Nonlinear arcsin-electrodynamics and asymptotic Reissner-Nordström black holes

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Abstract

A model of nonlinear electrodynamics with the Lagrangian density $\mathcal{L} = -(1/\beta)\arcsin(\beta F_{\mu\nu}F^{\mu\nu}/4)$ is considered. The scale invariance and dual invariance of electromagnetic fields are broken in the model. In the limit $\beta \to 0$ one comes to Maxwell's electrodynamics and the scale and dual invariance are recovered. We investigate the effect of coupling electromagnetic fields with the gravitation field. The asymptotic black hole solution is found which is similar to the Reissner-Nordström solution. We obtain corrections to Coulomb's law and the Reissner-Nordström solution in the model proposed.

1 Introduction

The attractive feature of nonlinear electrodynamics (NLE) is that the self-energy of charged particles is finite in some models and there is no singularity of the electric field at the origin of charged particles [1], [2], [3], [4], [5]. Thus, the total electromagnetic energy of charges is also finite. Therefore, one can treat the mass of the electron as a pure electromagnetic energy. NLE models may be viewed as (effective) models that take into consideration quantum corrections. It is known that one-loop quantum corrections to the Maxwell electrodynamics lead to nonlinear Heisenberg-Euler Lagrangian [6]. Born-Infeld's model [1] is NLE which smoothes the electric field singularity and may be explored for strong electromagnetic fields. NLE for weak fields leads to the Maxwell electrodynamics that can be considered as an approximation. Classical electrodynamics should be modified for strong electromagnetic fields [7] when the self-interaction of photons are important. NLE whith electric permittivity and magnetic permeability depending on fields may be considered as an effective electrodynamics [8].

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In cosmology the problems of the initial Big Bang singularity and inflation may be solved using NLE. The Einstein equation and NLE are classical theories and they can be investigated in the theory of gravity. Electromagnetic fields and gravitational fields in the early time of the universe creation are very strong, and nonlinear effects play very important role. In general relativity (GR) NLE is of interest as processes of vacuum polarizations should be taken into consideration. Instead of introducing the dark energy one can use nonlinear electromagnetic fields that influences on the evolution of the early universe near the Planck era and leads to inflation [9], [10]. Some models of NLE were considered [11]-[14] to describe accelerated expansion of the universe. The cosmological constant Λ, in the Λ-Cold Dark Matter (ΛCDM) model, drives the present cosmic acceleration. At the same time the trace anomaly of NLE can mimic the cosmological constant [15], [16]. The Einstein-Born-Infeld equations that take into consideration nonlinear effects were studied in [17]. In this paper we consider new NLE model, depending on a dimensional constant β, where electromagnetic fields coupled to the gravitational field. At the limit $\beta \to 0$ the black hole geometry approaches into Einstein-Maxwell geometry. We study the Einstein-NLE solution which gives corrections to the Reissner-Nordström (RN) black hole solution and modifies the RN geometry. In [18] the static and spherically symmetric spacetime of black hole with the Heisenberg-Euler NLE coupled to gravity was studied.

The paper is organized as follows. In Sec. 2 we consider a model of NLE with the dimensional parameter β and obtain the energy-momentum tensor. The electric permittivity, ε, and the magnetic permeability, μ, depending on the electromagnetic fields are found. We show that the scale invariance and dual invariance are broken in the model. In Sec. 3 nonlinear electromagnetic fields coupled with the gravity was studied. The corrections to Coulomb law are found. It is shown that the electric field possesses the maximum at the origin of charges. We obtain the black hole solution having the asymptotic Reissner-Nordström solution and find corrections to Reissner-Nordström solution in Sec. 4. In Sec. 5 we make the conclusion.

We use the units with $c = \hbar = 1$.

2 Nonlinear arcsin-electrodynamics

We propose NLE with the Lagrangian density

$$\mathcal{L} = -\frac{1}{\beta}\arcsin(\beta\mathcal{F}), \tag{1}$$

where β is dimensional parameter with the dimension of (length)^4 and $\beta\mathcal{F}$ is dimensionless, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength and $\mathcal{F} = (1/4)F_{\mu\nu}F^{\mu\nu}$. One can introduce the parameter $L = \beta^{1/4}$ with the dimension of the length which probably goes from the fundamental theory (string/M theory or quantum gravity). This parameter L or β defines the electric field maximum. For weak electromagnetic fields $\beta\mathcal{F} \ll 1$, using the Taylor series, we obtain from Eq. (1) the approximate expression

$$\mathcal{L} \approx -\mathcal{F} - \frac{\beta^2}{6}\mathcal{F}^3 + \mathcal{O}(\beta^4\mathcal{F}^5).$$

At $\beta\mathcal{F} \to 0$ (or $\beta \to 0$) the Lagrangian density (1) approaches to the Maxwell Lagrangian density, $\mathcal{L} \to -\mathcal{F}$. We find the symmetric energy-momentum tensor by varying the action corresponding to the Lagrangian density (1) with respect to the metric tensor $g^{\mu\nu}$ [19]

$$T^{\mu\nu} = H^{\mu\lambda}F^{\nu}{}_{\lambda} - g^{\mu\nu}\mathcal{L}, \tag{2}$$

where

$$H^{\mu\lambda} = \frac{\partial\mathcal{L}}{\partial F_{\mu\lambda}} = \frac{\partial\mathcal{L}}{\partial\mathcal{F}}F^{\mu\lambda} = -\frac{F^{\mu\lambda}}{\sqrt{1-(\beta\mathcal{F})^2}}. \tag{3}$$

The symmetric energy-momentum tensor obtained from Eqs. (2),(3) is given by

$$T^{\mu\nu} = -\frac{F^{\mu\lambda}F^{\nu}{}_{\lambda}}{\sqrt{1-(\beta\mathcal{F})^2}} - g^{\mu\nu}\mathcal{L}. \tag{4}$$

One can find the trace of the energy-momentum tensor (4),

$$\mathcal{T} \equiv T_\mu{}^\mu = \frac{4}{\beta}\arcsin(\beta\mathcal{F}) - \frac{4\mathcal{F}}{\sqrt{1-(\beta\mathcal{F})^2}}. \tag{5}$$

If $\beta \to 0$ we come to Maxwell's electrodynamics, and trace (5) becomes zero, $\mathcal{T} \to 0$. The scale invariance is broken because of non-zero trace of the energy-momentum tensor. The dilatation current is given by the expression $D_\mu = x_\nu T_\mu{}^\nu$ and the divergence becomes $\partial_\mu D^\mu = \mathcal{T}$. The electric displacement field can be obtained from the expression $\mathbf{D} = \partial\mathcal{L}/\partial\mathbf{E}$. Then from Eq. (1) one finds the electric displacement field

$$\mathbf{D} = \frac{\mathbf{E}}{\sqrt{1-(\beta\mathcal{F})^2}}. \tag{6}$$

With the help of the relation $\mathbf{D} = \varepsilon\mathbf{E}$, we obtain the electric permittivity

$$\varepsilon = \frac{1}{\sqrt{1-(\beta\mathcal{F})^2}}. \tag{7}$$

From the definition $\mathbf{H} = -\partial\mathcal{L}/\partial\mathbf{B}$ one finds the magnetic field

$$\mathbf{H} = \frac{\mathbf{B}}{\sqrt{1-(\beta\mathcal{F})^2}}. \tag{8}$$

From the equality $\mathbf{B} = \mu\mathbf{H}$ the magnetic permeability is equal to $\mu = 1/\varepsilon$. From Eqs. (6),(8) we observe that the relation $\mathbf{D}\cdot\mathbf{H} = \varepsilon^2\mathbf{E}\cdot\mathbf{B}$ holds. As a result, $\mathbf{D}\cdot\mathbf{H} \neq \mathbf{E}\cdot\mathbf{B}$, and according to [20] we make a conclusion that the dual symmetry is violated in our model. With the help of Eqs. (6),(8) the Lagrange-Euler equations can be represented in the form of the first pair of the Maxwell equations

$$\nabla\cdot\mathbf{D} = 0, \qquad \frac{\partial\mathbf{D}}{\partial t} - \nabla\times\mathbf{H} = 0. \tag{9}$$

The Bianchi identity $\partial_\mu \widetilde{F}_{\mu\nu} = 0$, where $\widetilde{F}_{\mu\nu}$ is a dual tensor, leads to the second pair of Maxwell's equations

$$\nabla\cdot\mathbf{B} = 0, \qquad \frac{\partial\mathbf{B}}{\partial t} + \nabla\times\mathbf{E} = 0. \tag{10}$$

Eqs. (6), (8), (9), (10) represent the nonlinear Maxwell equations because the electric permittivity ε and the magnetic permittivity μ depend on the electromagnetic fields $\mathbf{E}$, $\mathbf{B}$. The vacuum in this model is similar to a medium with the specific polarization characteristics.

3 Nonlinear electromagnetic fields and black holes

Let us consider NLE described by the Lagrangian density (1) coupled with the gravitation field,

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}R + \mathcal{L}\right]. \tag{11}$$

The R is the Ricci scalar and $\kappa^{-1} = M_{Pl}$ (M_{Pl} is the reduced Planck mass). From Eq. (11) we obtain the Einstein equation and equations for electromagnetic fields

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2 T_{\mu\nu}, \tag{12}$$

$$\partial_\mu\left(\frac{\sqrt{-g}F^{\mu\nu}}{\sqrt{1-(\beta\mathcal{F})^2}}\right) = 0. \tag{13}$$

To find the static charged black hole solutions to Eqs. (12),(13) we use the spherically symmetric line element in $(3+1)$-dimensional spacetime

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\phi^2). \tag{14}$$

We suppose that the vector-potential has non-zero component $A_0 = h(r)$ so that $\mathcal{F} = -[h'(r)]^2/2$, and the prime is the derivative with respect to the argument. As a result, the electric field is $E = h'(r)$ and Eq. (13) is rewritten as

$$\partial_r\left(\frac{2r^2h'(r)}{\sqrt{4-\beta^2[h'(r)]^4}}\right) = 0. \tag{15}$$

We obtain after the integration of Eq. (15) the equation

$$2r^2h'(r) = Q\sqrt{4-\beta^2[h'(r)]^4}, \tag{16}$$

where Q is the integration constant. One can introduce dimensionless values

$$y = \sqrt{\frac{\beta}{2}}h'(r), \qquad x = \frac{r}{(\beta Q^2)^{1/4}}. \tag{17}$$

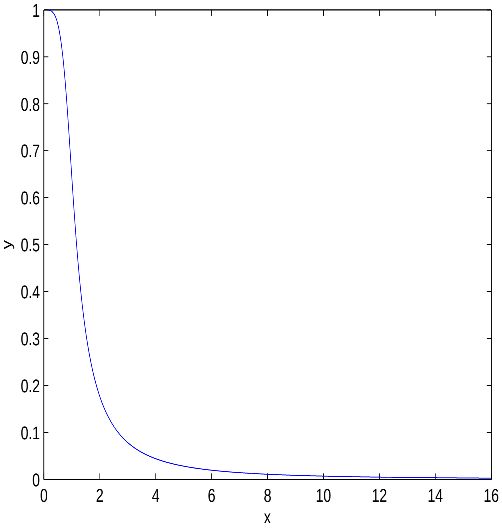

Figure 1: The function y versus x.

With the aid of Eqs. (16),(17) we find the equation as follows

$$y = \sqrt{\sqrt{x^8+1} - x^4}. \tag{18}$$

The plot of the function $y(x)$ is represented by Fig. 1. One obtains the finite value $E(0) = \sqrt{2/\beta}$ ($y = 1$) at the origin, and as a result, there is not singularity of the electric field. The Taylor series of the function $y(x)$ at $x \to \infty$ reads

$$y = \frac{1}{\sqrt{2}x^2} - \frac{1}{8\sqrt{2}x^{10}} + \mathcal{O}(x^{-13}). \tag{19}$$

The asymptotic value of the electric field at $r \to \infty$ follows from Eqs. (17),(19)

$$E = \frac{Q}{r^2} - \frac{Q^5\beta^2}{8r^{10}} + \mathcal{O}(r^{-13}). \tag{20}$$

The first term in Eq. (20) corresponds to the Coulomb law with Q being the charge. The second term in Eq. (20) represents the correction to the Coulomb law at $r \to \infty$. We see that at big distances the correction to the Coulomb law is negligible. If $\beta = 0$ we come to the Maxwell electrodynamics and the Coulomb law $E = Q/r^2$ holds. The electric potential follows from

Eqs. (17),(18)

$$A_0(r) \equiv h(r) = \frac{\sqrt{2}}{Q\beta}\int dr\sqrt{\sqrt{r^8+Q^4\beta^2}-r^4}$$

$$= \frac{\sqrt{2}}{3Q\beta}\left[-2^{7/4}Q\sqrt{\beta}\sqrt[4]{\sqrt{r^8+Q^4\beta^2}-r^4}F\left(\frac{1}{8},\frac{3}{4};\frac{9}{8};\frac{(r^4-\sqrt{r^8+Q^4\beta^2})^2}{Q^4\beta^2}\right)\right.$$

$$\left.+r\sqrt{\frac{r^8+Q^4\beta^2}{\sqrt{r^8+Q^4\beta^2}-r^4}}-\frac{r^5}{\sqrt{\sqrt{r^8+Q^4\beta^2}-r^4}}\right]+C, \qquad (21)$$

and $F(a,b;c;z)$ is the hypergeometric function (see [21]). To simplify expression (21) we can use the approximate value of the potential $A_0(r)$ at $r\to\infty$. The Taylor series of $A_0(r)$ at $r\to\infty$ (putting the integration constant to be zero, $C=0$) gives

$$A_0(r) \equiv h(r) = -\frac{Q}{r}+\frac{Q^5\beta^2}{72r^9}+\mathcal{O}(r^{-12}). \qquad (22)$$

From Eq. (22), after taking the derivative with the respect to r, we obtain the expression for the electric field, Eq. (20). Eq. (22) shows that corrections to the electric potential, due to nonlinear effects, are small at big r. Supposing $r=0$ in Eq. (21), one finds the electric potential at the origin

$$A_0(0) \equiv h(0) = -\frac{2^{9/4}\sqrt{Q}}{3\beta^{1/4}}F\left(\frac{1}{8},\frac{3}{4};\frac{9}{8};1\right) = -\frac{\sqrt{Q}}{2^{3/4}3\beta^{1/4}}B\left(\frac{1}{8},\frac{1}{4}\right), \qquad (23)$$

where $B(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the beta-function [21]. One can use the approximate value $B(1/8,1/4)\approx 11.523252$. Then the electric potential (23) becomes

$$A_0(0) \equiv h(0) \approx -\frac{2.28\sqrt{Q}}{\beta^{1/4}}. \qquad (24)$$

Thus, the electric potential, as well as the electric field, is finite at the origin of the charged object. As a result, there are not singularities of the electric field and the electric potential at the origin of charged objects. The similar situation occurs in well-known BI electrodynamics.

4 Asymptotic Reissner-Nordström black holes

We can obtain the function $f(r)$ for the spherically symmetric metric (14) from the relation [22]

$$f(r) = 1 + \frac{k_1}{r} + \frac{k_2}{r^2} + \frac{1}{r^2}\int dr\left[\int r^2 R(r) dr\right], \qquad (25)$$

with k_1, k_2 being the integration constants. To find the Ricci scalar we use the relation that follows from Einstein equation (12)

$$R = -\kappa^2 \mathcal{T}, \qquad \mathcal{T} = g^{\mu\nu} T_{\mu\nu}, \qquad (26)$$

and the trace of the energy-momentum tensor is given by Eq. (5). Then from Eqs. (5),(26) and the relation $\mathcal{F} = -(1/2)(h'(r))^2$, we find the Ricci scalar

$$R = \kappa^2 \left[\frac{4}{\beta}\arcsin\left(\beta[h'(r)]^2/2\right) - \frac{2[h'(r)]^2}{\sqrt{1-\beta^2[h'(r)]^4/4}}\right]. \qquad (27)$$

The plot of the function $R\beta/\kappa^2$ vs $r/(\beta Q^2)^{1/4}$ is given by Fig. 2. One

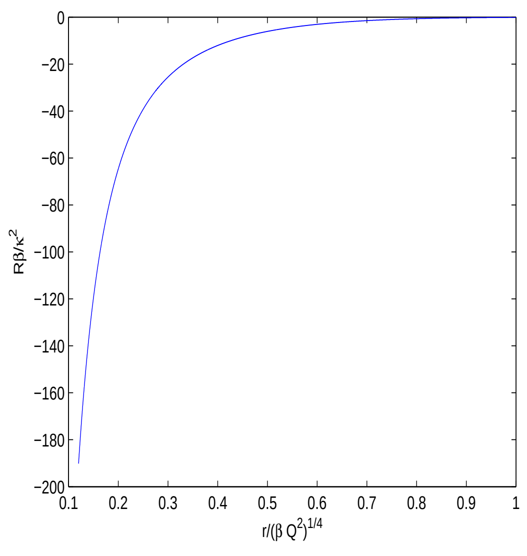

Figure 2: The function $R\beta/\kappa^2$ versus $r/(\beta Q^2)^{1/4}$.

obtains the approximate value of R using the Taylor series at $r \to \infty$ and taking $E = h'(r) << 1$,

$$R = -\kappa^2 \left(\frac{\beta^2 Q^6}{6r^{12}} + \mathcal{O}(r^{-13}) \right). \tag{28}$$

Integrating Eq. (25), with the help of Eq. (28), we find the asymptotic value of the metric function

$$f(r) = 1 + \frac{k_1}{r} + \frac{k_2}{r^2} - \frac{\kappa^2 \beta^2 Q^6}{792 r^{10}} + \mathcal{O}(r^{-11}). \tag{29}$$

The last terms in Eq. (29) represent corrections to the Reissner-Nordström solution. The constants k_1, k_2 are connected with the mass of the black hole M and the gravitation constant G, $k_1 = -2GM$ $k_2 = G^2 Q^2$. Then the function $f(r)$ at $r \to \infty$ becomes

$$f(r) = 1 - \frac{2GM}{r} + \frac{G^2 Q^2}{r^2} - \frac{\kappa^2 \beta^2 Q^6}{792 r^{10}} + \mathcal{O}(r^{-11}). \tag{30}$$

Eq. (30) shows that spacetime asymptotically (at $r \to \infty$) is the Minkowski spacetime (flat). We have Maxwell's electrodynamics at $\beta = 0$ and then solution (30) is converted to the Reissner-Nordström solution. One can see from Eq. (30) that corrections to Reissner-Nordström solution are very small at $r \to \infty$. Another model of NLE [23] also gives asymptotic Reissner-Nordström black hole solution. We notice that corrections to the Reissner-Nordström solution lead to the change of the event horizon r_+ and the Cauchy horizon r_-.

5 Conclusion

We have investigated a model of NLE with a dimensional parameter β and obtained the energy-momentum tensor and its non-zero trace. The scale invariance and the dual symmetry of the electromagnetic fields are violated because of the non-zero trace of the energy-momentum tensor. Nonlinear electromagnetic fields coupled with the gravitation field have been studied. We obtain the static spherically symmetric solutions that correspond to the charged black holes. It was demonstrated that the electric field has maximum at the origin of charged objects and there is no singularity. The electric potential also is finite at the origin. We show that the black hole solution obtained possesses some corrections to the Reissner-Nordström solution and asymptotically approaches to it. At $\beta \to 0$ NLE becomes the Maxwell electrodynamics, and we have the Reissner-Nordström solution. The Minkowskian limit follows at $r \to \infty$. The investigation of Maxwell's electromagnetic fields coupled with gravitation fields non-linearly was done in [24]. One may study non-minimal coupling nonlinear electromagnetic fields considered with the gravitation fields. The investigation of the instabilities of charged black holes and anti-evaporation effects in our model can be studied (see the investigation of these effects in the Maxwell-$F(R)$ theory [25]).

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