$$

From (2.21) and (2.22) we see that the lines AL and AK are isogonal. Similarly, BL and BK are isogonal, proving that L and K are isogonal points in $\Delta\, ABC$. This concludes the proof of (iii).

To prove (iv), we start by observing that the dilation of center O and factor $\frac{SQ}{DE}$ (i.e., the dilation which takes ΔDEF into ΔSQR) transforms the lines DK, EK and FK into the lines ST, RT and QT, respectively. (Indeed, this is a direct corollary of the fact that dilations map parallel lines into parallel lines.) Thus, under such a transformation, the point K is mapped into T and, as a consequence, the points O, K and T are collinear. The proof of Theorem 2.4 is now finished. □

The result below essentially says the that a pedal triangle (relative to ΔABC) is area-minimizing amongst all triangles of a similar shape inscribed in ΔABC.

Theorem 2.5. *Let P be a point contained inside the circumcircle of a given triangle ABC. Let D, E, F be the projections of P onto the sides BC, AC and AB, respectively, and set*

$$\alpha := \measuredangle EDF, \quad \beta := \measuredangle DEF, \quad \gamma := \measuredangle EFD. \tag{2.23}$$

Then ΔDEF is the area-minimizing triangle in the class $\mathcal{I}_{\alpha\beta\gamma}$ (cf. (ii) of Definition 2.2).

Proof. Let K be the isogonal of P (relative to ΔABC) and denote by ΔQRS the antipedal of K with respect to ΔABC. Since $\Delta\, DEF$ is the pedal triangle of the point P with respect to $\Delta\, ABC$, then quadrilateral $AFPE$ is inscribable. This implies that $\measuredangle PEF + \measuredangle PAF = \frac{\pi}{2}$. Next, since K is the isogonal of P, this further entails $\measuredangle PEF + \measuredangle KAE = \frac{\pi}{2}$. This means that $AK \perp EF$, i.e. $RS \| EF$. Via a similar reasoning, we can show that the two other pairs of corresponding sides in ΔQRS and ΔDEF are parallel. This shows that ΔQRS and ΔDEF are homotopic. In particular, the angles of ΔQRS are α, β, γ.

Recall the (three, concurrent) circles (O_1), (O_2), (O_3) introduced in the proof of Theorem 2.4. Then $S \in (O_1)$, $R \in (O_2)$, $Q \in (O_3)$. Given that the quadrilateral $AKCR$ is inscribable, it follows that $K \in (O_1)$. Similarly, $K \in (O_2)$, $K \in (O_3)$. Hence, K is the common point of (O_1), (O_2), (O_3), which implies that ΔQRS is homotopic to $\Delta O_1O_2O_3$. As pointed out already in the proof of Theorem 2.4, this necessarily implies that ΔQRS is a maximizer in the class $\mathcal{C}_{\alpha\beta\gamma}$. In turn, by (iii) in Theorem 2.4 this forces ΔDEF to be an area minimizer in $\mathcal{I}_{\alpha\beta\gamma}$. □

We end with a result which appears to be folklore (see [1]; a new proof has been given in [3]).

Theorem 2.6. *Assume ABC to be a given triangle and denote by O and R the center and the radius of the circumcircle, respectively. Let M be an arbitrary point in ΔABC and let $\Delta\alpha\beta\gamma$ be the pedal triangle of M with respect to ΔABC. Then*

$$\frac{|\Delta\alpha\beta\gamma|}{|\Delta ABC|} = \frac{|R^2 - OM^2|}{4R^2}. \tag{2.24}$$

3 The Main Result

This section is devoted to stating and proving the main result in this paper, Theorem 3.2. First, we take care of a number of prerequisites, starting with the result below.

Theorem 3.1. *Let M be a point contained in the circumcircle of $\Delta\, ABC$, which is assumed to have center O and radius R. In addition, let $N \in OM$ be such that $OM \cdot ON = R^2$. Then the pedal triangles, $\Delta M_1M_2M_3$ and $\Delta N_1N_2N_3$, of the points M and N with respect to $\Delta\, ABC$ are similar (with pairs of equal angles having vertices located on the same sides of $\Delta\, ABC$). Furthermore,*

$$|\Delta M_1M_2M_3| \le |\Delta N_1N_2N_3|. \tag{3.25}$$

Conversely, if M and $N \in OM$ are two points such that their pedal triangles with respect to $\Delta\, ABC$ are similar (with pairs of equal angles having vertices on the same sides of $\Delta\, ABC$), then $OM \cdot ON = R^2$.

Proof. Let X, Y and Z be the centers of the circles AMB, BMC and AMC, respectively (see Figure 5). By an inversion of center O and modulus R, the triangle ABC and the circle (O) remain unchanged. The circles (X), (Y) and (Z) are transformed in the circles (X'), (Y') and (Z'), respectively, passing through A and B, B and C, A and C, respectively, and have a common point N, which is the inverse transform of M (see Figure 6).

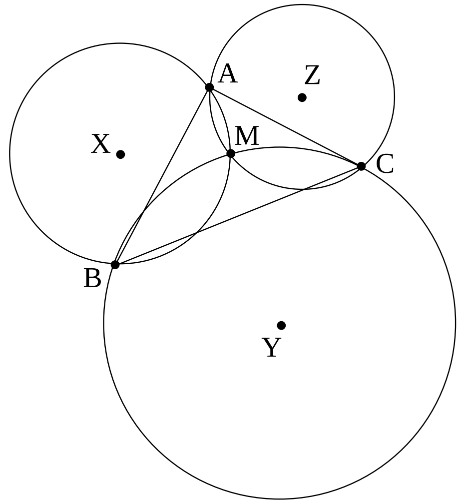

Figure 5

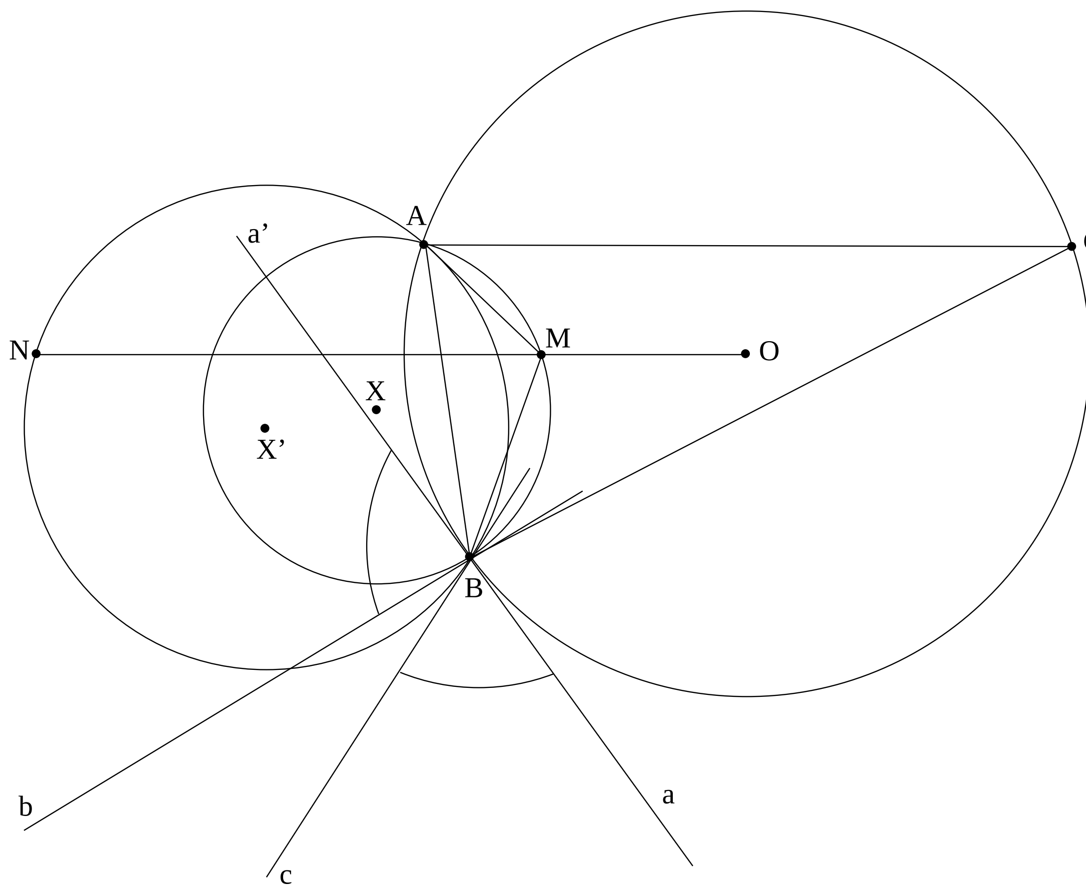

Figure 6

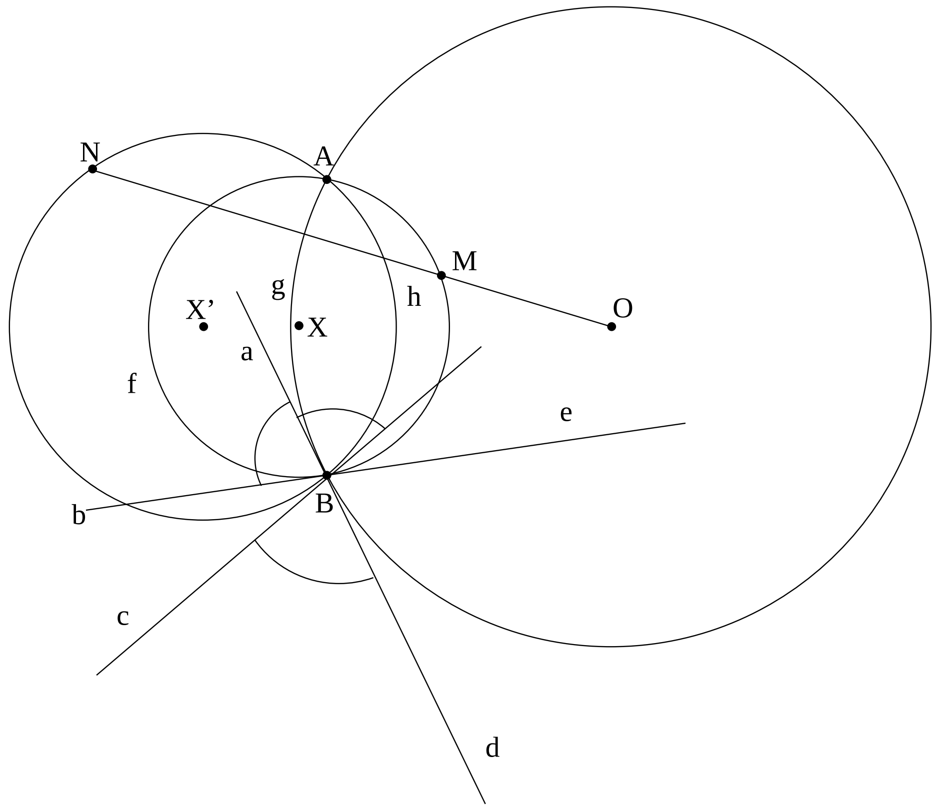

Figure 7

Consider the lines aB, bB and cB, which are tangent to the circles (O), (X) and (X'), respectively (cf. Figure 7). As the inversion preserves the angle between two curves, we obtain that $\measuredangle((O),(X)) = \measuredangle((O),(X'))$, i.e., $\measuredangle(bBa) = \measuredangle(cBa)$. We can write $\measuredangle bBa$ as the difference of $\measuredangle ABb$ and $\measuredangle ABa$. This difference in turn can be re-written as $\frac{\overset{\frown}{f}}{2} - \frac{\overset{\frown}{g}}{2}$. We have $\measuredangle cBd$ equal to $\measuredangle aBe$, which is further equal to $\measuredangle ABe + \measuredangle aBA = \frac{\overset{\frown}{h}}{2} + \frac{\overset{\frown}{g}}{2}$. Using the fact that $\measuredangle bBa = \measuredangle cBd$, we obtain that

$$\measuredangle BNA = \measuredangle BMA - 2\measuredangle ACB. \tag{3.26}$$

If $M_1M_2M_3$ and $N_1N_2N_3$ are the pedal triangles of M and N respectively, with

$$M_1, N_1 \in AB, M_2, N_2 \in BC, M_3, N_3 \in AC, \tag{3.27}$$

then

$$\begin{aligned} \measuredangle ACB &= \pi - \measuredangle AN_3N_2 - \measuredangle BN_2N_3 \\ &= \pi - \measuredangle AN_3N_1 - \measuredangle N_3 - \measuredangle BN_2N_1 - \measuredangle N_2. \end{aligned} \tag{3.28}$$

In addition, the quadrilaterals N_3NN_1A and N_2NN_1B can be inscribed in a circle, therefore

$$\measuredangle N_1NA = \measuredangle N_1N_3A \text{ and } \measuredangle N_1N_2B = \measuredangle BNN_1. \tag{3.29}$$

Furthermore,

$$\measuredangle BNA = \measuredangle N_1NA + \measuredangle BNN_1, \quad \text{and} \quad \measuredangle N_1 = \pi - \measuredangle N_2 - \measuredangle N_3, \tag{3.30}$$

Combining (3.26), (3.28), (3.29), and (3.30), we arrive at the conclusion that

$$\measuredangle N_1 = \measuredangle BNA + \measuredangle ACB. \tag{3.31}$$

It is also possible to write

$$\begin{aligned}
\measuredangle BMA &= \pi - \measuredangle MAM_1 - \measuredangle MBM_1, &(3.32)\\
\measuredangle ACB &= \pi - \measuredangle CAB - \measuredangle CBA, &(3.33)\\
\measuredangle CAB &= \measuredangle MAC + \measuredangle MAM_1, &(3.34)\\
\measuredangle CBA &= \measuredangle MBC + \measuredangle MBM_1. &(3.35)
\end{aligned}$$

Therefore, by combining (3.32)-(3.35), we obtain

$$\measuredangle BMA - \measuredangle ACB = \measuredangle MAC + \measuredangle MBC. \tag{3.36}$$

Since the quadrilaterals M_3AM_1M and MM_1BM_2 can be inscribed in a circle,

$$\measuredangle MAC = \measuredangle MM_1M_3 \text{ and } \measuredangle MBC = \measuredangle MM_1M_2. \tag{3.37}$$

However,

$$\measuredangle M_1 = \measuredangle MM_1M_3 + \measuredangle MM_1M_2 \tag{3.38}$$

so therefore, by (3.36), (3.37), and (3.38),

$$\measuredangle M_1 = \measuredangle BMA - \measuredangle ACB. \tag{3.39}$$

The identities (3.39) and (3.31) imply that $\measuredangle N_1 = \measuredangle M_1$. A similar reasoning can be used to show that $\measuredangle N_2 = \measuredangle M_2$, and that $\measuredangle N_3 = \measuredangle M_3$, proving that $\Delta N_1N_2N_3$ and $\Delta M_1M_2M_3$ are similar, with their corresponding angles' vertices located on the same sides of the triangle ABC.

Next we will prove (3.25). By Theorem 2.6,

$$\frac{|\Delta M_1M_2M_3|}{|\Delta N_1N_2N_3|} = \frac{R^2 - OM^2}{ON^2 - R^2} \tag{3.40}$$

and the last fraction is ≤ 1, as it can be seen from the fact that $OM \leq R \leq ON$ and $OM \cdot ON = R^2$. Hence, (3.25) follows.

Now we will prove the converse statement referred to in Theorem 3.1. In order for $\Delta N_1N_2N_3$ and $\Delta M_1M_2M_3$ to be similar, it is a necessary condition that $\measuredangle BNA = \measuredangle BMA - 2\measuredangle ACB$, which is equivalent to $\measuredangle BNA = \frac{\overset{\frown}{f}}{2} - \frac{\overset{\frown}{g}}{2}$. This can only occur when $\measuredangle BNA = \frac{\overset{\frown}{h}}{2}$ which implies that N is on the circle X'. A similar reasoning can be applied to show that N must be on the circles Y' and Z', proving that N is the inversion of M. This shows that the converse statement is also valid, completing the proof of Theorem 3.1. □

Remark 3. (i) From the reasoning above we see that there exist precisely six points in the interior of the circumscribed circle of $\Delta\, ABC$ such that their antipedal triangles with respect to $\Delta\, ABC$ are similar to a given reference triangle MNP. We shall call these points *the interior points of* ΔMNP *with respect to* ΔABC.

(ii) Likewise, there exist precisely six points in the exterior of the circumcircle of $\Delta\, ABC$ with the same property as above. We shall call these points *the exterior points of* ΔMNP *with respect to* ΔABC.

After this preamble, we are ready to state and prove our main result:

Theorem 3.2. *The pedal points of the six triangles, $\Delta_{\alpha\beta\gamma}, \Delta_{\alpha\gamma\beta}, \Delta_{\beta\alpha\gamma}, \Delta_{\beta\gamma\alpha}, \Delta_{\gamma\alpha\beta}, \Delta_{\gamma\beta\alpha}$, that are area-minimizers in the classes $\mathcal{I}_{\alpha\beta\gamma}$, $\mathcal{I}_{\alpha\gamma\beta}$, $\mathcal{I}_{\beta\alpha\gamma}$, $\mathcal{I}_{\beta\gamma\alpha}$, $\mathcal{I}_{\gamma\alpha\beta}$, $\mathcal{I}_{\gamma\beta\alpha}$, respectively, all lie on a circle.*

Proving Theorem 3.2 is equivalent to proving the following. Consider ABC, triangle of reference, and $\Delta\, D_0E_0F_0$ a fixed fundamental triangle (in the terminology of Definition 2.2).

Let M_i, $i = 1, 2, ..., 6$, be six points such that, for each $i \in \{1, 2, ..., 6, \}$, the pedal triangle of M_i with respect to $\Delta\, ABC$ is $\Delta\, D_iE_iF_i$ with

$$\measuredangle D_i = \measuredangle D_0, \quad \measuredangle E_i = \measuredangle E_0, \quad \measuredangle F_i = \measuredangle F_0, \tag{3.41}$$

and

$$\begin{aligned} &D_1, D_2, E_3, E_4, F_5, F_6 \in AB, \\ &D_3, D_5, E_1, E_6, F_2, F_4 \in BC, \\ &D_4, D_6, E_2, E_5, F_1, F_3 \in AC, \end{aligned} \tag{3.42}$$

(see Figure 8). Then the points M_i, $i = 1, 2, ..., 6$, lie on the same circle.

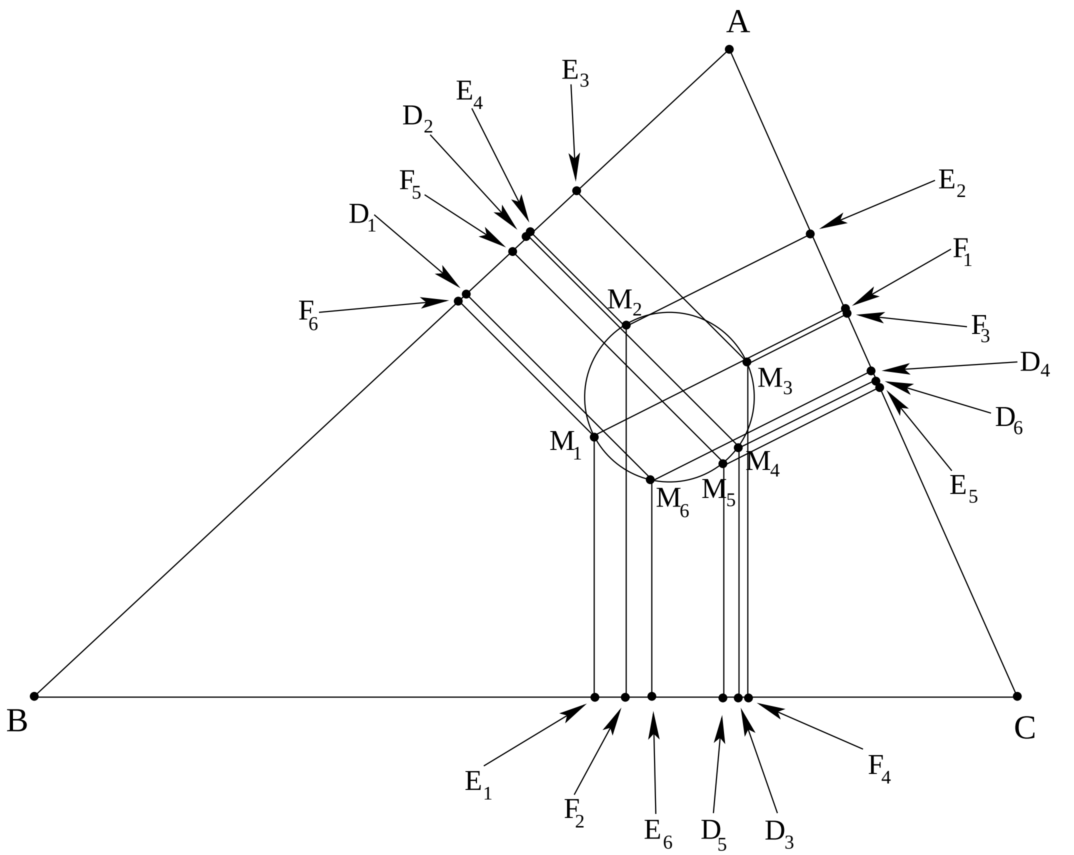

Figure 8

Proof. Since $\Delta\, D_1E_1F_1$ is the antipedal triangle of M_1, with respect to $\Delta\, ABC$, we can obtain (reasoning similarly to what was done in the proof of Theorem 3.1 to show that $\measuredangle M_1 = \measuredangle BMA - \measuredangle ACB$) that

$$\measuredangle E_1 = \measuredangle BM_1C - \measuredangle A, \quad \measuredangle D_1 = \measuredangle AMB - \measuredangle C, \quad \measuredangle F_1 = \measuredangle AM_1C - \measuredangle B. \tag{3.43}$$

Similarly we obtain $\measuredangle D_2 = \measuredangle BM_2A - \measuredangle C$. From $\measuredangle D_0 = \measuredangle D_1 = \measuredangle D_2$, it follows that $\measuredangle BM_1A = \measuredangle BM_2A$, i.e.,

$$\begin{aligned} &M_1, M_2 \text{ lie on a circle } (O_D)\text{, passing through } A \text{ and } B \\ &\text{and with the property that } \tfrac{1}{2}\overset{\frown}{AB} = \measuredangle AM_1B = \measuredangle C + \measuredangle D_0. \end{aligned} \tag{3.44}$$

Likewise (see Figure 9),

$$\begin{aligned} &M_3, M_4 \text{ lie on a circle } (O_E)\text{, passing through } A \text{ and } B \\ &\text{and with the property that } \tfrac{1}{2}\overset{\frown}{AB} = \measuredangle C + \measuredangle E_0, \end{aligned} \tag{3.45}$$

$$\begin{aligned} &M_5, M_6 \text{ lie on a circle } (O_F)\text{, passing through } A \text{ and } B \\ &\text{and with the property that } \tfrac{1}{2}\overset{\frown}{AB} = \measuredangle C + \measuredangle F_0, \end{aligned} \tag{3.46}$$

$$\begin{aligned} &M_6, M_4 \text{ lie on a circle } (O'_D)\text{, passing through } A \text{ and } C \\ &\text{and with the property that } \tfrac{1}{2}\overset{\frown}{AC} = \measuredangle B + \measuredangle D_0, \end{aligned} \tag{3.47}$$

$$\begin{aligned} &M_5, M_2 \text{ lie on a circle } (O'_E)\text{, passing through } A \text{ and } C \\ &\text{and with the property that } \tfrac{1}{2}\overset{\frown}{AC} = \measuredangle B + \measuredangle E_0, \end{aligned} \tag{3.48}$$

$$\begin{aligned} &M_1, M_3 \text{ lie on a circle } (O'_F)\text{, passing through } A \text{ and } C \\ &\text{and with the property that } \tfrac{1}{2}\overset{\frown}{AC} = \measuredangle B + \measuredangle F_0. \end{aligned} \tag{3.49}$$

The key step in the proof is to make at this stage an inversion of center A and arbitrary modulus. Since the lines AB and AC pass through A, they will remain unchanged. The points B, C become B' and C', respectively. The circles (O_D), (O_E), (O_F) will be transformed into the lines d_D, d_E and d_F respectively, which are concurrent at B'. Meanwhile, the circles (O'_D), (O'_E) and (O'_F) become the lines d'_D, d'_E and d'_F, respectively, which are concurrent at C'. The points M_1, M_2, M_3, M_4, M_5, M_6 will be transformed into M'_1, M'_2, M'_3, M'_4, M'_5 and M'_6,respectively, and $M'_1, M'_2 \in d_D$, $M'_3, M'_4 \in d_E$, $M'_5, M'_6 \in d_F$, $M'_6, M'_4 \in d'_D$, $M'_5, M'_2 \in d'_E$, $M'_1, M'_3 \in d'_F$.

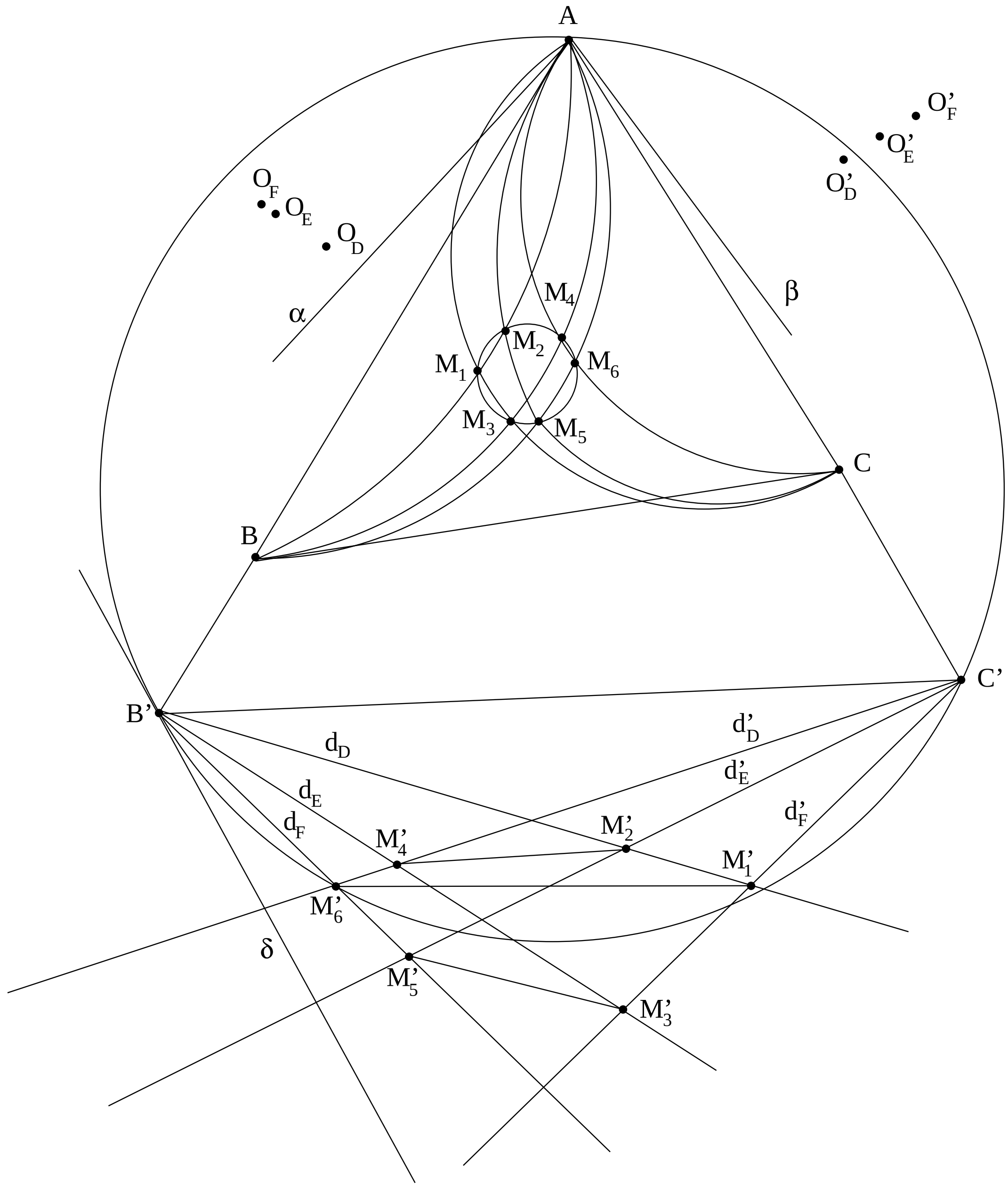

Figure 9

The goal is to show that the points M_1', M_2', M_3', M_4', M_5', M_6' lie on the same circle. We note that since under an inversion the angle between two curves is preserved, we have $\measuredangle(d_D, d_F') = \measuredangle((O_D),(O_F'))$, as the angle between two circles is defined to be the angle between the tangents at one of their common points. Therefore, $\measuredangle(d_D, d_F') = \measuredangle(\alpha A \beta) = \pi - \measuredangle(O_D A O_F')$. However,

$$\measuredangle(O_D A O_F') = \measuredangle(O_D AB) + \measuredangle A + \measuredangle(CAO_F'). \tag{3.50}$$

On the other hand $\measuredangle(O_D AB) = \frac{\pi - \measuredangle(AO_D B)}{2} = \measuredangle C + \measuredangle D_0 - \frac{\pi}{2}$ and $\measuredangle(CAO_F') = \measuredangle F_0 + \measuredangle B - \frac{\pi}{2}$. Hence,

$$\measuredangle(O_D A O_F') = (\measuredangle A + \measuredangle B + \measuredangle C) + \measuredangle D_0 + \measuredangle F_0 - \pi = \measuredangle D_0 + \measuredangle F_0, \tag{3.51}$$

and $\measuredangle(d_D, d_F') = \pi - \measuredangle(O_D A O_F') = \measuredangle E_0$.

Let $B'\delta$ be the tangent at B to the circle determined by the points A', B', C'. Then $\measuredangle(d_E, B'C') = \measuredangle(\delta B'C') - \measuredangle(\delta B', d_E)$. However, $\measuredangle(\delta B'C') = \measuredangle A$, and $\measuredangle(\delta B', d_E) = \measuredangle(C(A, B', C'), d_E) = \measuredangle(BC, (O_E)) = -\measuredangle B + (\pi - \measuredangle C - \measuredangle E_0)$, where $C(A, B, C)$ denotes the circle containing A, B, C. Therefore

$$\measuredangle(d_E, B'C') = \measuredangle E_0. \tag{3.52}$$

We have the pairs of lines (d_D, d'_D), (d_E, d'_E), (d_F, d'_F) having the same slope as the line BC, and by symmetry, we see that $M'_4M'_2 \parallel M'_6M'_1 \parallel M'_5M'_3 \parallel B'C'$. Hence, $\measuredangle(M'_4M'_2B') = \measuredangle(M'_2B'C')$ and $\measuredangle(B'M'_3C') = \pi - \measuredangle(M'_3B'C') - \measuredangle(M'_3C'B') = \pi - \measuredangle E_0 - \measuredangle F_0 = \measuredangle D_0$. As a consequence, the quadrilateral $M'_4M'_2M'_1M'_3$ is inscribable. Since $\measuredangle(B'M'_5C') = \measuredangle(B'M'_3C') = \measuredangle D_0$, it follows that the quadrilateral $M'_4M'_2M'_1M'_6$ is also inscribable. The quadrilateral $M'_4M'_2M'_1M'_6$ is an isosceles trapezoid, hence it is inscribable as well. As a result, M'_6 lies on the circumscribed circle to the quadrilateral $M'_3M'_1M'_2M'_4$. Since M'_5 lies on the circumscribed circle to the quadrilateral $M'_6M'_1M'_3M'_5$, it follows that all the points M_i, $i = 1, 2, ..., 6$, lie on the same circle. □

Remark 4. Since the interior points lie on a circle, it follows that the exterior points lie on a circle (the transformed under the inversion of center O and modulus R of the former circle).

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