FROM EINSTEIN TO STOBBE-SAUTER PHOTOEFFECT AND RELATED PROBLEMS

Miroslav Pardy
Department of Physical Electronics
Masaryk University,
Kotlářská 2, 611 37 Brno, Czech Republic
e-mail:pamir@physics.muni.cz

October 12, 2017

Abstract

We consider photoelectric effect including phonon emission and the initial dressed photon. We include the polychromatic form of the photoeffect, and the photoeffect in the two-dimensional electron gas in magnetic field. We consider the nonrelativistic and relativistic quantum theory of ionization as the extension of the old theory of photoeffect. As the related problem, we calculate the H-atom in the black body sea, which is equivalent to the Gibbons-Hawking thermal bath. We include the problem of the velocity of sound in the relic photon sea, thermal Casimir effect, dielectric crystal immersed in the black-body sea and the Cherenkov radiation in the two-dimensional dielectric medium.

1 Introduction

The photoelectric effect is a quantum electromagnetic phenomenon in which electrons are emitted from matter after the absorption of energy from electromagnetic radiation. Frequency of radiation must be above a threshold frequency, which is specific to the type of surface and material. No electrons are emitted for radiation with a frequency below that of the threshold. These emitted electrons are also known as photoelectrons in this context. The photoelectric effect was theoretically explained by Einstein in his paper in 1905 (Einstein, 1905; 1965) and the term "light quanta" called "photons" was introduced by chemist G. N. Lewis, in 1926. Einstein writes (Einstein, 1905; 1965): *In accordance with the assumption to be considered here, the energy of light ray spreading out from point source is not continuously distributed over an increasing space but consists of a finite number of energy quanta which are localized at points in space, which move without dividing, and which can only be produced and absorbed as complete units.*

It is well known statement that the free electron in vacuum cannot absorb photon. It follows from the special theory of relativity. Namely: if p_1, p_2 are the initial and final 4-momenta of electron with rest mass m and k is the 4-momentum of photon, then after absorption of photon by electron we write $k + p_1 = p_2$, which gives when squared $k^2 + 2kp_1 + p_1^2 = p_2^2$. Then, with $p_1^2 = p_2^2 = -m^2$ and $k^2 = 0$, we get for the rest electron with $\mathbf{p}_1 = 0$, the elementary relation $m\omega = 0$, Q.E.D..

The linear dependence on the frequency was experimentally determined in 1915, when Robert Andrews Millikan showed that Einstein formula

$$\hbar\omega = \frac{mv^2}{2} + W \tag{1}$$

was correct. Here $\hbar\omega$ is the energy of the impinging photon, v is electron velocity measured by the magnetic spectrometer and W is the work function of concrete material. The work function for Aluminium is 4.3 eV, for Beryllium 5.0 eV, for Lead 4.3 eV, for Iron 4.5 eV, and so on (Rohlf, 1994). More information on the work function is possible to find in the book by Lide (Lide, 2008). The work function concerns the surface photoelectric effect, where the photon is absorbed by an electron in a band. The theoretical determination of the work function is the problem of the solid state physics. On the other hand, there is the so called atomic photoeffect (Amusia, 1987), where the ionization energy plays the role of the work function. The system of the ionization energies is involved in the tables of the solid state physics. The work function is the one of the prestige problem of the contemporary experimental and theoretical crystal physics.

In case of the volume photoeffect, the ionization work function is defined in many textbooks on quantum mechanics. Or,

$$W = \int_{x_1}^{x_2} \left(\frac{dE}{dx}\right) dx \tag{2}$$

where E is the energy loss of moving electron.

The formula (1) is the law of conservation of energy. The classical analogue of the equation (1) is the motion of the Robins ballistic pendulum in the resistive medium.

The Einstein ballistic principle is not valid inside of the blackbody. The Brownian motion of electrons in this cavity is caused by the repeating Compton process $\gamma + e \to \gamma + e$ and not by the ballistic collisions. The diffusion constant for electrons must be calculated from the Compton process and not from the ballistic process. The same is valid for electrons immersed into the cosmic relic photon sea.

The idea of the existence of the Compton effect is also involved in the Einstein article. He writes (Einstein, 1905; 1965): *The possibility should not be excluded, however, that electrons might receive their energy only in part from the light quantum.* However, Einstein was not sure, a priori, that his idea of such process is realistic. Only Compton proved the reality of the Einstein statement.

At energies $\hbar\omega < W$, the photoeffect is not realized. However, the photo-conductivity is the real process. The photoeffect is realized only in medium and with low energy photons, but with energies $\hbar\omega > W$, which gives the Compton effect negligible. Compton effect can be realized with electrons in medium and also with electrons in vacuum. For $\hbar\omega \gg W$ the photoeffect is negligible in comparison with the Compton effect. At the same time it is necessary to say that the Feynman diagram of the Compton effect cannot be reduced to the Feynman diagram for photoeffect. In case of the high energy gamma rays, it is possible to consider the process called photoproduction of elementary particles on protons in LHC, or photo-nuclear reactions in nuclear physics (Levinger, 1960). Such processes are energetically far from the photoelectric effect in solid state physics.

Eq. (1) represents so called one-photon photoelectric effect, which is valid for very weak electromagnetic waves. At present time of the laser physics, where the strong electromagnetic intensity is possible, we know that so called multiphoton photoelectric effect is possible. Then, instead of equation (1) we can write

$$\hbar\omega_1 + \hbar\omega_2 + ...\hbar\omega_n = \frac{mv^2}{2} + W. \tag{3}$$

The time lag between the incidence of radiation and the emission of a photoelectron is very small, less than 10^{-9} seconds.

As na analogue of the equation (3), the multiphoton Compton effect is also possible: $\gamma_1 + \gamma_2 + ...\gamma_n + e \to \gamma + e$ and two-electron, three-electron,... n-electron photoelectric effect is also possible (Amusia, 1987). To our knowledge the Compton process with the entangled photons was still not discovered and elaborated. On the other hand, there is the deep inelastic Compton effect in the high energy particle physics.

In *the second part* of the chapter we consider elementary explanation of the photoeffect ivolving the emission of phonon.

In the *third section* we consider the nonrelativistic quantum field theory of photoeffect in the form of ionization of atom involving the emission of phonon.

In the *4-th part*, we discuss the relativistic quantum field theory (QFT) of photoeffect in the form of ionization of atom involving the emission of phonon.

In the *5-th part* of the chapter we consider the polychromatic photoeffect to get the generalized Einstein formula (Pardy, 2009a).

The *6-th part* deals with photoelectric effect in the two-dimensional system in homogeneous magnetic field (Pardy, 2010).

The generalization of the photoeffect to the situation with the dressed photon is expressed in the *7-th part* of the chapter.

The *8-th part* deals with the H-atom immersed in the black-body photon sea. The situation is equivalent to the H-atom in the Gibbons-Hawking thermal bath and it is expected the important astrophysical meaning (Pardy, 2016a).

The *9-th part* consider the dielectric crystal immersed in the black-body which is equivalent to the influence of the index of refraction on the spectral formula of the black-body (Pardy, 2015a).

The thermal physics problem is also the situation of the Casimir effect at temperature T. The *10-th part* of the chapter deals with such situation (Pardy, 2016b).

The *11-th part* of the chapter is devoted to the Cherenkov radiation in the two-dimensional dielectric medium (Pardy, 2015b).

The *12-th part* of the chapter concerns the calculation of the velocity of sound in the relic photon sea which is the relic astrophysical black-body (Pardy, 2013a, 2013b).

The *13-th part* of the chapter is conclusion.

2 The photoelectric effect with the emission of phonon

A phonon is a collective excitation in a periodic, elastic arrangement of atoms, or, molecules in condensed matter, often designated a quasiparticle. It is an quantum mechanical excited state of the modes of vibrations of elastic structures of interacting particles. They play a major role in thermal conductivity and electrical conductivity. The concept of phonons was introduced in 1932 by Russian physicist Igor Tamm. The long-wavelength phonons give rise to sound. The higher-frequency phonons are responsible for the majority of the thermal capacity of solids.

Phonons have particle-like properties forming the wave particle duality known from quantum mechanics.

Acoustic phonons are coherent movements of atoms of the lattice out of their equilibrium positions similarly to the acoustic waves. They exhibit a linear relationship between frequency and phonon wave vector for long wavelengths. Optical phonons are out-of-phase movements of the atoms in the lattice, one atom moving to the left, and its neighbor to the right.

By analogy to photons and matter waves, phonon has wave vector k and momentum k, however, k is not actually a physical momentum; it is called pseudomomentum, because k is only determined up to addition of constant vectors (the reciprocal lattice vectors and integer multiples thereof).

A phonon with wavenumber k is thus equivalent to an infinite family of phonons with wavenumbers $k \pm 2\pi/a, k \pm 4\pi/a$, and so on, with a being the lattice constant.

The thermodynamic properties of a solid are directly related to phonons. The phonon density of states determines the heat capacity of a crystal. Phonons generated by the temperature of the lattice are called thermal phonons.

The behavior of thermal phonons is similar to the photon gas in a cavity, wherein photons may be emitted or absorbed by the cavity walls. Einstein has considered such model to obtain the heat capacity and Debye performed the brilliant generalization of the Einstein model.

The impossibility of photon absorption by free electron can be demonstrated using the relativistic equations

$$\hbar\omega = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \tag{1}$$

and

$$\frac{\hbar\omega}{c} = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{2}$$

where the second equation (2) is the expression of the conservation of momentum of the system pf particles photon and electron. After division of eq. (1) by eq. (2), ((1)/(2)), we get after elementary modification $1 = c/v$, which is logical contradiction.

Now, let us consider the situation, where electron is located in some medium where the Einstein work function is the necessary physical reality and the emission of phonon of the energy $\hbar\Omega$ is also the physical reality. Then instead of equations (1) and (2) we write

$$\hbar(\omega - \Omega) - W = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{3}$$

and

$$\frac{\hbar\omega}{c} = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} + P = \frac{mv + P\sqrt{1 - \frac{v^2}{c^2}}}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{4}$$

where we have introduced P as the momentum of phonon. After division of eq. (3) by eq. (4), ((3)/(4)), we get after elementary modification

$$\frac{\hbar(\omega - \Omega) - W}{\omega} = \frac{mc}{mv + P\sqrt{1 - \frac{v^2}{v^2}}}, \tag{5}$$

where there is no contradiction.

3 The QED photoelectric effect with phonon emission

The main idea of the quantum mechanical description of the photoeffect is the process of atom ionization. In case with the no phonon ejection it must be described by the appropriate S-matrix element involving the interaction of atom with the impinging photon with the simultaneous generation of the electron, the motion of which can be described approximately by the plane wave

$$\psi_{\mathbf{q}} = \frac{1}{\sqrt{V}} e^{i\mathbf{q}\cdot\mathbf{x}}, \quad \mathbf{q} = \frac{\mathbf{p}}{\hbar}, \tag{1}$$

where $\mathbf{p}$ is the momentum of the ejected electron.

The standard approach consists in the definition of the cross-section by the quantum mechanical equation (Berestetzky et al., 1989):

$$d\sigma = \frac{2\pi}{\hbar}|V_{fi}|\delta(-I + \hbar\omega - \varepsilon)\frac{d^3p}{(2\pi)^3}, \tag{2}$$

where I is the ionization energy of an atom and $\varepsilon = E_f$ is the the final energy of the emitted electron, $|V_{fi}|$ is the matrix element of the transition of electron from the initial bound state to the final state. The matrix element follows from the perturbation theory and it involves the first order term of the interaction between electron and photon.

In case that the electro-process is accompanied by the phonon emission with the energy $E = \hbar\Omega$, the last formula is presented with very small modification, leading however to the interesting experimental result.

$$d\sigma = \frac{2\pi}{\hbar}|V_{fi}|\delta(-I + \hbar\omega - \hbar\Omega - \varepsilon)\frac{d^3p}{(2\pi)^3}. \tag{3}$$

We suppose in a sufficient distance from atom the wave function is of the form of the plane wave (1) which is the classical atomic situation discussed in monograph (Davydov, 1976). However, if the photon energy only just exceeds the ionization energy I of atom, then we cannot used the plane wave approximation but the wave function of the continuous spectrum.

The probability of the emission of electron by the electromagnetic wave is of the well-known form (Berestetzky et al., 1989) (we use nomenclature with $\hbar = 1$):

$$V_{fi} = -e\mathbf{A}\mathbf{j} = -e\sqrt{4\pi}\frac{1}{\sqrt{2\omega}}M_{fi}, \tag{4}$$

where

$$M_{fi} = \int \psi'^*(\boldsymbol{\alpha}\mathbf{e})e^{i\mathbf{k}\cdot\mathbf{r}}\psi d^3x \tag{5}$$

where $\psi = \psi_i, \psi' = \psi_f$ is the initial and final wave function of electron (Berestetzky et al. 1989).

Using (Berestetzky et al., 1989)

$$d^3p = \mathbf{p}^2 d|\mathbf{p}|do = \varepsilon|\mathbf{p}|d\varepsilon do, \tag{6}$$

we get after integration of the δ-function over ε

$$d\sigma = e^2\frac{\varepsilon|\mathbf{p}|}{2\pi\omega}|M_{fi}|^2. \tag{7}$$

Let us consider the case with $I \ll \omega \ll m$. It follows from $\omega \ll m$ that the velocity of electron is very small and it means that matrices $\boldsymbol{\alpha}_k$ can be replaced by the operators (Berestetzky et al., 1989 § 45)

$$\boldsymbol{\alpha}_k \to -i\nabla_k/m. \tag{8}$$

At the same time we use the dipole approximation with $\exp(i\mathbf{kr}) \approx 1$. Then we get

$$d\sigma = e^2\frac{\varepsilon|\mathbf{p}|}{2\pi\omega}|\mathbf{e}\mathbf{v}_{fi}|^2 do, \tag{9}$$

where,

$$\mathbf{v}_{fi} = -\frac{i}{m}\int \psi'^*\nabla\psi d^3x. \tag{10}$$

Let us consider the photoeffect from the basic level of atom, then $\psi = \psi_i$, or

$$\psi = \frac{(Ze^2m)^{3/2}}{\sqrt{\pi}}e^{-Ze^2mr}. \tag{11}$$

(In the standard units $me^2 \to 1/a_0$ with $a_0 = \hbar^2/me^2$, a_0 being the Bohr radius). Function ψ' is taken in such a way that its asymptotic form is the exponential form (1) and together with this form it involves the convergent spherical wave $\psi_{\mathbf{p}}^-$. According to (Landau et al., 1991, § 36) we write

$$\psi_{\mathbf{p}}^- = \frac{1}{p}\sum_{l=0}^{l=\infty} i^l(2l+1)a^{-i\delta_l}R_{pl}(r)P_l(\mathbf{n}\mathbf{n}_1), \tag{12}$$

where $\mathbf{n} = \mathbf{p}/p, \mathbf{n_l} = \mathbf{r}/r, p = |\mathbf{p}|$, and ψ' describes the transition from the s-state to the p-state according to the selection rule (the dipole case), which means that it is possible to put $l = 1$ (Landau et al., 1991, § 36).

Ignoring the nonsubstantional coefficients, we write

$$\psi' = \frac{3}{2p}(\mathbf{n}\mathbf{n}_1)R_{p1}(r). \tag{13}$$

We get with function from (11) and (13) the following expression

$$\mathbf{e}\mathbf{v}_{fi} = \frac{3(Ze^2m)^{5/2}}{2\sqrt{\pi}mp}\int\int(\mathbf{n}\mathbf{n}_1)(\mathbf{n}_1\mathbf{e})e^{-Ze^2mr}R_{p1}(r)do_1r^2dr =$$

$$\frac{\sqrt{2\pi}(Ze^2m)^{5/2}}{pm}(\mathbf{ne})\int_0^\infty r^2e^{-Ze^2mr}R_{p1}(r)dr. \tag{14}$$

We get with (Landau et al., 1991, § 36, eq. 36.18) and (Landau et al., 1991, § 36, eq. 36.24) for R_{p1}:

$$R_{p1} = \frac{\sqrt{8\pi 2e^2m}}{3}\sqrt{\frac{1+\nu^2}{\nu(1-e^{-2\pi\nu})}}pr.e^{-ipr}F(2+i\nu, 4, 2ipr) \tag{15}$$

with

$$\nu = \frac{2e^2m}{p} = \frac{Ze^2}{\hbar v}. \tag{16}$$

Now, it is necessary to calculate the integral in (14). To realize the goal, we use the following identity:

$$\int_0^\infty e^{-\lambda z}z^{\gamma-1}F(\alpha,\gamma,kz) = \Gamma(\gamma)\lambda^{\alpha-\gamma}(\lambda-k)^{-\alpha}. \tag{17}$$

Using the elementary relation

$$\left(\frac{\nu+i}{\nu-i}\right)^{i\nu} = e^{-2\nu\arctan\nu} \tag{18}$$

we get

$$\mathbf{e}\mathbf{v}_{ij} = \frac{2^{7/2}\pi\nu^3(\mathbf{ne})}{\sqrt{p}m(1+\nu^2)^{3/2}}\frac{e^{-2\nu\arctan\nu}}{\sqrt{1-e^{-2\pi\nu}}}. \tag{19}$$

The δ-function function involves the conservation law in the form

$$\omega = \frac{p^2}{2m} + I = \frac{p^2}{2m}(1+\nu^2). \tag{20}$$

Using the last equation, we get

$$d\sigma = 2^7\pi\alpha a^2\left(\frac{I}{\hbar\omega}\right)^4(\mathbf{ne})^2\frac{e^{-4\nu\arctan\nu}}{1-e^{-2\pi\nu}}do, \tag{21}$$

where

$$a = \frac{\hbar^2}{mZe^2} = a_0/Z. \tag{22}$$

In case of the nonpolarized photon, $d\sigma$ must be averaged in $\mathbf{e}$, which leads to transition (Berestetzky et al., 1989; § 45, eq. 45. 4b):

$$(\mathbf{ne})^2 = \frac{1}{2}(\mathbf{n}_0 \times \mathbf{e})^2; \quad \mathbf{n}_0 = \mathbf{k}/k. \tag{23}$$

After integration over all angles in $d\sigma$, we get the Stobbe formula (Stobbe, 1930)

$$\sigma = (2^9/3)\pi^2\alpha a^2 \left(\frac{I}{\hbar\omega}\right)^4 \frac{e^{-4\nu\arctan\nu}}{1 - e^{-2\pi\nu}}. \tag{24}$$

In case $\hbar\omega \gg I$ and at the same time $\hbar\omega \ll mc^2$ we get

$$\sigma = (2^8/3)\pi\alpha a_0^2 Z^5 \left(\frac{I_0}{\hbar\omega}\right)^{7/2}; \quad I_0 = \frac{e^4 m}{2\hbar^2}. \tag{25}$$

Now, let as consider the QED photoelectric effect with phonon emission with the the conservation law

$$\hbar\omega = \varepsilon + I + \hbar\Omega, \tag{26}$$

which can be physically interpreted in such a way as the photoelectric effect with the initial energy $\hbar\omega - \hbar\Omega$. It mathematically means that the final formulas for the photoeffect (24) and (25) must be modified by the relation $\omega \to \omega - \Omega$, or,

$$\sigma = (2^9/3)\pi^2\alpha a^2 \left(\frac{I}{\hbar\omega - \hbar\Omega}\right)^4 \frac{e^{-4\nu\arctan\nu}}{1 - e^{-2\pi\nu}}. \tag{27}$$

In case $\hbar\omega \gg I$ and at the same time $\hbar\omega \ll mc^2$ we get

$$\sigma = (2^8/3)\pi\alpha a_0^2 Z^5 \left(\frac{I_0}{\hbar\omega - \hbar\Omega}\right)^{7/2}; \quad I_0 = \frac{e^4 m}{2\hbar^2}. \tag{28}$$

The last formula can be experimentally verified in the analogue with the Einstein formula.

4 The relativistic QED photoelectric effect with phonon emission

Let us consider the case with

$$\omega \gg I. \tag{1}$$

In this case $\varepsilon = \omega - I \gg I$ and then the influence of Coulomb field of nucleus on the wave function of the photoelectron ψ' can be determined by the theory of perturbation. So we write (Berestetzky et al., 1989):

$$\psi' = \frac{1}{\sqrt{2\varepsilon}}(u' e^{i\mathbf{pr}} + \psi^{(1)}). \tag{2}$$

The relativistic motion of the electron is involved in the plane wave of electron in formula (2).

The function ψ is take according to § 39 in the form:

$$\psi = \left(1 - \frac{i}{2m}\gamma^0 \boldsymbol{\gamma}\nabla\right)\frac{u}{\sqrt{2m}}\psi_{nonrel}, \tag{3}$$

where ψ_{nonrel} is the nonrelativistic function of the bound state (11) sect. 3 , and u is the bispinor amplitude of the rest electron with the normalization $\bar{u}u = 2m$.

Now, let us insert functions ψ, ψ' into the matrix element

$$M_{fi} = \int \psi'^*(\boldsymbol{\alpha}\mathbf{e})e^{i\mathbf{k}\cdot\mathbf{r}}\psi dx^3. \tag{4}$$

Then, we get

$$M_{fi} = \frac{1}{2\sqrt{m\varepsilon}}\times$$

$$\int \left\{\bar{u}'(\boldsymbol{\gamma}\mathbf{e})\left[\left(1 - \frac{i}{2m}\gamma^0\boldsymbol{\gamma}\nabla\right)u\psi_{nonrel}\right]e^{-i(\mathbf{p}-\mathbf{k})\mathbf{r}} + \bar{\psi}^{(1)}(\boldsymbol{\gamma}\mathbf{e})e^{i\mathbf{kr}}u\psi_{nonrel}\right\}d^3x. \tag{5}$$

Now, we approximate the wave function in [..] by constant as follows (Berestetzky et al., 1989):

$$\psi_{nonrel} = \frac{(Ze^2m)^{3/2}}{\sqrt{\pi}}. \tag{6}$$

Then, after integration by per partes of the first term in eq. (5) in order to get the exponential term, we get

$$M_{fi} = \frac{(Ze^2m)^{3/2}}{2\sqrt{\pi m\varepsilon}}\left\{\bar{u}'(\boldsymbol{\gamma}\mathbf{e})\left[1 + \frac{1}{2m}\gamma^0\boldsymbol{\gamma}(\mathbf{p}-\mathbf{k})\right]\left(e^{-Ze^2mr}\right)_{\mathbf{p}-\mathbf{k}} + \bar{\psi}^{(1)}_{-\mathbf{k}}(\boldsymbol{\gamma}\mathbf{e})u\right\}, \tag{7}$$

where the vector component is as follows in approximation in term Ze^2:

$$\left(e^{-Ze^2mr}\right)_{\mathbf{p}-\mathbf{k}} = \frac{8\pi Ze^2m}{(\mathbf{p}-\mathbf{k})^4}. \tag{8}$$

After insertion of ψ' from eq. (2) into the Dirac equation

$$[\gamma(p - eA) - m]\psi = 0 \tag{9}$$

we get the following equation for $\psi^{(1)}$:

$$\left(\gamma^0\varepsilon + i\gamma^0\boldsymbol{\gamma}\nabla - m\right)\psi^{(1)} = e(\gamma^\mu A_\mu)u'e^{i\mathbf{pr}} = -\frac{(Ze^2}{r}\gamma^0u'e^{i\mathbf{pr}}. \tag{10}$$

After application of the operator

$$\left(\gamma^0\varepsilon + i\gamma^0\boldsymbol{\gamma}\nabla + m\right) \tag{11}$$

to the last equation, we get

$$(\Delta+\mathbf{p}^2)\psi^{(1)}_{\mathbf{k}} = -Ze^2\left(\gamma^0\varepsilon+i\boldsymbol{\gamma}\nabla+m\right)(\gamma^0u')\frac{1}{r}e^{i\mathbf{pr}}. \tag{12}$$

Now, let us multiply the last equation by $e^{-i\mathbf{kr}}$ and perform the integration in d^3x. We perform the integration per partes in terms with Δ and ∇. We get:

$$(\mathbf{p}^2+\mathbf{k}^2)\psi^{(1)}_{\mathbf{k}} = -Ze^2\left(\gamma^0\varepsilon-\boldsymbol{\gamma}\mathbf{k}+m\right)(\gamma^0u')\left(\frac{1}{r}\right)_{\mathbf{k}-\mathbf{p}} =$$

$$-Ze^2\left(2\gamma^0\varepsilon-\boldsymbol{\gamma}(\mathbf{k}-\mathbf{p})+m\right)(\gamma^0u')\left(\frac{4\pi}{(\mathbf{k}-\mathbf{p})^2}\right). \tag{13}$$

We used the following equations

$$\left(\gamma^0\varepsilon-\mathbf{p}\boldsymbol{\gamma}-m\right)u'=0;\quad \left(\gamma^0\varepsilon+\mathbf{p}\boldsymbol{\gamma}-m\right)\gamma^0u'=0 \tag{14}$$

in the last line of eq. (13)
So, we get:

$$\bar\psi^{(1)}_{-\mathbf{k}} = \psi^{(1)*}_{\mathbf{k}}\gamma^0 = 4\pi Ze^2\bar u'\frac{2\gamma^0\varepsilon+\boldsymbol{\gamma}(\mathbf{k}-\mathbf{p})}{(\mathbf{k}^2-\mathbf{p}^2)(\mathbf{k}-\mathbf{p})^2}\gamma^0. \tag{15}$$

After insertion of eqs. (8) and (15) into matrix element (7), we get

$$M_{fi} = \frac{4\pi^{1/2}(Ze^2m)^{5/2}}{(\varepsilon m)^{1/2}(\mathbf{k}-\mathbf{p})^2}\bar u'Au, \tag{16}$$

where

$$A = a(\boldsymbol{\gamma}\mathbf{e})+(\boldsymbol{\gamma}\mathbf{e})\gamma^0(\boldsymbol{\gamma}\mathbf{b})+(\boldsymbol{\gamma}\mathbf{c})\gamma^0\boldsymbol{\gamma}\mathbf{e} \tag{17}$$

with

$$a=\frac{1}{(\mathbf{k}-\mathbf{p})^2}+\frac{\varepsilon}{m}\frac{1}{(\mathbf{k}^2-\mathbf{p}^2)},\quad \mathbf{b}=-\frac{\mathbf{k}-\mathbf{p}}{2m(\mathbf{k}-\mathbf{p})^2},\quad \mathbf{c}=\frac{\mathbf{k}-\mathbf{p}}{2m(\mathbf{k}^2-\mathbf{p}^2)}. \tag{18}$$

Now, the cross-section is of the form:

$$d\sigma = \frac{8e^2(Ze^2m)^5|\mathbf{p}|}{\omega m(\mathbf{k}-\mathbf{p})^4}(\bar u'Au)(\bar u\bar A u')do, \tag{19}$$

where $\bar A=\gamma^0A^+\gamma^0$. The derived cross-section must be summed through the final direction of spins and averaged through the final spin directions. Such operations can be easily performed using the polarization matrices of the initial and final states as follows:

$$\varrho=\frac{m}{2}(\gamma^0+1),\quad \varrho'=\frac{m}{2}(\gamma^0\varepsilon-\boldsymbol{\gamma}\mathbf{p}+m). \tag{20}$$

Let us remark that in the initial state is $\mathbf{p}=0, \varepsilon=m$. Using eq. (20), we get the cross-section is the form:

$$d\sigma = \frac{16e^2(Ze^2m)^5|\mathbf{p}|}{\omega m(\mathbf{k}-\mathbf{p})^4}\mathrm{Sp}(\varrho'A\varrho\bar A)do, \tag{21}$$

The spur of the mathematical object is according to Berestetzky et al. (1989) as follows:

$$\mathrm{Sp}(\varrho' A \varrho \bar{A}) =$$

$$\frac{m}{\varepsilon + m}[a\mathbf{p} - (\mathbf{b} - \mathbf{c})(\varepsilon + m)]^2 + 4m(\mathbf{be})[(\varepsilon + m)a(\mathbf{ce}) + a(\mathbf{pe})], \tag{22}$$

where vector $\mathbf{e}$ is real for the linear polarization.

Now, let us introduce the polar angle φ, azimut angle θ of the direction $\mathbf{p}$ with regard to $\mathbf{k}$ as the z-axis is the plane $\mathbf{k}, \mathbf{e}$ forming the xz plain (which means that $\mathbf{pe}$) $|\mathbf{p}| = \cos\varphi \sin\theta$. For $\omega \gg I$ there is the conservation energy in the form $\varepsilon - m = \omega$.

It may be easy to see that

$$(\mathbf{k}^2 - \mathbf{p}^2) = -2m(\varepsilon - m), \quad (\mathbf{k} - \mathbf{p})^2 = 2\varepsilon(\varepsilon - m)(1 - v\cos\theta), \tag{23}$$

where $\mathbf{v}\mathbf{p}/\varepsilon$ is the velocity of photon. After some mathematical operation we get the final form of the differential cross-section:

$$d\sigma = Z^5\alpha^4 r_e^2 \frac{v^3(1 - v^2)^3 \sin^2\theta}{(1 - \sqrt{1 - v^2})^5 (1 - v\cos\theta)^4} \quad \times$$

$$\left\{ \frac{(1 - \sqrt{1 - v^2})^2 (1 - v\cos\theta)}{2(1 - v^2)^{3/2}} \quad + \right.$$

$$\left. \left[2 - \frac{(1 - \sqrt{1 - v^2})(1 - v\cos\theta)}{(1 - v^2)} \right] \cos^2\varphi \right\} do, \tag{24}$$

where $r_e = e^2/m$.

It is $\varepsilon \gg m$ for the ultrarelativistic situation and the photoeffect has the sharp maximum for small angles $\theta \sim (1 - v^2)^{1/2}$, which means that electrons are emitted maximally in the direction of motion of photon.

We have in the vicinity of maximum:

$$1 - v\cos\theta \approx \frac{1}{2}[(1 - v^2) + \theta^2] \tag{25}$$

Then, the main terms in eq. (24) gives

$$d\sigma \approx 4Z^5\alpha^4 r_e^2 \frac{(1 - v^2)^{3/2} + \theta^3}{(1 - v^2 + \theta^2)^3} d\theta d\varphi \tag{26}$$

After elementary but long integration of eq. (26) we get the total differential cross-section of the photoelectric effect (Sauter, 1931; Berestetzky et al., 1989):

$$d\sigma = 4\pi Z^5\alpha^4 r_e^2 \frac{(\gamma^2 - 1)^{3/2}}{(\gamma - 1)^5} \quad \times$$

$$\left\{ \frac{4}{3} + \frac{\gamma(\gamma - 2)}{\gamma + 1} \left(1 - \frac{1}{2\gamma\sqrt{\gamma^2 - 1}} \ln \frac{\gamma + (\gamma^2 - 1)^{1/2}}{\gamma - (\gamma^2 - 1)^{1/2}} \right) \right\}, \tag{27}$$

where we introduced the Lorentz factor

$$\gamma = (1 - v^2)^{-1/2} = \frac{\varepsilon}{m} \approx \frac{m + \omega}{m}. \tag{28}$$

In case of the ultrarelativistic situation, we get the most simple expression

$$d\sigma = 2\pi Z^5 \alpha^4 r_e^2 / \gamma. \tag{29}$$

In case $I \ll \omega \ll m$, we have in the limiting case for small $\gamma - 1$, the known result (25), in section 3.

5 The polychromatic photoelectric effect

The physical meaning of the Einstein equation (1) (sect. 1) is in the interaction of the monochromatic photon beam with energy $\hbar\omega$ with an electron in matter. The possible generalization of the Einstein equation is, to consider the situation where the metal film absorbs the photons with the Planckian energy distribution of photons of the blackbody:

$$\varrho(\omega) = \frac{\omega^2}{\pi^2 c^3} \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1}, \tag{1}$$

or, the synchrotron radiation with the photon density (Jackson, 1999) (in the asymptotic limit case)

$$P(\omega) = \frac{I}{\hbar\omega_c} \frac{9\sqrt{3}}{8\pi} \int_y^\infty K_{5/3} x) dx; \qquad y = \frac{\omega}{\omega_c};\ \omega_c = \frac{3}{2}\left(\frac{E}{mc^2}\right)^3 \frac{c}{R} \tag{2}$$

with

$$I = \frac{4\pi^2 e^2 \gamma^4}{3R};\ \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \tag{3}$$

where R is the radius of the curvature, v is the relativistic velocity of an electron moving along curved trajectory and $K_{5/3}$ is the modified McDonald function of the index 5/3.

In the first case with the blackbody situation, we multiply the Einstein original equation by the density of photons

$$n(\omega) = \frac{\omega^2}{\pi^2 c^3} \frac{1}{e^{\frac{\hbar\omega}{kT}} - 1} \tag{4}$$

and integrate from the threshold frequency $\omega_0 = W/\hbar$ to infinity to get the polychromatic photoelectric equation:

$$\int_{\omega_0}^\infty n(\omega)\hbar\omega d\omega = \int_{\omega_0}^\infty n(\omega) d\omega \frac{mv^2}{2} + W \int_{\omega_0}^\infty n(\omega) d\omega. \tag{5}$$

The last equation is the generalization of the original Einstein equation from 1905 to the situation that matter is irradiated by the photons from the blackbody cavity.

In case that the matter is irradiated by the laser field with the known spectral distribution, the the symbol $n(\omega)$ in the last equation is of the physical meaning of the spectral distribution of photons in the laser beam.

Function

$$\int_{\omega_0}^{\infty} n(\omega)d\omega\frac{mv^2}{2} = E_{kin} \tag{6}$$

has the physical meaning of the total energy of the emitted electrons of different velocities during the photoeffect and it can be determined by the adequate experimental technique.

Function

$$\int_{\omega_0}^{\infty} n(\omega)d\omega = N(W,T) \tag{7}$$

is the total number of photons emitted by the blackbody in the interval (ω_0, ∞). It depends on the work function W an on the temperature of the thermal bath which is in our case the blackbody.

We can write the polychromatic photoelectric equation in the following form:

$$\int_0^{\infty} n(\omega)\hbar\omega d\omega - \int_0^{\omega_0} n(\omega)\hbar\omega d\omega = E_{kin} + WN(W,T), \tag{8}$$

or, in the modified form

$$aT^4 - \int_0^{\omega_0} n(\omega)\hbar\omega d\omega = E_{kin} + WN(W,T), \tag{9}$$

where the term aT^4 was obtained by the obligate mathematical procedure

$$\int_0^{\infty} \varrho(\omega)d\omega = \int_0^{\infty} n(\omega)\hbar\omega d\omega = \int_0^{\infty} \frac{\omega^2}{\pi^2c^3}\frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}}-1}d\omega = \frac{\pi^2}{15}\frac{k^4T^4}{c^3\hbar^3} = aT^4. \tag{10}$$

We know from the textbooks that

$$a = \frac{\pi^2k^4}{15c^3\hbar^3} = 7.56 \times 10^{-13}\mathrm{erg.cm}^{-3}.\mathrm{grad}^{-4}. \tag{11}$$

The equation (9) is of the two scientific meaning. The first meaning is the mathematical. Namely, if we obtain from the experiment the quantity E_{kin}, then the equation (9) is the mathematical equation for the determination of the work function W, where, however the work function W is also inbuilt in the integral. In other words, it is the new and original mathematical problem of elimination of some quantity from the nontrivial equation.

The next physical meaning of the equation (9) is, that the work function W is defined by the quantum collective motion of electrons and we know that the quantum collective motion of electrons is not the sum of the individual motion of electrons along the individual trajectories. So, the work function obtained from the polychromatic photoelectric equation (9) differs from the work function obtained from the monochromatic Einstein equation (1). The theoretical determination of the two different work functions represents the basic, the fundamental and the crucial problem of the quantum theory of the solid state physics and this problem was not till this time solved.

The same procedure can be performed using the distribution function of photons of the synchrotron radiation, where instead the blackbody density of photons is the synchrotron density of photons $P(\omega)$.

$$\int_{\omega_0}^{\infty} P(\omega)\hbar\omega d\omega = E_{kin} + WN_{synchro}. \qquad (12)$$

We can easily determine the work function only by the measurement of the total energy of the emitted electron during the photoeffect.

Let us remark, that the main motivation of the Einstein approach was the solid state proof of the existence of the light quanta. The possible next step was the generalization of the photoelectric effect for the situation where the absorption of photons is polychromatic, generated for instance by the blackbody, or by the synchrotron. In time of the Einstein photoelectric derivation, the blackbody radiation was under discussion and the Schott formula for the synchrotron radiation was not derived. So, the Einstein motivation to go beyond his photoelectric equation was not sufficiently strong. Now, the polychromatic form of the Einstein photoelectric equation is physically meaningful.

It is not excluded, a priori, that the collective motion of electrons in multiphoton experiment influences the work function in such a way that it is different from the work function in case where we use only the monochromatic light generating the individual motion of electrons. The measurement and investigation of eqs. (9), (12) can be considered as crucial and leading to the new discoveries in the photonic physics, elementary particle physics and solid state physics.

The information following from the polychromatic photoelectric effect is necessary not only in the solid state physics, but also in the elementary particle physics where multiphoton beams play the substantial role of the particle detectors.

6 The photoelectric effect in the 2D electron gas in strong magnetic field

The S-matrix element involving the interaction of an atom with the impinging photon and with the ejected electron with the final plane wave

$$\psi_{\mathbf{q}} = \frac{1}{\sqrt{V}} e^{i\mathbf{q}\cdot\mathbf{x}}, \quad \mathbf{q} = \frac{\mathbf{p}}{\hbar}, \qquad (1)$$

where $\mathbf{p}$ is the momentum of the ejected electron, gives the quantum mechanical cross-section

$$d\sigma = \frac{2\pi}{\hbar} |V_{fi}| \delta(-I + \hbar\omega - E_f) \frac{d^3p}{(2\pi)^3}, \qquad (2)$$

where I is the ionization energy of an atom and E_f is the the final energy of the emitted electron, $|V_{fi}|$ is the matrix element of the transition of electron from the initial bound state to the final state. The matrix element foollows from the perturbation theory and it involves the first order term of the interaction between electron and photon. We follow here the Davydov elementary approach (Davydov, 1976).

We suppose here that magnetic field is applied locally to 2D sheet of electrons , so, in a sufficient distance from it the wave function is of the form of the plane wave (1). Let us remark that if the photon energy only just exceeds the ionization energy I of atom, then we cannot used the plane wave approximation but the wave function of the continuous spectrum.

The probability of the emission of electron by the electromagnetic wave is of the well-known form (Davydov, 1976):

$$dP = \frac{e^2 p}{8\pi^2\varepsilon_0\hbar m\omega}\left|\int e^{i(\mathbf{k}-\mathbf{q})\cdot\mathbf{x}}(\mathbf{e}\cdot\nabla)\psi_0 dxdydz\right|^2 d\Omega = C|J|^2 d\Omega, \tag{3}$$

where the interaction for absorption of the electromagnetic wave is normalized to *one photon in the unit volume*, $\mathbf{e}$ is the polarization of the impinging photon, ε_0 is the dielectric constant of vacuum, ψ_0 is the basic state of and atom. We have denoted the integral in $||$ by J and the constant before $||$ by C.

We consider the case with electrons in magnetic field as an analog of the Landau diamagnetism. So, we take the basic function ψ_0 for one electron in the lowest Landau level, as

$$\psi_0 = \left(\frac{m\omega_c}{2\pi\hbar}\right)^{1/2}\exp\left(-\frac{m\omega_c}{4\hbar}(x^2+y^2)\right), \tag{4}$$

which is solution of the Schrödinger equation in the magnetic field with potentials $\mathbf{A} = (-Hy/2, -Hx/2, 0, 0)$, (Drukarev, 1988):

$$\left[\frac{p_x^2}{2m} + \frac{p_y^2}{2m} - \frac{m}{2}\left(\frac{\omega_c}{2}\right)^2(x^2+y^2)\right]\psi = E\psi. \tag{5}$$

We have supposed that the motion in the z-direction is zero and it means that the wave function $\exp[(i/\hbar)p_z z] = 1$.

So, the main problem is to calculate the integral

$$J = \int e^{i(\mathbf{K}\cdot\mathbf{x})}(\mathbf{e}\cdot\nabla)\psi_0 dxdydz; \quad \mathbf{K} = \mathbf{k} - \mathbf{q}. \tag{6}$$

with the basic Landau function ψ_0 given by the equation (4).

Operator $(\hbar/i)\nabla$ is Hermitean and it means we can rewrite the last integrals as follows:

$$J = \frac{i}{\hbar}\mathbf{e}\cdot\int\left[\left(\frac{\hbar}{i}\nabla\right)e^{i(\mathbf{K}\cdot\mathbf{x})}\right]^*\psi_0 dxdydz, \tag{7}$$

which gives

$$J = i\mathbf{e}\cdot\mathbf{K}\int e^{-i(\mathbf{K}\cdot\mathbf{x})}\psi_0 dxdydz, \tag{8}$$

The integral in eq. (8) can be transformed using the cylindrical coordinates with

$$dxdydz = \varrho d\varrho d\varphi dz, \quad \varrho^2 = x^2 + y^2 \tag{9}$$

which gives for vector $\mathbf{K}$ fixed on the axis z with $\mathbf{K}\cdot\mathbf{x} = Kz$ and with physical condition $\mathbf{e}\cdot\mathbf{k} = 0$, expressing the physical situation where polarization is perpendicular to the direction of the wave propagation. So,

$$J = (i)(\mathbf{e}\cdot\mathbf{q})\int_0^\infty \varrho d\varrho\int_{-\infty}^\infty dz\int_0^{2\pi} d\varphi e^{-iKz}\psi_0. \tag{10}$$

Using

$$\psi_0 = A\exp\left(-B\varrho^2\right); \quad A = \left(\frac{m\omega_c}{2\pi\hbar}\right)^{1/2}; \quad B = \frac{m\omega_c}{4\hbar}. \tag{11}$$

The integral (12) is then

$$J = (-\pi i)\frac{A}{B}(\mathbf{e}\cdot\mathbf{q})\int_{-\infty}^{\infty} e^{-iKz}dz = (-\pi i)\frac{A}{B}(\mathbf{e}\cdot\mathbf{q})(2\pi)\delta(K). \tag{12}$$

Then,

$$dP = C|J|^2 d\Omega = 4\pi^4\frac{A^2}{B^2}C(\mathbf{e}\cdot\mathbf{q})^2\delta^2(K)d\Omega. \tag{13}$$

Now, let be the angle Θ between direction $\mathbf{k}$ and direction $\mathbf{q}$, and let be the angle Φ between planes $(\mathbf{k},\mathbf{q})$ and $(\mathbf{e},\mathbf{k})$. Then,

$$(\mathbf{e}\cdot\mathbf{q})^2 = q^2\sin^2\Theta\cos^2\Phi. \tag{14}$$

So, the differential probability of the emission of photons from the graphene (Pardy, 2010) in the strong magnetic field is as follows:

$$dP = \frac{4e^2p}{\pi\varepsilon_0 m^2\omega\omega_c}\left[q^2\cos^2\Theta\sin^2\Phi\right]\delta^2(K)d\Omega; \quad \omega_c = \frac{|e|H}{mc}. \tag{15}$$

We can see that our result differs form the result for the original photoelectric effect which involves still the term

$$\frac{1}{(1-\frac{v}{c}\cos\Theta)^4}, \tag{16}$$

which means that the most intensity of the classical photoeffect is in the direction of the electric vector of the electromagnetic wave ($\Phi = \pi/2, \Theta = 0$). While the nonrelativistic solution of the photoeffect in case of the Coulomb potential was performed by Stobbe (1930) and the relativistic calculation by Sauter (Sauter, 1931), the general magnetic photoeffect (with electrons moving in the magnetic field and forming atom) was not still performed in a such simple form. The delta term $\delta\cdot\delta$ represents the conservation law $|\mathbf{k}-\mathbf{q}| = 0$ in our approximation.

So, we have calculated only the process which can be approximated by the Schrödinger equation for an electron orbiting in magnetic field.

6.1 The photoelectric effect with Volkov solution

It is valuable from the pedagogical point of view (Berestetzky et al., 1989) to remember the Volkov solution, where the motion of the Dirac electron is considered in the following four potential

$$A_\mu = a_\mu\varphi; \quad \varphi = kx; \quad k^2 = 0. \tag{17}$$

From equation (23), it follows that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = a_\nu k_\mu - a_\mu k_\nu = const.$, which means that electron moves in the constant electromagnetic field with the components $\mathbf{E}$ and $\mathbf{H}$. The parameters a and k can be chosen in a such a way that $\mathbf{E} = 0$. So, the motion of electron is performed in the constant magnetic field.

The Volkov (1935) solution of the Dirac equation for an electron moving in a field of a plane wave was derived in the form (Berestetzky et al., 1989; Pardy, 2004):

$$\psi_p = \frac{u(p)}{\sqrt{2p_0}}\left[1 + e\frac{(\gamma k)(\gamma A(\varphi))}{2kp}\right]\exp\left[(i/\hbar)S\right] \tag{18}$$

and S is an classical action of an electron moving in the potential $A(\varphi)$ (Berestetzky et al., 1989).

$$S = -px - \int_0^{kx} \frac{e}{(kp)}\left[(pA) - \frac{e}{2}(A)^2\right] d\varphi. \tag{19}$$

It was shown that for the potential (17) the Volkov wave function is (Berestetzky et al., 1989):

$$\psi_p = \frac{u(p)}{\sqrt{2p_0}}\left[1 + e\frac{(\gamma k)(\gamma a)}{2kp}\varphi\right]\exp\left[(i/\hbar)S\right] \tag{20}$$

with

$$S = -e\frac{ap}{2kp}\varphi^2 + e^2\frac{a^2}{6kp}\varphi^3 - px. \tag{21}$$

We used $c = \hbar = 1$.

However, the relativistic wave function can be obtained by solving the Dirac equation in magnetic field. It was derived in the form (Sokolov et al., 1983).

$$\Psi(\mathbf{x}, t) = \frac{1}{L}\exp\{-\frac{i}{\hbar}\epsilon Et + ik_2 y + ik_3 z\}\psi; \quad \psi = \begin{pmatrix} C_1 u_{n-1}(\eta) \\ iC_2 u_n(\eta) \\ C_3 u_{n-1}(\eta) \\ iC_4 u_n(\eta) \end{pmatrix}, \tag{22}$$

where $\epsilon = \pm 1$ and the spinor components are given by the following formulas:

$$u_n(\eta) = \sqrt{\frac{\sqrt{2\gamma}}{2^n n!}\sqrt{\pi}}\; e^{-\eta^2/2} H_n(\eta) \tag{23}$$

with

$$H_n(\eta) = (-1)^n e^{\eta^2}\left(\frac{d}{d\eta}\right)^n e^{-\eta^2}, \tag{24}$$

$$\eta = \sqrt{2\gamma}\, x + k_2/\sqrt{2\gamma}; \quad \gamma = eH/2c\hbar. \tag{25}$$

The coefficients C_i are defined in the Sokolov et al. monograph (Sokolov et al., 1983). So, our approach can be generalized.

7 The photoeffect with the dressed photon

We define here the dressed photon as a such with the additional radiative corrections, where we take the radiative correction in the form of the virtual electron-positron pair. We have shown that such approach to the photon leads to the modification he photon propagator. According to Dittrich (1978) and Schwinger (1973), the photon propagator with radiative correction is in the momentum representation of the form:

$$\tilde{D}(k) = D(k) + \delta D(k), \tag{1}$$

or,

$$\tilde{D}(k) = \frac{1}{|\mathbf{k}|^2 - n^2(k^0)^2 - i\epsilon} +$$

$$+ \int_{4m^2}^{\infty} dM^2 \frac{a(M^2)}{|\mathbf{k}|^2 - n^2(k^0)^2 + \frac{M^2c^2}{\hbar^2} - i\epsilon}, \tag{2}$$

where the last term in equation (2) is derived on the virtual photon condition

$$|\mathbf{k}|^2 - n^2(k^0)^2 = -\frac{M^2c^2}{\hbar^2}, \tag{3}$$

where n is the index of refraction of the medium. The weight function $a(M^2)$ has been derived in the following form (Dittrich, 1978; Schwinger, 1973):

$$a(M^2) = \frac{\alpha}{3\pi}\frac{1}{M^2}\left(1 + \frac{2m^2}{M^2}\right)\left(1 - \frac{4m^2}{M^2}\right)^{1/2}. \tag{4}$$

The x-representation of $D(k)$ in eq. (1) is as follows:

$$D_+(x - x') = \int \frac{(dk)}{(2\pi)^4} e^{ik(x-x')} D(k). \tag{5}$$

Or,

$$D_+(x - x') = \int \frac{(dk)}{(2\pi)^4} \frac{e^{ik(x-x')}}{|\mathbf{k}^2| - n^2(k^0)^2 - i\epsilon} =$$

$$= \frac{i}{c}\frac{1}{4\pi^2}\int_0^{\infty} d\omega \frac{\sin \frac{n\omega}{c}|\mathbf{x} - \mathbf{x}'|}{|\mathbf{x} - \mathbf{x}'|} e^{-i\omega|t-t'|}. \tag{6}$$

Now, with regard to the definition of x-representation (5) and (6) of the $D_+(x - x')$, we get the x-representation of the δD_+ in the following form:

$$\delta D_+(x - x') = \frac{i}{c}\frac{1}{4\pi^2}\int_{4m^2}^{\infty} dM^2 a(M^2) \times$$

$$\times \int d\omega \, \frac{\sin[\frac{n^2\omega^2}{c^2} - \frac{M^2c^2}{\hbar^2}]^{1/2}|\mathbf{x} - \mathbf{x}'|}{|\mathbf{x} - \mathbf{x}'|} e^{-i\omega|t-t'|}. \tag{7}$$

The function (7) differs from the the original function D_+ especially by the factor

$$\gamma = \left(\frac{\omega^2 n^2}{c^2} - \frac{M^2c^2}{\hbar^2}\right)^{1/2} \tag{8}$$

and by the additional mass-integral which involves the radiative corrections to the original photon processes. It was easily shown in case of the Čerenkov effect by author (Pardy, 1994a).

So, to involve the photoelectric effect with the dressed photon with electron positron pair we replace the wave function of photon $\exp(i\mathbf{k}\cdot\mathbf{x})$ by the function involving the radiative correction factor as follows:

$$e^{i\mathbf{k}\cdot\mathbf{x}} \to \int_{4m^2}^{\infty} dM^2 a(M^2) e^{i\boldsymbol{\kappa}\cdot x}, \tag{9}$$

where $\boldsymbol{\kappa}\cdot\mathbf{x} = \lambda|k||x|\cos\varphi$.

Let us consider here the alternative approach to the photoeffect which differs formally from Berestetzky approach. We mean the Davydov textbook approach. The probability of the emission of electron by the electromagnetic wave is then of the well-known form (Davydov, 1976):

$$dP = \frac{e^2 p}{8\pi^2\varepsilon_0\hbar m\omega}\left|\int e^{i(\boldsymbol{\kappa}-\mathbf{q})\cdot\mathbf{x}}(\mathbf{e}\cdot\nabla)\psi_0 dxdydz\right|^2 d\Omega = C|J|^2 d\Omega, \tag{10}$$

where the interaction for absorption of the electromagnetic wave is normalized to *one photon in the unit volume*, $\mathbf{e}$ is the polarization of the impinging photon, ε_0 is the dielectric constant of vacuum, ψ_0 is the basic state of and atom. We have denoted the integral in $||$ by J and the constant before $||$ by C.

So, the main problem is to calculate the integral

$$J = \int e^{i(\boldsymbol{\kappa}\cdot\mathbf{x})}(\mathbf{e}\cdot\nabla)\psi_0 dxdydz; \quad \mathbf{K} = \boldsymbol{\kappa} - \mathbf{q}. \tag{11}$$

with the basic Landau function ψ_0 given by the equation (4).

Operator $(\hbar/i)\nabla$ is Hermitean and it means we can rewrite the last integrals as follows:

$$J = \frac{i}{\hbar}\mathbf{e}\cdot\int\left[\left(\frac{\hbar}{i}\nabla\right)e^{i(\mathbf{K}\cdot\mathbf{x})}\right]^* \psi_0 dxdydz, \tag{12}$$

which gives

$$J = i\mathbf{e}\cdot\mathbf{K}\int e^{-i(\mathbf{K}\cdot\mathbf{x})}\psi_0 dxdydz, \tag{13}$$

Let us consider the problem in the magnetic field where the motion of electron is in plain. Then, the integral in eq. (13) can be transformed using the cylindrical coordinates with

$$dxdydz = \varrho d\varrho d\varphi dz, \quad \varrho^2 = x^2 + y^2 \tag{14}$$

which gives for vector $\mathbf{K}$ fixed on the axis z with $\mathbf{K}\cdot\mathbf{x} = Kz$ and with physical condition $\mathbf{e}\cdot\kappa = 0$, expressing the physical situation where polarization is perpendicular to the direction of the wave propagation. So,

$$J = (i)(\mathbf{e}\cdot\mathbf{q})\int_0^\infty \varrho d\varrho\int_{-\infty}^\infty dz\int_0^{2\pi} d\varphi e^{-iKz}\psi_0. \tag{15}$$

Using

$$\psi_0 = A\exp\left(-B\varrho^2\right); \quad A = \left(\frac{m\omega_c}{2\pi\hbar}\right)^{1/2}; \quad B = \frac{m\omega_c}{4\hbar}; \qquad \omega_c = \frac{|e|H}{mc}. \tag{16}$$

The integral (15) is then

$$J = (-\pi i)\frac{A}{B}(\mathbf{e}\cdot\mathbf{q})\int_{-\infty}^{\infty} e^{-iKz}dz = (-\pi i)\frac{A}{B}(\mathbf{e}\cdot\mathbf{q})(2\pi)\delta(K). \tag{17}$$

Then,

$$dP = C|J|^2 d\Omega = 4\pi^4\frac{A^2}{B^2}C(\mathbf{e}\cdot\mathbf{q})^2\delta^2(\kappa)d\Omega. \tag{18}$$

Now, let be the angle Θ between direction $\boldsymbol{\kappa}$ and direction $\mathbf{q}$, and let be the angle Φ between planes $(\boldsymbol{\kappa}, \mathbf{q})$ and $(\mathbf{e}, \boldsymbol{\kappa})$. Then,

$$(\mathbf{e}\cdot\mathbf{q})^2 = q^2\sin^2\Theta\cos^2\Phi. \tag{19}$$

So, the differential probability of the emission of photons from the plane in the strong magnetic field is as follows:

$$dP = \frac{4e^2p}{\pi\varepsilon_0 m^2\omega\omega_c}\int_{4m^2}^{\infty} dM^2 a(M^2)\left[q^2\cos^2\Theta\sin^2\Phi\right]\delta^2(K)d\Omega; \quad \omega_c = \frac{|e|H}{mc}. \tag{20}$$

We can see that our result differs form the result for the original photoelectric effect which involves still the term

$$\frac{1}{(1-\frac{v}{c}\cos\Theta)^4}, \tag{21}$$

which means that the most intensity of the classical photoeffect is in the direction of the electric vector of the electromagnetic wave ($\Phi = \pi/2, \Theta = 0$). While the nonrelativistic solution of the photoeffect in case of the Coulomb potential was performed by Stobbe (1930) and the relativistic calculation by Sauter (Sauter, 1931), the general magnetic photoeffect (with electrons moving in the magnetic field and forming atom) was not still performed in a such simple form. The delta term $\delta\cdot\delta$ represents the conservation law $|\boldsymbol{\kappa} - \mathbf{q}| = 0$ in our approximation.

8 H-aton in the Gibbons-Hawking thermal bath

The Gibbons-Hawking effect is the statement that a temperature can be associated to each solution of the Einstein field equations that contains a causal horizon. It is named after Gary Gibbons and Stephen William Hawking.

Schwarzschild spacetime contains an event horizon and so can be associated with temperature. In the case of Schwarzschild spacetime this is the temperature T of a black hole of mass M, satisfying T/M .

De Sitter space which contains an event horizon has the temperature T proportional to the Hubble parameter H. We consider here the influence of the heat bath of the Gibbons-Hawking photons on the energy shift of H-atom electrons.

The considered problem is not in the scientific isolation, because some analogical problems are solved in the scientific respected journals. At present time it is a general conviction that there is an important analogy between black hole and the hydrogen atom. The similarity between black hole and the hydrogen atom was considered for instance by

Corda (2015a), who discussed the precise model of Hawking radiation from the tunneling mechanism. In this article an elegant expression of the probability of emission is given in terms of the black hole quantum levels. So, the system composed of Hawking radiation and black hole quasi-normal modes introduced by Corda (2015b) is somewhat similar to the semiclassical Bohr model of the structure of a hydrogen atom.

The time dependent Schrödinger equation was derived for the system composed by Hawking radiation and black hole quasi-normal modes (Corda, 2015c). In this model, the physical state and the correspondent wave function are written in terms of an unitary evolution matrix instead of a density matrix. Thus, the final state is a pure quantum state instead of a mixed one and it means that there is no information loss. Black hole can be well defined as the quantum mechanical systems, having ordered, discrete quantum spectra, which respect 't Hooft's assumption that Schrödinger equations can be used universally for all dynamics in the universe.

Thermal photons by Gibbons and Hawking form so called blackbody, which has the distribution law of photons derived in 1900 by Planck (1900, 1901), (Schöpf, 1978). The derivation was based on the investigation of the statistics of the system of oscillators inside of the blackbody. Later Einstein (1917) derived the Planck formula from the Bohr model of atom where electrons have the discrete energies and the energy of the emitted photons are given by the Bohr formula $\hbar\omega = E_i - E_f$, E_i, E_f are the initial and final energies of electrons.

Now, let us calculate the modified Coulomb potential due to blackbody. The starting point of the determination of the energy shift in the H-atom is the potential $V_0(\mathbf{x})$, which is generated by nucleus of the H-atom. The potential at point $V_0(\mathbf{x} + \delta\mathbf{x})$, evidently is (Akhiezer, et al., 1953; Welton, 1948):

$$V_0(\mathbf{x} + \delta\mathbf{x}) = \left\{1 + \delta\mathbf{x}\nabla + \frac{1}{2}(\delta\mathbf{x}\nabla)^2 + ...\right\} V_0(\mathbf{x}). \tag{1}$$

If we average the last equation in space, we can eliminate so called the effective potential in the form

$$V(\mathbf{x}) = \left\{1 + \frac{1}{6}(\delta\mathbf{x})^2_T\Delta + ...\right\} V_0(\mathbf{x}), \tag{2}$$

where $(\delta\mathbf{x})^2_T$ is the average value of te square coordinate shift caused by the thermal photon fluctuations. The potential shift follows from eq. (2):

$$\delta V(\mathbf{x}) = \frac{1}{6}(\delta\mathbf{x})^2_T\Delta V_0(\mathbf{x}). \tag{3}$$

The corresponding shift of the energy levels is given by the standard quantum mechanical formula (Akhiezer, et al., 1953)

$$\delta E_n = \frac{1}{6}(\delta\mathbf{x})^2_T(\psi_n\Delta V_0\psi_n). \tag{4}$$

In case of the Coulomb potential, which is the case of the H-atom, we have

$$V_0 = -\frac{e^2}{4\pi|\mathbf{x}|}. \tag{5}$$

Then for the H-atom we can write

$$\delta E_n = \frac{2\pi}{3}(\delta\mathbf{x})^2_T \frac{e^2}{4\pi}|\psi_n(0)|^2, \tag{6}$$

where we used the following equation for the Coulomb potential

$$\Delta\frac{1}{|\mathbf{x}|} = -4\pi\delta(\mathbf{x}). \tag{7}$$

Motion of electron in electric field is evidently described by elementary equation

$$\delta\ddot{\mathbf{x}} = \frac{e}{m}\mathbf{E}_T, \tag{8}$$

which can be transformed by the Fourier transformation into the following equation

$$|\delta\mathbf{x}_{T\omega}|^2 = \frac{1}{2}\left(\frac{e^2}{m^2\omega^4}\right)\mathbf{E}^2_{T\omega}, \tag{9}$$

where the index ω concerns the Fourier component of above functions.

On the basis of the Bethe idea of the influence of vacuum fluctuations on the energy shift of electron (Bethe, 1947), the following elementary relations was used by Welton (1948), Akhiezer et al. (1953) and Berestetzky et al. (1989):

$$\frac{1}{2}\mathbf{E}^2_\omega = \frac{\hbar\omega}{2} \tag{10}$$

and in case of the thermal bath of the blackbody, the last equation is of the following form (Isihara, 1971):

$$\mathbf{E}^2_{T\omega} = \varrho(\omega) = \left(\frac{\hbar\omega^3}{\pi^2c^3}\right)\frac{1}{e^{\frac{\hbar\omega}{kT}}-1}, \tag{11}$$

because the Planck law in eq. (11) was written as

$$\varrho(\omega) = G(\omega) < E_\omega >= \left(\frac{\omega^2}{\pi^2c^3}\right)\frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}}-1}, \tag{12}$$

where the term

$$< E_\omega >= \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}}-1} \tag{13}$$

is the average energy of photons in the blackbody and

$$G(\omega) = \frac{\omega^2}{\pi^2c^3} \tag{14}$$

is the number of electromagnetic modes in the interval $\omega, \omega + d\omega$.

Then,

$$(\delta\mathbf{x}_{T\omega})^2 = \frac{1}{2}\left(\frac{e^2}{m^2\omega^4}\right)\left(\frac{\hbar\omega^3}{\pi^2c^3}\right)\frac{1}{e^{\frac{\hbar\omega}{kT}}-1}, \tag{15}$$

where $(\delta\mathbf{x}_{T\omega})^2$ involves the number of frequencies in the interval $(\omega, \omega + d\omega)$.

So, after some integration, we get

$$(\delta \mathbf{x})^2_T = \int_{\omega_1}^{\omega_2} \frac{1}{2}\left(\frac{e^2}{m^2\omega^4}\right)\left(\frac{\hbar\omega^3}{\pi^2 c^3}\right)\frac{d\omega}{e^{\frac{\hbar\omega}{kT}}-1} = \frac{1}{2}\left(\frac{e^2}{m^2}\right)\left(\frac{\hbar}{\pi^2 c^3}\right)F(\omega_2-\omega_1), \tag{16}$$

where $F(\omega)$ is the primitive function of the omega-integral

$$J = \frac{1}{\omega}\frac{1}{e^{\frac{\hbar\omega}{kT}}-1}, \tag{17}$$

which cannot be calculated by the elementary integral methods and it is not involved in the tables of integrals.

Frequencies ω_1 and ω_2 will be determined with regard to the existence of the fluctuation field of thermal photons. It was determined in case of the Lamb shift (Bethe, 1947 ; Welton, 1947) by means of the physical analysis of the interaction of the Coulombic atom with the surrounding fluctuation field. We suppose here that the Bethe and Welton arguments are valid and so we take the frequencies in the Bethe-Welton form. In other words, electron cannot respond to the fluctuating field if the frequency which is much less than the atom binding energy given by the Rydberg constant (Rohlf, 1994) $E_{Rydberg} = \alpha^2 mc^2/2$. So, the lower frequency limit is

$$\omega_1 = E_{Rydberg}/\hbar = \frac{\alpha^2 mc^2}{2\hbar}, \tag{18}$$

where $\alpha \approx 1/137$ is so called the fine structure constant.

The specific form of the second frequency follows from the elementary argument, that we expect the effective cutoff, since we must neglect the relativistic effect in our non-relativistic theory. So, we write

$$\omega_2 = \frac{mc^2}{\hbar}. \tag{19}$$

If we take the thermal function of the form of the geometric series

$$\frac{1}{e^{\frac{\hbar\omega}{kT}}-1} = q(1+q^2+q^3+.....); \quad q = e^{-\frac{\hbar\omega}{kT}}, \tag{20}$$

$$\int_{\omega_1}^{\omega_2} q(1+q^2+q^3+.....)\frac{1}{\omega}d\omega = \ln|\omega| + \sum_{k=1}^{\infty}\frac{(-\frac{\hbar\omega}{kT})^k}{k!k} + ...; \quad q = e^{-\frac{\hbar\omega}{kT}} \tag{21}$$

and the first thermal contribution is

$$Thermal\ contribution = \ln\frac{\omega_2}{\omega_1} - \frac{\hbar}{kT}(\omega_2-\omega_1), \tag{22}$$

Then, with eq. (6)

$$\delta E_n \approx \frac{2\pi}{3}\left(\frac{e^2}{m^2}\right)\left(\frac{\hbar}{\pi^2 c^3}\right)\left(\ln\frac{\omega_2}{\omega_1} - \frac{\hbar}{kT}(\omega_2-\omega_1)\right)|\psi_n(0)|^2, \tag{23}$$

where (Sokolov et al., 1962)

$$|\psi_n(0)|^2 = \frac{1}{\pi n^2 a_0^2} \tag{24}$$

with

$$a_0 = \frac{\hbar^2}{me^2}. \tag{25}$$

Let us only remark that the numerical form of eq. (23) has deep experimental astrophysical meaning.

In article by author (Pardy, 1994), which is the continuation of author articles on the finite-temperature Cherenkov radiation and gravitational Cherenkov radiation (Pardy, 1989a; ibid., 1989b), the temperature Green function in the framework of the Schwinger source theory (Schwinger, 1970) was derived in order to determine the Coulomb and Yukawa potentials at finite-temperature using the Green functions of a photon with and without radiative corrections, and then by considering the processes expressed by the Feynman diagrams.

The determination of potential at finite temperature is one of the problems which form the basic ingredients of the quantum field theory (QFT) at finite temperature. This theory was formulated some years ago by Dolan and Jackiw (1974), Weinberg (1974) and Bernard (1974) and some of the first applications of this theory were the calculations of the temperature behavior of the effective potential in the Higgs sector of the standard model.

Information on the systematic examination of the finite temperature effects in quantum electrodynamics (QED) at one-loop order was given by Donoghue, Holstein and Robinett (1985). Partovi (1994) discussed the QED corrections to Planck's radiation law and photon thermodynamics,

A similar discussion of QED was published by Johansson, Peressutti and Skagerstam (1986) and Cox et al. (1984).

So, We considered here the thermal gas corresponding to the Gibbons-Hawking theory of space-time (at temperature T) as the preamble for new experiments for the determination of the energy shift of H-atom electrons interacting with the Gibbons-Hawking thermal gas. It is not excluded, that the observations performed by the well educated astro-experts will be the crucial ones.

9 The Planck formula in dielectric crystal

It is physically meaningful to consider, in quantum theory of light and quantum theory of solids, dielectric crystalline medium with phonons which is inserted in the Planck blackbody photon gas. It means that photon gas of the blackbody surrounding the dielectric crystalline medium with with index of refraction n flows into such crystal and initiate the quantum osmotic pressure of photons as solvent and phonons as solute.

The classical osmosis is the spontaneous passage of solvent molecules through a partially permeable membrane separating two solutions of different concentration into a region of higher solute concentration of solute, in order to equalize the solute concentrations on the two sides. The physical law which controll the osmotic pressure is so called van't Hoff's equation (published in 1885):

$$p = i\frac{C}{\mu}RT, \tag{1}$$

where p, i, C, μ, R, T are pressure, van't Hoff factor, concentration of solute, mollar mass, thermodynamic gas constant and temperature, and concentration is defined by formula $C = m/V$, where m is mass of solute in volume V. We consider here the quantum osmosis with photons and phonons and with the semi-permeable membrane for photons which is the surface of the dielectric crystal.

The derivation of the van't Hoff formula using the thermodynamic potential can be found in the textbooks on thermodynamics and statistical physics (Landau et al., 1980). The derivation of the osmotical pressure from rigorous statistical physics was given by Isihara (1971). On the other hand, the quantum theory of osmosis was not published. A Duth physical and organic chemist van't Hoff presented his Nobelian theory long time before the introduction of photons into physics by Max Planck, Lewis and Einstein and before the introduction of phonons into solid state physics by Einstein and Debye. So, the problem of the osmotic pressure in the Planck blackbody with the dielectric medium arises as the problem of modern physics.

The dielectric crystal with photons is called here by term Planck dielectric blackbody. Inside of the dielectric medium with index of refraction n, the spectral radiation formula is modified and we derive in the next part mathematical form of the spectrum of such dielectric blackbody. The derivation of the spectral formula is based on the original Planck spectral formula which was rederived by Einstein (1917).

The spectral distribution of the blackbody does not depend on the specific atomic composition of the blackbody and it means the formula (7) must be so called the Planck formula:

$$\varrho_\omega = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{e^{\frac{\hbar\omega}{kT}} - 1}. \tag{2}$$

The internal density energy of the blackbody gas is given by integration of the last equation over all frequencies ω, or

$$u = \int_0^\infty \varrho(\omega)d\omega = aT^4; \quad a = \frac{\pi^2 k^4}{15\hbar^3 c^3}. \tag{3}$$

and the pressure of photons inside the blackbody follows from the electrodynamic situation inside blackbody as follows:

$$p = \frac{u}{3} \qquad (4).$$

We suppose here that inside of the Planck blackbody there is the dielectric crystal with the index of refraction $n(\omega)$. Then, the wave vector of photon inside the dielectric medium is given by known formula

$$q = n(\omega)\frac{\omega}{c}. \tag{5}$$

The number of light modes in the interval $q, q+dq$ inside of the dielectric in the volume V is Vq^2dq/π^2. After differentiation of formula (5) we get with $d\ln\omega = d\omega/\omega$

$$dq = \frac{1}{c}[n(\omega) + \omega\frac{dn(\omega)}{d\omega}]d\omega = \frac{n(\omega)}{c}\frac{d\ln[n(\omega)\omega]}{d\ln\omega}d\omega. \tag{6}$$

Then, it is easy to see that the number of states in the interval $\omega, \omega + d\omega$ of the electromagnetic vibrations in the volume V is

$$Vg(\omega)d\omega = \frac{V}{\pi^2}\left(\frac{n(\omega)}{c}\right)^3 \frac{d\ln[n(\omega)\omega]}{d\ln\omega}d\omega. \tag{7}$$

If we multiply the last formula by the average energy of the harmonic oscillator,

$$< E_\omega >= \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1}, \tag{8}$$

we get the Planck formula for the blackbody with dielectric medium:

$$\varrho(\omega) = \frac{n^3(\omega)\omega^2}{\pi^2 c^3}\frac{d\ln[n(\omega)\omega]}{d\ln\omega}\frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1}, \tag{9}$$

where for $n = 1$, we get exactly formula (2).

9.1 The oscillator model of the index of refraction

This model follows from the classical theory of dispersion, which is based on the vibration equation of electron in an atom

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{e}{m}E_0\cos\omega t, \tag{10}$$

where γ is the oscillator constant and ω_0 is the basic frequency of oscillator. The symbol ω is the frequency of the applied electric field. The index of refraction following from eq (17) is given by the formula (Garbuny, 1965)

$$n = 2\pi N\frac{e^2}{m}\frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}, \tag{11}$$

where N is number of electrons in the unit of volume.

In case of electrons with basic frequencies $\omega_1, \omega_2, \omega_3, \omega_4 ... \omega_n$, the last refraction index can be generalized to form more complex mathematical object. We consider here, to be pedagogical clear, only one oscillator with one basic frequency. Nevertheless it is possible consider arbitrary dielectric material with the phenomenological index of refraction.

Now the question arises, if the dielectric blackbody can be considered as the solution composed from atoms, phonons and photons where the osmotic pressure play some role. We had accepted this hypothesis as the correct one.

9.2 The osmosis in dielectric blackbody

Phonons were introduced in the crystal physics by Einstein in order to derive the adequate formula for he specific heat. The Einstein formula was generalized and improved by Debye who derived the formula for the average energy of phonons in a crystal in the interval of temperatures $\Theta - \delta < T < \Theta + \delta$ (δ is some parameter) as follows (Rumer et al., 1977):

$$U = N\varepsilon_0 + 3NTD\left(\frac{\Theta}{T}\right), \tag{12}$$

where $\varepsilon_0 = (9/8)\hbar\omega_{max}$, where

$$\omega_{max} = 2\pi v \left(\frac{3N}{4\pi V}\right)^{1/3} \tag{13}$$

and $D(x)$ is so called the Debye wave function of the following structure:

$$D(x) = \frac{3}{x^3}\int_0^x \frac{y^3}{e^y - 1}dy, \tag{14}$$

and the critical temperature Θ was derived by Debye in the following form:

$$\Theta = v\left(\frac{6\pi^2 N}{V}\right)^{1/3}, \tag{15}$$

with v being velocity of sound waves defined in the theory of elasticity of the crystal.

Let us compare the internal energies of the pure blackbody and dielectric blackbody and then let us compare the pressure inside of the pure blackbody and inside the dielectric blackbody.

For pure blackbody, we have $u = aT^4$ and for model with n given by eq. (11) we have

$$u = \int_0^\infty \varrho_n(\omega)d\omega = \int_0^\infty \varrho_n(\omega)\frac{n^3(\omega)\omega^2}{c^3}\frac{d\ln[n(\omega)\omega]}{d\ln\omega}\frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1}d\omega. \tag{16}$$

Because the dielectric medium is permeable for photons and not for phonons (the photon osmosis), the outer pressure is equal to the photon gas pressure in the dielectric blackbody, or $p(n) = u(n)/3 = u/3$. So,

$$\int_0^\infty \varrho_n(\omega)d\omega = u/3 = \frac{aT^4}{3}, \tag{17}$$

or,

$$\int_0^\infty \frac{n^3(\omega)\omega^2}{\pi^2c^3}\frac{d\ln[n(\omega)\omega]}{d\ln\omega}\frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT_{diel}}} - 1}d\omega = \frac{aT^4}{3}, \tag{18}$$

where we introduced the dielectric temperature T_{diel}, which physically means that the temperature of dielectric blackbody is not the same as the temperature of the bath of vacuum blackbody photons. The last equation is the integral equation for function T_{diel} and in general represents very difficult mathematical problem of the future physics of the dielectric blackbody. The experimental verification of the last equation will be also the crucial problem of photon physics.

In the most simple case with $n = const$, we get after some algebraic operation, that the temperature dielectric blackbody surrounded by the vacuum blackbody is given by the formula

$$T_{diel} = \frac{T}{\sqrt[4]{n^3}}. \tag{19}$$

The last formula can form the goal of the experimenters working in the blackbody radiation physics. The dielectric as the osmotic membrane plays the role of the Maxwell demonic refrigerator. The second possibility is to put $n = n(T)$ in order to get the integral equation for the dependence of the index of refraction on temperature. However, it seems that this assumption is not physically adequate.

In case of the dielectric Debye crystal, the equation of state is (Rumer et al., 1977)

$$p = \left(\frac{U_{phon}}{\Theta} - \frac{9}{4}N\right)\frac{d\Theta}{dV}, \tag{20}$$

where V and N is volume and number of oscillators in crystal. The difference $\Delta p = p(T) - p(T_{diel})$ is the osmotic pressure caused by the photon flow.

In case of the two-dimensional crystal, the internal phonon energy is (Rumer et al., 1977)

$$U_{2D-phon} = \frac{4}{3}N\Theta\left[1 + \left(\frac{T}{\Theta}\right)^3 \int_0^{\Theta/T} \frac{y^2}{e^y - 1}dy\right]. \tag{21}$$

and

$$\Theta = 2\pi v\left(\frac{N}{\pi\sigma}\right)^{1/2}, \tag{22}$$

where σ is the area of the 2D crystal (e. g. graphene, which is the carbon sheet), instead of $d\Theta/dV$ is $d\Theta/d\sigma$ and 9/4 must be replaced by the adequate constant. The osmotic temperature of the 2-dimensional and 1-dimensional dielectric crystal is an analogue of the 3-dimensional case and can be derived from the formulas by author article (Pardy, 2015b).

We know that the classical osmosis is the physical phenomenon in the system with solute, solvent, solution and semi-permeable membrane. It plays fundamental role in biological and physiological systems, where for instance the photosynthesis in plants is not possible without water and photon osmosis and human being does not exist without liquid osmosis.

Isihara (1971) derived from the statistical physics the following formula for the osmotic pressure of the two-component statistical system:

$$p = kT\frac{\partial[\ln(\Xi/\Xi_0)]}{\partial V}, \tag{23}$$

where Ξ and Ξ_0 are the big statistical sums of solute and solvent. The explicit mathematical form of the formula is sophisticated and the derivation of the van't Hoff formula is not elementary.

The theory of phonon-photon dielectric blackbody is the preamble for experiments for the determination of the osmotic process as the consequence of the quantum properties of the phonon-photon gas. The role of phonon-photon osmosis in biological and physiological systems is crucial. The phonon-photon osmotic pressure plays probably substantial negative role in the formation and in the development of skin cancer.

It is not excluded, that the experiments with the quantum osmosis in plasma with magnetic field as semi-permeable osmotic membrane, will play crucial role in the fusion reactor physics.

10 The Casimir effect at finite temperature

The Casimir effect, or, Casimir-Polder force are physical forces arising from a quantized field. They are named after the Dutch physicist Hendrik Casimir who predicted it in

1948.

The Casimir effect is an interaction between disjoint neutral bodies caused by the fluctuations of the electrodynamic vacuum. It can be explained by considering the normal modes of electromagnetic fields, which explicitly depend on the boundary (or matching) conditions on the interacting bodies surfaces. At the most basic level, the field at each point in space is a simple quantum harmonic oscillator. Excitations of the field (oscillator) correspond to the elementary particles of particle physics. However, even the vacuum has a complex structure, all calculations must be made in relation to such model of the vacuum.

The Casimir effect at finite temperature is the integral part of the finite-temperature ($T \neq 0$) QED, QFT and also quantum chromodynamics (QCD) which usually deal with the specific processes in the heat bath of photons or other particles (Donoghue et al., 1985). The heat bath can be formed by different kinds of elementary particles and so such different hot media have a different influence on the same specific physical process developing in the media. We consider here the influence of the heat bath photons on the energy shift inside of the thermal box, leading to the attraction of the capacitor plates with a separation a.

10.1 Casimir effect at zero temperature

In order to understand the Casimir effect at zero temperature, we follow Holstein (1992) and imagine two capacitor plates with a separation a. The field modes permitted by the boundary condition have the electrical intensity vanishing on the surface on the plates. If the normal to the surface defines the z-direction, then for the propagation in this direction wavelength varies from zero to a. If the zero point energy of the oscillators representing the quantum field is $\hbar\omega_k/2$ (Berestetsky et al., 1989), then the total energy between the plates is given by the formula

$$U(a) = \sum_k \frac{1}{2}\hbar\omega_k. \tag{1}$$

When the plate separation is increased, more modes are permitted so the energy is increasing function of separation a. In case that the separation a is lowered, then the energy is also lowered which means that the change of energy is force of the form:

$$F = -\frac{\partial U(a)}{\partial a}. \tag{2}$$

The force has been detected for instance by Sparnay (1958) and represents the macroscopic manifestation of the validity of quantum field theory.

The quantitative evaluation of the Casimir force is as follows. Let be wave numbers k_x, k_z in the x, y direction. Then the density of states is given by the formula

$$A\int \frac{d^2k}{(2\pi)^2}, \tag{3}$$

where A is the area of the plates.

In the z-direction, on the other hand, the boundary conditions $\mathbf{E}(0) = \mathbf{E}(a) = 0$ requires

$$E \sim \sin(k_z z) \tag{4}$$

with

$$k_z = \frac{n\pi}{a}; \qquad n = 1, 2, ... \tag{5}$$

The frequencies are

$$\omega_k = \sqrt{k_x^2 + k_y^2 + \left(\frac{n\pi}{a}\right)^2}. \tag{6}$$

The total vacuum energy of photons (with two polarizations) between plates is evidently as follows:

$$U(a) = 2\sum_{n=1}^{\infty} A \int \frac{d^2k}{(2\pi)^2} \frac{1}{2}\omega_k. \tag{7}$$

Defining

$$k = \sqrt{k_x^2 + k_y^2}, \tag{8}$$

we have from eq. (5)

$$kdk = \omega d\omega \tag{9}$$

and the new mathematical form of the total intermediate vacuum energy is

$$U(a) = A\sum_{n=1}^{\infty} \frac{1}{2\pi} \int_{\frac{n\pi}{a}}^{\infty} d\omega\omega^2. \tag{10}$$

Using the cutoff operation with $\exp(-\varepsilon\omega)$, we get the following formulas:

$$U(a) = \frac{A}{2\pi}\sum_{n=1}^{\infty} \int_{\frac{n\pi}{a}}^{\infty} d\omega\omega^2 e^{-\varepsilon\omega} = \frac{A}{2\pi}\frac{d^2}{d\varepsilon^2}\sum_{n=1}^{\infty} \int_{\frac{n\pi}{a}}^{\infty} d\omega e^{-\varepsilon\omega} =$$

$$\frac{A}{2\pi}\frac{d^2}{d\varepsilon^2}\sum_{n=1}^{\infty} \frac{1}{\varepsilon} e^{-\frac{n\pi\varepsilon}{a}} = \frac{A}{2\pi}\frac{d^2}{d\varepsilon^2}\frac{1}{\varepsilon}\left(\frac{1}{1-e^{\frac{\varepsilon\pi}{a}}} - 1\right). \tag{11}$$

After application the formula with the Bernoulli numbers B_n (Prudnikov et al., 1984)

$$\frac{1}{1-e^{-t}} = -\sum_{n=1}^{\infty} B_n \frac{t^{n-1}}{n!}, \tag{12}$$

we get for $\varepsilon \to 0$ the final formula for the attraction of two plates immersed in the quantum vacuum (Holstein, 1992):

$$\frac{1}{A}F = -\frac{\partial}{\partial a}\frac{1}{A}U(a) = -\frac{\pi^2}{240a^4}. \tag{13}$$

Now, we can approach the calculation of the attractive force due to the photons of the blackbody sea.

10.2 The thermal Casimir effect due to blackbody photons

The blackbody photons are supposed in the box with the edges l_l, l_2, l_3 and the situation is the analogue of the quantum mechanical particle inside such box. However with regard to the fact that the photon gas has the temperature T, it is necessary to perform the following transformation to the thermodynamical system in the box:

$$U(a) = \sum_k \frac{1}{2}\hbar\omega_k \to \sum_k \left(\frac{\omega_k^2}{\pi c^3}\right) \frac{\hbar\omega_k}{e^{\frac{\hbar\omega_k}{k_B T}} - 1} \tag{14}$$

with

$$\omega_k = \omega_{n_1,n_2,n_3} = \sqrt{\left(\frac{n_1\pi}{l_1}\right)^2 + \left(\frac{n_2\pi}{l_2}\right)^2 + \left(\frac{n_3\pi}{l_3}\right)^2}. \tag{15}$$

So, the energy of photons in the photon sea is

$$U(a) = \sum_{n_1,n_2,n_3} \left(\frac{\omega_{n_1,n_2,n_3}^2}{\pi c^3}\right) \frac{\hbar\omega_{n_1,n_2,n_3}}{e^{\frac{\hbar\omega_{n_1,n_2,n_3}}{k_B T}} - 1}. \tag{16}$$

It is elementary statement that if $l_1 \to \infty, l_2 \to \infty, l_3 \to \infty$, we get the classical Planck distribution

$$\varrho(\omega) \to \left(\frac{\omega^2}{\pi c^3}\right) \frac{\hbar\omega}{e^{\frac{\hbar\omega}{k_B T}} - 1} \tag{17}$$

with (Feynman, 1972; Isihara, 1971)

$$U(blackbody) = \int_0^\infty \varrho(\omega) d\omega = \sigma T^4; \quad \sigma = \frac{\pi^2 (k_B T)^4}{15\hbar^3 c^3}. \tag{18}$$

The force in the x-direction is

$$F_x = -\frac{\partial U(l_1, l_2, l_3}{\partial l_1} = \sum_{n_1,n_2,n_3} \left(\frac{\hbar}{\pi c^3}\right) \left(\frac{n_1\pi}{l_1}\right)^2 \frac{1}{l_1} \quad \times$$

$$\left[\frac{3\omega}{e^{\frac{\hbar\omega}{k_B T}} - 1} - \frac{\omega^2 e^{\frac{\hbar\omega}{k_B T}}}{\left(e^{\frac{\hbar\omega}{k_B T}} - 1\right)^2} \frac{\hbar}{k_B T}\right]. \tag{19}$$

The force in the y-direction is

$$F_y = -\frac{\partial U(l_1, l_2, l_3)}{\partial l_2} = \sum_{n_1,n_2,n_3} \left(\frac{\hbar}{\pi c^3}\right) \left(\frac{n_2\pi}{l_2}\right)^2 \frac{1}{l_2} \quad \times$$

$$\left[\frac{3\omega}{e^{\frac{\hbar\omega}{k_B T}} - 1} - \frac{\omega^2 e^{\frac{\hbar\omega}{k_B T}}}{\left(e^{\frac{\hbar\omega}{k_B T}} - 1\right)^2} \frac{\hbar}{k_B T}\right] \tag{20}$$

and the force in the z-direction is

$$F_z = -\frac{\partial U(l_1,l_2,l_3)}{\partial l_3} = \sum_{n_1,n_2,n_3}\left(\frac{\hbar}{\pi c^3}\right)\left(\frac{n_3\pi}{l_3}\right)^2\frac{1}{l_3} \quad \times$$

$$\left[\frac{3\omega}{e^{\frac{\hbar\omega}{k_BT}}-1} - \frac{\omega^2 e^{\frac{\hbar\omega}{k_BT}}}{\left(e^{\frac{\hbar\omega}{k_BT}}-1\right)^2}\frac{\hbar}{k_BT}\right]. \tag{21}$$

The specific pressure on the unit area l_2l_3, l_1l_3, l_1l_2. is

$$p_{23} = \frac{1}{l_2l_3}F_x = -\frac{1}{l_2l_3}\frac{\partial U(l_1,l_2,l_3)}{\partial l_1}, \tag{22}$$

$$p_{13} = \frac{1}{l_1l_3}F_y = -\frac{1}{l_1l_3}\frac{\partial U(l_1,l_2,l_3)}{\partial l_2}, \tag{23}$$

$$p_{12} = \frac{1}{l_1l_2}F_z = -\frac{1}{l_1l_2}\frac{\partial U(l_1,l_2,l_3)}{\partial l_3}. \tag{24}$$

In case of the equal edges of the thermal bath i.e. $l_1 = l_2 = l_3 = l$, the specific pressures are equal and it means that

$$p = \frac{1}{3l^5}\sum_{n_1,n_2,n_3}\left(\frac{\hbar}{\pi c^3}\right)\left[\left(\frac{n_1\pi}{l}\right)^2+\left(\frac{n_2\pi}{l}\right)^2+\left(\frac{n_3\pi}{l}\right)^2\right] \quad \times$$

$$\left[\frac{3\omega}{e^{\frac{\hbar\omega}{k_BT}}-1} - \frac{\omega^2 e^{\frac{\hbar\omega}{k_BT}}}{\left(e^{\frac{\hbar\omega}{k_BT}}-1\right)^2}\frac{\hbar}{k_BT}\right]. \tag{25}$$

Let us remark that the three-dimensional sums in eqs. (16), (19–22), (23–25) is not easy to calculate because they are not considered as the integral part of the standard mathematics. So, we can simplify the calculation by the so called continual limit. In other words, we perform replacing of the the sum by the ω-integral and for eq. (25) we get:

$$p = \frac{1}{3l^5}\left(\frac{\hbar}{\pi c^3}\right)\int_0^\infty d\omega\omega^2\left[\frac{3\omega}{e^{\frac{\hbar\omega}{k_BT}}-1} - \frac{\omega^2 e^{\frac{\hbar\omega}{k_BT}}}{\left(e^{\frac{\hbar\omega}{k_BT}}-1\right)^2}\frac{\hbar}{k_BT}\right]. \tag{26}$$

Now, we are prepared to evaluate the ω-integral in the last formula. Putting

$$x = \frac{\hbar\omega}{k_BT}; \quad \omega = \frac{xk_BT}{\hbar}; \quad d\omega = dx\frac{k_BT}{\hbar}; \quad C = \frac{k_BT}{\hbar}, \tag{27}$$

we get equation in the following form:

$$p = \frac{1}{3l^5}\left(\frac{\hbar}{\pi c^3}\right)\int_0^\infty dxC^5\left[\frac{3x^3}{e^x-1} - \frac{x^4e^x}{(e^x-1)^2}\right]. \tag{28}$$

According to textbook (Rumer et al., 1977)

$$\int_0^\infty dx \frac{x^n}{e^x - 1} = \Gamma(n+1)\zeta(n+1). \tag{29}$$

and (Prudnikov et al., 1984)

$$\int_0^\infty dx \frac{x^{2n} e^x}{(e^x - 1)^2} = 2^{2n-1}\pi^4 |B_{2n}|. \tag{30}$$

In case of the specification of n, we get (Rumer et al., 1977)

$$\int_0^\infty dx \frac{x^3}{e^x - 1} = \Gamma(4)\zeta(4) = 3! \left(\frac{\pi^4}{90}\right) \tag{31}$$

and (Prudnikov et al., 1984)

$$\int_0^\infty dx \frac{x^4 e^x}{(e^x - 1)^2} = 2^3\pi^4 \left|-\frac{1}{30}\right| = 2^3\pi^4 \frac{1}{30}, \tag{32}$$

where

$$|B_4| = \left|-\frac{1}{30}\right| = 1/30 \tag{33}$$

follows from the general formula (12).

So, the final formula for the so called Casimir effect at finite temperature is the numerical form of the formula (28). Or,

$$p = \frac{1}{3l^5}\left(\frac{\hbar}{\pi c^3}\right)\left(\frac{k_B T}{\hbar}\right)^5 \left[3.3!\left(\frac{\pi^4}{90}\right) - 2^3\left(\frac{\pi^4}{30}\right)\right]. \tag{34}$$

The last author formula is the original one and it was not published in the scientific physical research journals. The submitted approach can be easily generalized to phonon thermal bath, magnon thermal bath and and so on, or astrophysical thermal bath.

10.3 The quantum pressure

We have seen how the thermal photons with the Planck blackbody statistics generated the Casimir effect at finite temperature. The motivation for considering such problem can be seen in quantum mechanics with the electron confined in the box with the infinite barriers at point 0 and l. Then, the energy levels of electron inside the box is (Sokolov et al. 1962)

$$E_n = \frac{\pi^2\hbar^2 n^2}{2ml^2} \tag{35}$$

and the corresponding wave function is

$$\psi_n = \sqrt{\frac{2}{l}}\sin\left(\pi n \frac{x}{l}\right). \tag{36}$$

The quantum pressure caused by the quantum mechanical motion of particle is obtained by the same operation as in the Casimir effect. Or,

$$F = -\frac{\partial E_n}{\partial l} = \frac{\pi^2\hbar^2 n^2}{ml^3}. \quad (37)$$

In case that the thermal box is three dimensional, we get (Sokolov et al., 1962)

$$E_{n_1,n_2,n_3} = \frac{\pi^2\hbar^2}{2m}\left[\left(\frac{n_1}{l_1}\right)^2 + \left(\frac{n_2}{l_2}\right)^2 + \left(\frac{n_3}{l_3}\right)\right] \quad (38)$$

and the corresponding wave function is

$$\psi_{n_1,n_2,n_3} = \sqrt{\frac{8}{l_1l_2l_3}}\sin\left(\pi n_1\frac{x}{l_1}\right)\sin\left(\pi n_2\frac{x}{l_2}\right)\sin\left(\pi n_3\frac{x}{l_3}\right). \quad (39)$$

The corresponding pressures are

$$p_{23} = -\frac{1}{l_2l_3}\frac{\partial E_{n_1,n_2,n_3}}{\partial l_1} \quad (40)$$

$$p_{13} = -\frac{1}{l_1l_3}\frac{\partial E_{n_1,n_2,n_3}}{\partial l_2} \quad (41)$$

$$p_{12} = -\frac{1}{l_1l_2}\frac{\partial E_{n_1,n_2,n_3}}{\partial l_3}. \quad (42)$$

Let us only remark that the quantum pressure derived here is the perfect proof that the wave function in quantum mechanics is physical reality independent on the human mind, and not only mathematical object. The wave function is in such a way the objective form of matter, where matter is continuum which forms Universe.

The article is the continuation of the previous and related problems in the finite-temperature physics published by author (Pardy, 1989a, 1989b, 1994b, 2013a, 2013b).

Information on the systematic examination of the finite temperature effects in quantum electrodynamics (QED) at one-loop order was given by Donoghue, Holstein and Robinett (1985). They have treated the calculation of mass, charge, wave function renormalization and so on, and demonstrated the running of the coupling constant at finite temperature and discussed the normalized vertex function and the energy momentum tensor.

11 Cherenkov effect in the two-dimensional medium

The fast moving charged particle in a medium when its speed is faster than the speed of light in this medium produces electromagnetic radiation which is called the Vavilov-Čerenkov radiation.

The prediction of Cherenkov radiation came long ago. Heaviside (1889) investigated the possibility of a charged object moving in a medium faster than electromagnetic waves in the same medium becomes a source of directed electromagnetic radiation. Kelvin (1901) presented an idea that the emission of particles is possible at a speed greater than that of light. Somewhat later, Sommerfeld (1904) proposed the hypothetical radiation with a sharp angular distribution. However, in fact, from experimental point of view, the electromagnetic Cherenkov radiation was first observed in the early 1900's by experiments developed by Marie and Pierre Curie when studying radioactivity emission. In essence they observed the emission of a bluish-white light from transparent substances in the neighborhood of strong radioactive source. But the first attempt to understand the origin of this was made by Mallet (1926, 1929a, 1929b) who observed that the light emitted by a variety of transparent bodies placed close to a radioactive source always had the same bluish-white quality, and that the spectrum was continuous, with no line or band structure characteristic of fluorescence.

Unfortunately, these investigations were forgotten for many years. Cherenkov (or, Čerenkov) experiments was performed at the suggestion of Vavilov who opened a door to the true physical nature of this effect[1] (Bolotovsky, 2009).

This radiation was first theoretically interpreted by Tamm and Frank (1937) in the framework of the classical electrodynamics. The source theoretical description of this effect was given by Schwinger et al. (1976) at the zero temperature regime and the classical spectral formula was generalized to the finite temperature situation and for the massive photons by autor (Pardy, 1989a, 1997b). The Vavilov-Cherenkov effect was also used by author (Pardy, 1997) to possible measurement of the Lorentz contraction.

We derive, in the following text, by the Schwinger source theory method (Schwinger, 1970), the power spectrum of photons, generated by charged particle moving within 2D sheet, with index of refraction n. Some graphene-like structures, for instance graphene with implanted ions, or, also 2D-glasses, are dielectric media, enabling the experimental realization of the Vavilov-Cherenkov radiation. The relation of the Vavilov-Cherenkov radiation to LED, where the 2D the additional dielectric sheet is the integral part of LED, is discussed. It is not excluded that LEDs with the 2D dielectric sheets will be the crucial components of detectors in experimental particle physics.

11.1 Source theory of the Vavilov-Cherenkov effect

Let us start with the three dimensional source theory formulation of the problem. Source theory (Schwinger et al., 1976) is the theoretical construction which uses quantum-mechanical particle language. Initially it was constructed for description of the particle physics situations occurring in the high-energy physics experiments. However, it was found that the original formulation simplifies the calculations in the electrodynamics and gravity where the interactions are mediated by photon or graviton respectively.

The basic formula in the source theory is the vacuum to vacuum amplitude:

$$< 0_+|0_- >= e^{\frac{i}{\hbar}W(S)}, \tag{1}$$

where the minus and plus tags on the vacuum symbol are causal labels, referring to any time before and after space-time region where sources are manipulated. The exponential form is introduced with regard to the existence of the physically independent experimental arrangements which has a simple consequence that the associated probability amplitudes multiply and corresponding W expressions add.

The electromagnetic field is described by the amplitude (1) with the action

$$W(J) = \frac{1}{2c^2}\int (dx)(dx')J^{\mu}(x)D_{+\mu\nu}(x-x')J^{\nu}(x'), \tag{2}$$

[1]So, the adequate name of this effect is the Vavilov-Cherenkov effect, (or, Čerenkov effect). In the English literature, however, it is usually called the Cherenkov effect.

where the dimensionality of $W(J)$ is the same as the dimensionality of the Planck constant $\hbar$. J_μ is the charge and current densities, where quantity J_μ is conserved. The symbol $D_{+\mu\nu}(x-x')$, is the photon propagator and its explicit form will be determined later.

It may be easy to show that the probability of the persistence of vacuum is given by the following formula (Schwinger et al., 1976):

$$|<0_+|0_->|^2 = \exp\{-\frac{2}{\hbar}\mathrm{Im}\,W\} \stackrel{d}{=} \exp\{-\int dtd\omega\frac{P(\omega,t)}{\hbar\omega}\}, \tag{3}$$

where we have introduced the so called power spectral function $P(\omega,t)$ (Schwinger et al., 1976). In order to extract this spectral function from $\mathrm{Im}\,W$, it is necessary to know the explicit form of the photon propagator $D_{+\mu\nu}(x-x')$.

The electromagnetic field is described by the four-potentials $A^\mu(\varphi,\mathbf{A})$ and it is generated, including a particular choice of gauge, by the four-current $J^\mu(c\varrho,\mathbf{J})$ according to the differential equation, (Schwinger et al., 1976):

$$\left(\Delta - \frac{\mu\varepsilon}{c^2}\frac{\partial^2}{\partial t^2}\right)A^\mu = \frac{\mu}{c}\left(g^{\mu\nu} + \frac{n^2-1}{n^2}\eta^\mu\eta^\nu\right)J_\nu \tag{4}$$

with the corresponding Green function $D_{+\mu\nu}$:

$$D_+^{\mu\nu} = \frac{\mu}{c}(g^{\mu\nu} + \frac{n^2-1}{n^2}\eta^\mu\eta^\nu)D_+(x-x'), \tag{5}$$

where $\eta^\mu \equiv (1,\mathbf{0})$, μ (in the fraction μ/c)is the magnetic permeability of the dielectric medium with the dielectric constant ε, c is the velocity of light in vacuum, n is the index of refraction of this medium, and $D_+(x-x')$ was derived by (Schwinger et al., 1976) in the following form:

$$D_+(x-x') = \frac{i}{4\pi^2 c}\int_0^\infty d\omega\frac{\sin\frac{n\omega}{c}|\mathbf{x}-\mathbf{x}'|}{|\mathbf{x}-\mathbf{x}'|}e^{-i\omega|t-t'|}. \tag{6}$$

Using formulas (2), (3), (5) and (6), we get for the power spectral formula the following expression (Schwinger et al., 1976) :

$$P(\omega,t) = -\frac{\omega}{4\pi^2}\frac{\mu}{n^2}\int d\mathbf{x}d\mathbf{x}'dt'\frac{\sin\frac{n\omega}{c}|\mathbf{x}-\mathbf{x}'|}{|\mathbf{x}-\mathbf{x}'|}\cos[\omega(t-t')]\times$$
$$\times\left\{\varrho(\mathbf{x},t)\varrho(\mathbf{x}',t') - \frac{n^2}{c^2}\mathbf{J}(\mathbf{x},t)\cdot\mathbf{J}(\mathbf{x}',t')\right\}. \tag{7}$$

11.2 The two-dimensional Vavilov-Cherenkov effect

Now, we apply the last formula to the situations of the two-dimensional dielectric medium. We derive here the power spectrum of photons generated by charged particle moving within the plane of the graphene-like structure with index of refraction n. However, we cannot immediately apply the formula (7) to the graphene-like 2D structures because the index of refraction n is $n(x,y,z) = 1, z > 0$, $n(x,y,z) = const > 1, z = 0$ and $n(x,y,z) = 1, z < 0$. It means that the situation is not the Vavilov-Cherenkov problem but the problem with the transition radiation which was solved by Ginzburg and Tsytovich (1984) for thin dielectric film. The problem of the transition radiation when electron is moving with the arbitrary angle with respect to the boundary is discussed by Bass et al. (1965). Our goal is to solve only the Vavilov-Cherenkov radiation of charge when moving within the plane of dielectric sheet. So, it needs some modified approach.

While the graphene sheet is conductive, some graphene-like structures, for instance graphene with implanted ions, or, also 2D-glasses, are dielectric media, and it means that it enables the experimental realization of the Vavilov-Cherenkov radiation. Some graphene-like structure can be represented by graphene-based polaritonic crystal sheet (Bludov et al., 2012) which can be used to study the Vavilov-Cherenkov effect. We calculate it from the viewpoint of the Schwinger theory of sources (Schwinger, 1970).

The charge and current density of electron moving with the velocity $\mathbf{v}$ and charge e is as it is well known:

$$\varrho = e\delta(\mathbf{x} - \mathbf{v}t) \tag{8}$$

$$\mathbf{J} = e\mathbf{v}\delta(\mathbf{x} - \mathbf{v}t). \tag{9}$$

In case of the the two-dimensional Vavilov-Cherenkov radiation by source theory formulation, the form of equations (2) and (3) is the same with the difference that $\eta^\mu \equiv (1, \mathbf{0})$ has two space components, or $\eta^\mu \equiv (1, 0, 0)$, and the Green function D_+ as the propagator must be determined by the two-dimensional procedure. In other words, the Fourier form of this propagator is with $(dk) = dk^0 d\mathbf{k} = dk^0 dk^1 dk^2 = dk^0 k dk d\theta$

$$D_+(x - x') = \int \frac{(dk)}{(2\pi)^3} \frac{1}{\mathbf{k}^2 - n^2(k)^2} e^{ik(x-x')}, \tag{10}$$

or, with $R = |\mathbf{x} - \mathbf{x}'|$

$$D_+(x - x') = \frac{1}{(2\pi)^3} \int_0^{2\pi} d\theta \int_0^\infty k dk \int_{-\infty}^\infty \frac{d\omega}{c} \frac{e^{ikR\cos\theta - i\omega(t-t')}}{k^2 - \frac{n^2\omega^2}{c^2} - i\varepsilon}. \tag{11}$$

Using $\exp(ikR\cos\theta) = \cos(kR\cos\theta) + i\sin(kR\cos\theta)$ and $(z = kR)$

$$\cos(z\cos\theta) = J_0(z) + 2\sum_{n=1}^\infty (-1)^n J_{2n}(z) \cos 2n\theta \tag{12}$$

and

$$\sin(z\cos\theta) = \sum_{n=1}^\infty (-1)^n J_{2n-1}(z) \cos(2n-1)\theta, \tag{13}$$

where $J_n(z)$ are the Bessel functions (Kuznetsov, 1962), we get after integration over θ:

$$D_+(x - x') = \frac{1}{(2\pi)^2} \int_0^\infty k dk \int_{-\infty}^\infty \frac{d\omega}{c} \frac{J_0(kR)}{k^2 - \frac{n^2\omega^2}{c^2} - i\varepsilon} e^{-i\omega(t-t')}. \tag{14}$$

The ω-integral in (14) can be performed using the residuum theorem after integration in the complex half ω-plane.

The result of such integration is the propagator D_+ in the following form:

$$D_+(x - x') = \frac{i}{2\pi c} \int_0^\infty d\omega J_0\left(\frac{n\omega}{c}|\mathbf{x} - \mathbf{x}'|\right) e^{-i\omega|t-t'|}. \tag{15}$$

The spectral formula for the two-dimensional Vavilov-Cherenkov radiation is the analogue of the formula (7), or,

$$P(\omega,t) = -\frac{\omega}{2\pi}\frac{\mu}{n^2}\int d\mathbf{x}d\mathbf{x}'dt' J_0\left(\frac{n\omega}{c}|\mathbf{x}-\mathbf{x}'|\right)\cos[\omega(t-t')]\times$$

$$\times\left\{\varrho(\mathbf{x},t)\varrho(\mathbf{x}',t') - \frac{n^2}{c^2}\mathbf{J}(\mathbf{x},t)\cdot\mathbf{J}(\mathbf{x}',t')\right\}, \tag{16}$$

where the charge density and current involves only two-dimensional velocities and integration is also only two-dimensional.

The difference is in the replacing mathematical formulas as follows:

$$\frac{\sin\frac{n\omega}{c}|\mathbf{x}-\mathbf{x}'|}{|\mathbf{x}-\mathbf{x}'|} \quad\longrightarrow\quad J_0\left(\frac{n\omega}{c}|\mathbf{x}-\mathbf{x}'|\right). \tag{17}$$

So, After insertion the quantities (8) and (9) into (16), we get:

$$P(\omega,t) = \frac{e^2}{2\pi}\frac{\mu\omega v}{c^2}\left(1-\frac{1}{n^2\beta^2}\right)\int dt' J_0\left(\frac{nv\omega}{c}|t-t'|\right)\cos[\omega(t-t')], \quad \beta = v/c, \tag{18}$$

where the t'-integration must be performed. Putting $\tau = t'-t$, we get the final formula:

$$P(\omega,t) = \frac{e^2}{2\pi}\frac{\mu\omega v}{c^2}\left(1-\frac{1}{n^2\beta^2}\right)\int_{-\infty}^{\infty} d\tau J_0\left(n\beta\omega\tau\right)\cos(\omega\tau), \quad \beta = v/c. \tag{19}$$

The integral in formula (19) is involved in the tables of integrals (Gradshteyn et al., 1962). Or,

$$J = \int_0^{\infty} dx J_0\left(ax\right)\cos(bx) = \frac{1}{\sqrt{a^2-b^2}}, \quad 0 < b < a,$$

$$J = \infty,\ a = b;\ J = 0, \quad 0 < a < b, \tag{20}$$

In our case we have $a = n\beta\omega$ and $b = \omega$. So, the power spectrum in eq. (19) is as follows with $J_0(-z) = J_0(z)$:

$$P = \frac{e^2}{\pi}\frac{\mu v}{c^2}\left(1-\frac{1}{n^2\beta^2}\right)\frac{2}{\sqrt{n^2\beta^2-1}}, \quad n\beta > 1,\ \beta = v/c. \tag{21}$$

and

$$P = 0;\ n\beta < 1, \tag{22}$$

which means that the physical meaning of the quantity P is really the Vavilov-Cherenkov radiation. And it is in our case the two-dimensional form of this radiation.

The fundamental features of the 3D and 2D Vavilov-Cherenkov radiation are as follows:
1) The radiation arises only for particle velocity greater than the velocity of light in the dielectric medium.
2) It depends only on the charge and not on mass of the moving particles
3) The radiation is produced in the visible interval of the light frequencies and partly in the ultraviolet part of the frequency spectrum. The radiation does not exists for very

short waves, which follows from the dispersion theory of the index of refraction n, where $n < 1$.
4) The spectral dependency on the frequency is linear for the 3D homogeneous medium.
5) The radiation generated in the 3D medium at given point of the trajectory spreads on the surface of the Mach cone with the vertex at this point and with the axis identical with the direction of motion of the particle. The vertex angle of the cone is given by the relation $\cos\Theta = c/nv$.
6) There is no Mach cone in the 2D dielectric medium. There is only the Mach angle in the 2D sheet. It follows from the fact that Vavilov-Cherenkov effect is the result of the collective motion of the 2D dielectric medium and it also follows from the quantum definition of the Vavilov-Cherenkov effect in the 2D structures. The conservation laws of momentum and energy for the Vavilov-Cherenkov effect is as follows:

$$\mathbf{p}_i = \mathbf{p}_f + \hbar\mathbf{k}, \tag{23}$$

$$E_i = E_f + \hbar\omega, \tag{24}$$

where index i concerns the initial momentum and energy of an electron and index f concerns the final momentum and energy of an electron. Symbol $\mathbf{k}$ is the wave vector of emitted photon and $\hbar\omega$ is its energy. With regard to the situation that the motion of an electron is realized in the plane $x-y$, the 3D Mach cone cannot be realized (The existence of Mach cone in our situation is the nonphysical escape of photons from 2D plane to the extra-dimension). So, the nonexistence of the Mach cone in the 2D structures is not mysterious.

While the formula for the three dimensional (3D) Vavilov-Cherenkov radiation is well known the from textbooks and monographs, the two-dimensional (2D) form of the Vavilov-Cherenkov radiation was derived here. Zuev (2009) considers the Vavilov-Cherenkov phenomenon in nanofilms from Au, Ag, Cu, where the Vavilov-Cherenkov phenomenon is realized only as the surface plasmons which cannot escape the 2D medium.

12 Velocity of sound in the black body photon sea

We determine the velocity of sound in the blackbody gas of photons. Derivation is based on the thermodynamic theory of the photon gas and the Einstein relation between energy and mass. The spectral form for the n-dimensional blackbody is derived. The 1D, 2D and 3D blackbody radiation is specified.

The spectral form of the blackbody radiation was derived firstly by Planck. The original Planck derivation of the blackbody radiation was based on the relation between the entropy of the system and the internal energy of the blackbody denoted by Planck as U.

While from the postulation of the relation

$$\frac{d^2S}{dU^2} = -\frac{const}{U} \tag{1}$$

the Wien law follows, the a priori generalization of eq. (1) gives new physics. The generalization of the equation (1) to be in harmony with blackbody thermodynamics was postulated by Planck in the following form:

$$\frac{d^2S}{dU^2} = -\frac{k}{U(\varepsilon + U)}, \tag{2}$$

where ε has the dimensionality of energy, k is the Boltzmann constant, and formula (2) is the approximation of the more general formula $d^2S/dU^2 = \alpha/\sum_n a_n U^n$ leading to exotic statistics.

The first integration of eq. (2) can be performed using the integral

$$\int \frac{dx}{x(a + bx)} = -\frac{1}{a}\ln\left|\frac{a}{x} + b\right|. \tag{3}$$

After integration we get the following result:

$$\frac{1}{T} = \frac{dS}{dU} = \frac{k}{\varepsilon}\ln\left(\frac{\varepsilon}{U} + 1\right). \tag{4}$$

The solution of eq. (4) is

$$U = \frac{\varepsilon}{\mathrm{e}^{\varepsilon/kT} - 1}. \tag{5}$$

The general validity of the Wien law

$$\frac{dS}{dU} = \frac{1}{\nu} f\left(\frac{U}{\nu}\right) \tag{6}$$

confronted with the equation (4) gives the famous Planck formula $\varepsilon = h\nu$.

The next step of Planck was to find the appropriate physical statistical system (heuristic model) which led to the correct power spectrum of the blackbody. This model was the thermal reservoir of the independent electromagnetic oscillators with the discrete energies $\varepsilon = h\nu$.

Einstein introduced coefficients of spontaneous and stimulated emission A_{mn}, B_{mn}, B_{nm}. In case of spontaneous emission, the excited atomic state decays without external stimulus as an analog of the natural radioactivity decay. Later, quantum theory explained rigorously the process of spontaneous emission. The energy of the emitted photon is given by the Bohr formula. In the process of the stimulated emission the atom is induced by the external stimulus to make the same transition. The external stimulus is a black body photon that has an energy given by the Bohr formula.

The Planck power spectral formula is as follows:

$$P(\omega)d\omega = \hbar\omega G(\omega)\frac{d\omega}{\exp\frac{\hbar\omega}{k_BT} - 1}; \quad G(\omega) = \frac{\omega^2}{\pi^2 c^3}, \tag{7}$$

where $\hbar\omega$ is the energy of a blackbody photon and $G(\omega)$ is the number of electromagnetic modes inside of the blackbody, k is the Boltzmann constant, c is the velocity of light, T is the absolute temperature.

The internal density energy of the blackbody gas is given by integration of the last equation over all frequencies ω, or

$$u = \int_0^\infty P(\omega)d\omega = aT^4; \quad a = \frac{\pi^2 k^4}{15\hbar^3 c^3}. \tag{8}$$

12.1 The speed of sound in the blackbody photon gas

In order to understand the the derivation of speed of sound in gas and in the relic photon sea, we start with the derivation of the speed of sound in the real elastic rod.

Let A be the cross-section of the element Adx of a rod, where dx is the linear infinitesimal length on the abscissa x. The $\varphi(x,t)$ let be deflection of the element Adx at point x at time t. The shift of he element Adx at point $x + dx$ is evidently

$$\varphi + \frac{\partial \varphi}{\partial x} dx. \tag{9}$$

The relative prolongation is evidently $\partial\varphi(x,t)/\partial x$. The differential equation of motion of the rod can be derived by the following obligate way. We suppose that the force tension $F(x,t)$ acting on the element Adx of the rod is given by the Hook law:

$$F(x,t) = EA\frac{\partial \varphi}{\partial x}, \tag{10}$$

where E is the Young modulus of elasticity, A is the cross section of the rod. We easily derive that

$$F(x+dx) - F(x) \approx EA\frac{\partial^2 \varphi}{\partial x^2} dx \tag{11}$$

The mass of the element Adx is ϱAdx, where ϱ is the mass density of the rod and the dynamical equilibrium is expressed by the Newton law of force:

$$\varrho Adx\varphi_{tt} = EA\varphi_{xx}dx \tag{12}$$

or,

$$\varphi_{tt} - v^2\varphi_{xx} = 0, \tag{13}$$

where

$$v = \left(\frac{E}{\varrho}\right)^{1/2} \tag{14}$$

is the velocity of sound in the rod.

The complete solution of eq. (13) includes the initial and boundary conditions. We suppose that the velocity law (14) involving modulus of elasticity and mass density is valid also for gas intercalated in the rigid cylinder tube. According to the definition of the Young modulus of elasticity where $(\Delta L/L)$ is the relative prolongation of a rod, we have as an analogue for the tube of gas $\Delta V/V$, $F \to \Delta p$, where V is the volume of a gas and p is pressure of a gas. Then, the modulus of elasticity is defined as the analogue of eq. (10). Or,

$$E = -\frac{dp}{dV}V. \tag{15}$$

The process of the sound spreading in ideal gas is the adiabatic thermodynamic process with no heat exchange. We use it later as a model of the sound spreading in the gas of blackbody photons. Such process is described by the thermodynamical equation

$$pV^\kappa = const, \tag{16}$$

where κ is the Poisson constant defined as $\kappa = c_p/c_v$, with c_p, c_v being the specific heat under constant pressure and under constant volume.

After differentiation of eq. (16) we get the following equation

$$dpV^{\kappa} + \kappa V^{\kappa-1}dV = 0, \tag{17}$$

or,

$$\frac{dp}{dV} = -\kappa\frac{p}{V}. \tag{18}$$

After inserting of eq. (18) into eq. (15), we get from eq. (14) for the velocity of sound in gas the so called Newton-Laplace formula:

$$v = \sqrt{\kappa\frac{p}{\varrho}}, \tag{19}$$

where ϱ is the mass density of gas.

The density of the equilibrium radiation is given by the Stefan-Boltzmann formula

$$u = aT^4, ; \quad a = 7,5657.10^{-16}\frac{\mathrm{J}}{\mathrm{K^4m^3}}. \tag{20}$$

.

Then, with regard to the thermodynamic definition of the specific heat

$$c_v = \left(\frac{\partial u}{\partial T}\right)_V = 4aT^3. \tag{21}$$

Similarly, with regard to the general thermodynamic theory

$$c_p = c_v + \left[\left(\frac{\partial u}{\partial V}\right)_T + p\right]\left(\frac{\partial V}{\partial T}\right)_p = c_v, \tag{22}$$

because $\left(\frac{\partial V}{\partial T}\right)_T = 0$ for photon gas and in such a way, $\kappa = 1$ for photon gas. According to the theory of relativity, there is simple equivalence between mass and energy. Namely, $m = E/c^2$. At the same time, there is relation between pressure and the internal energy of the blackbody gas following from the electromagnetic theory of light $p = u/3$. So, in our case

$$\varrho = u/c^2 = \frac{aT^4}{c^2}; \quad p = \frac{u}{3}. \tag{23}$$

So, after insertion of formulas in equation (23) in to eq. (19), we get the final formula for the velocity of sound in three photon sea of the blackbody is as follows:

$$v = c\sqrt{\frac{\kappa}{3}} = \frac{c}{3}\sqrt{3}, \tag{24}$$

which is the result derived by Partovi (1994) using the QED theory applied to the photon gas. No energy signal can move with velocity greater than the speed of light. And we correctly derived $v/c < 1$.

So, we have seen in this section, that using the classical thermodynamical model of sound in the classical gas we can easily derive some properties of the black body gas, namely the velocity of sound in it and in the relic photon sea. It is not excluded that the relic sound can be detected by the special microphones of Bell laboratories. Let us still remark that if we use van der Waals equation of state, or, the Kamerlingh Onnes virial equation of state, the obtained results will be modified with regard to the basic results.

12.2 The n-dimensional blackbody

The problem of the n-dimensional blackbody is related to the dimensionality of space and some ideas on the dimensionality of space was also discussed by many authors. The experimental facts following from QED experiments, galaxy formation and formation of the molecules DNA, prove that the external space is 3-dimensional. With regard to the Russell philosophy of mathematics, there is no possibility to prove the dimensionality of space, or, space-time, by means of pure mathematics, because the statements of mathematics are non-existential. The existence of the external world cannot be also proved by pure mathematics. However, if there is an axiomatic system related adequately to the external world and reflecting correctly the external world, then, it is possible to do many predictions in the external world by pure logic. This is the substance of exact sciences.

In case of the n-dimensional blackbody, the number of modes can be determined (Al-Jaber, 2003). We use here alternative and elementary derivation. In case we consider instead of the three-dimensional blackbody the n-dimensional blackbody, the photon energy is defined by the same manner and at the same time the statistical factor is the same as in the three-dimensional case. Only number of the electromagnetic modes $G(\omega)$ depends on dimensionality of space. We determine in this article the Planck blackbody law for the n-dimensional space..

The blackbody radiation is composed from the electromagnetic waves corresponding to photons in such a way that every monochromatic wave is of the form:$A_\mu = \varepsilon_\mu e^{i\mathbf{kx}-i\omega t}$, where ε_μ is the polarization amplitude. If we take the blackbody in the form of cube with side L, then it is necessary to apply for the electromagnetic wave the boundary conditions. It is well known that the appropriate boundary conditions are so called periodic condition, which means for instance for x-coordinate $\exp(ik_1 0) = \exp(ik_1 L) = 1$, from which follows that only specific values of k_1 correspond to the boundary conditions, namely, $k_1 = \frac{2\pi N_1}{L}; \quad N_1 = 1, 2, 3...$. In case that the electromagnetic field is in a box of the volume L^n, the wave vector $\mathbf{k}$ is quantized and the elementary volume in the k-space is

$$\Delta_{0n} = (2\pi)^n / L^n \tag{25}$$

.

The elementary volume of the n-dimensional k-space is evidently the volume dV_n between spheres with radius k and $k + dk$ (Rumer et al., 1977):

$$dV_n = d\left(\frac{2\pi^{n/2}}{n\Gamma\left(\frac{n}{2}\right)}k^n\right) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}k^{n-1}dk, \tag{26}$$

where $\Gamma(n)$ is so called Euler gamma-function defined in the internet mathematics ($http://mathworld.wolfram.com/GammaFunction.html$) as

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt; \quad \Gamma(n/2) = \frac{(n-2)!!\sqrt{\pi}}{2^{(n-1)/2}}. \tag{27}$$

The number of electromagnetic modes involved inside the spheres between k and $k+dk$ is then, with $\omega = ck$, or $k = \omega/c$ and $dk = d\omega/c$,

$$G_n(\omega)d\omega = 2 \times \frac{dV_n}{\Delta_{0n}} = 2 \times \frac{1}{2^{(n-1)}} \frac{1}{\Gamma\left(\frac{n}{2}\right)} \frac{1}{\pi^{n/2}} L^n \frac{\omega^{n-1}}{c^n} d\omega, \tag{28}$$

where isolated number 2 expresses the fact that light has 2 polarizations.

For the energetic spectrum of the Planck law of the n-dimensional black body we have

$$P_n(\omega) = \hbar\omega G_n(\omega) \frac{1}{\exp(\frac{\hbar\omega}{kT}) - 1} = 2 \times \frac{1}{2^{(n-1)}} \frac{1}{\Gamma\left(\frac{n}{2}\right)} \frac{1}{\pi^{n/2}} \hbar \frac{\omega^n}{c^n} \frac{1}{\exp(\frac{\hbar\omega}{kT}) - 1}. \tag{29}$$

The energy density of the radiation of the n-dimensional blackbody is then

$$u_n = \int_0^\infty P_n(\omega) d\omega = A_n \int_0^\infty \frac{\omega^n}{\exp(\frac{\hbar\omega}{kT}) - 1} d\omega; \quad A_n = \frac{1}{2^{(n-1)}} \frac{2\hbar}{c^n \pi^{n/2}} \frac{1}{\Gamma\left(\frac{n}{2}\right)}. \tag{30}$$

The integral in the last formula can be evaluated using well-known relations (Dwight, 1961) (int. 860.39)

$$\int_0^\infty \frac{x^p}{e^{ax} - 1} dx = \frac{\Gamma(p+1)\zeta(p+1)}{a^{p+1}} = \frac{p!\zeta(p+1)}{a^{p+1}} = \frac{p!}{a^{p+1}} \left[1 + \frac{1}{2^{p+1}} + \frac{1}{3^{p+1}} + ...\right], \tag{31}$$

where $\zeta(p)$ is so called Riemann ζ-function and $a = \hbar/kT$.

Let us test the n-dimensional Planck law and density radiation in case of n = 1, 2, and 3.

$$P_1(\omega) = 2 \times \frac{1}{\Gamma(1/2)} \frac{1}{\sqrt{\pi}} \frac{\hbar\omega}{e^{(\frac{\hbar\omega}{kT})} - 1} \frac{1}{c} \tag{32}$$

$$P_2(\omega) = 2 \times \frac{1}{2} \frac{1}{\Gamma(2/2)} \frac{1}{\pi} \frac{\hbar\omega^2}{e^{(\frac{\hbar\omega}{kT})} - 1} \frac{1}{c^2} \tag{33}$$

$$P_3(\omega) = 2 \times \frac{1}{4} \frac{1}{\Gamma(3/2)} \frac{1}{\pi^{3/2}} \frac{\hbar\omega^3}{e^{(\frac{\hbar\omega}{kT})} - 1} \frac{1}{c^3}, \tag{34}$$

and so on.

Let us remark, that P_1 corresponds to the radiation of 1D blackbody and can be verified by long carbon nanotube at temperature T. P_2 corresponds to the radiation of 2D blackbody and can be verified by the graphene sheet (Pardy, 2007b, 2010, 2011) after some geometrical modification. P_4 and further formulas cannot be realized in the 3D space with the adequate blackbody.

$$u_1 = A_1 \int_0^\infty \frac{x}{e^{ax} - 1} dx = A_1 \left(\frac{kT}{\hbar}\right)^2 1!\zeta(2) = A_1 \left(\frac{kT}{\hbar}\right)^2 \frac{\pi^2}{6}; \quad A_1 = \frac{2\hbar}{c\pi^{1/2}} \frac{1}{\Gamma\left(\frac{1}{2}\right)} \tag{35}$$

$$u_2 = A_2 \int_0^\infty \frac{x^2}{e^{ax}-1} dx = A_2 \left(\frac{kT}{\hbar}\right)^3 2!\zeta(3) = A_2 \left(\frac{kT}{\hbar}\right)^3 2,4..; \quad A_2 = \frac{\hbar}{c^2\pi} \frac{1}{\Gamma(1)} \tag{36}$$

$$u_3 = A_3 \int_0^\infty \frac{x^3}{e^{ax}-1} dx = A_3 \left(\frac{kT}{\hbar}\right)^4 3!\zeta(4) = A_3 \left(\frac{kT}{\hbar}\right)^4 6\frac{\pi^4}{90}; A_3 = \frac{\hbar}{2c^3\pi^{3/2}} \frac{1}{\Gamma\left(\frac{3}{2}\right)} \tag{37}$$

and so on, where we used tables of Dwight (1961) with formulas 48.002, 48.003, 48.004 for $\zeta(2) = \pi^2/6, \zeta(3) = 1,2020569032, \zeta(4) = \pi^4/90$

Let us remark that the formula (37) is identical with formula (8) with regard to relation $\Gamma(x+1) = x\Gamma(x)$, or, $\Gamma(3/2) = \Gamma(1/2+1) = (1/2)\Gamma(1/2) = (1/2)\pi^{1/2}$, and it is the proof of the correctness of derived formula u_3.

We have seen that our derivation of the light velocity in the blackbody photon gas was based on the classical thermodynamical model with the adiabatic process ($\delta Q = 0$), controlling the spreading of sound in the gas. The problem was not solved by Einstein, because only QED, elaborated many years later was able to give motivation for the formulation of such problem. In other words, Einstein was not motivated for such activity. Partovi (1994) derived additional radiation corrections to the Planck distribution formula and the additional correction to the speed of sound in the relic photon sea. His formula is of the form:

$$v_{sound} = \left[1 - \frac{88\pi^2\alpha^2}{2025}\left(\frac{T}{T_e}\right)^4\right] \frac{c}{\sqrt{3}}, \tag{38}$$

where α is the fine structure constant and $T_e = 5.9$ G Kelvin. We see that our formula is the first approximation in the Partovi expression.

There is rigorous statistical theory of transport of sound energy in gas based on the Boltzmann equation (Uhlenbeck et al., 1963). After application of Boltzmann equation to the photon gas, or, relic photon gas we can expect the rigorous results with regard to fact that the cross-section of the photon-photon interaction is very small. Namely, (Berestetzky et al., 1989):

$$\sigma_{\gamma\gamma} = 4,7\alpha^4 \left(\frac{c}{\omega}\right)^2; \quad \hbar\omega \ll mc^2, \tag{39}$$

and

$$\sigma_{\gamma\gamma} = \frac{973}{10125\pi} \alpha^2 r_e^2 \left(\frac{\hbar\omega}{mc^2}\right)^6; \quad \hbar\omega \gg mc^2, \tag{40}$$

where $r_e = e^2/mc^2 = 2,818 \times 10^{-13}$ cm is the classical radius of electron and $\alpha = e^2/\hbar c$ is the fine structure constant with numerical value $1/\alpha = 137,04$.

No doubt, the solution of the Boltzmann equation gives the existence of sound waves in the statistical system of particles.

13 Conclusion

We have considered a quantum phenomenon in which electrons are emitted from matter after the absorption of energy from electromagnetic radiation. Or, in other words, we discussed so called the photoelectric effect.

We have considered the classical theory of photoeffect and its extension to the nonrelativistic and relativistic quantum theory of photoeffect in the form of ionization of atoms. We have investigated the problems concerning the photoelectric effect including phonon generation and process with the initial dressed photon. We have considered also the polychromatic form of the photoeffect and the photoeffect in two-dimensional electron gas in magnetic field. As the related problem, we have calculated the H-atom in the black body sea, being related to the Gibbons-Hawking thermal bath. The related problems such as the velocity of sound in the relic photon sea, thermal Casimir effect, dielectric crystal immersed in the black-body sea and the Cherenkov radiation in the two-dimensional dielectric medium were included.

The dressed photon is here considered as the photon composed from the electron-positron pair.

The H-aton imersed in the black-body photon sea is related problem to photoeffect. Such a case is an analogue of the H-atom in the Gibbons-Hawking thermal bath and it has the astrophysical meaning (Pardy, 2016a).

The dielectric crystal immersed in the black-body is equivalent to the influence of the index of refraction on the spectral formula of the blackbody (Pardy, 2015).

The Casimir effect at temperature finite temperature is the old problem and our approach was original.

The Cherenkov radiation in the two-dimensional dielectric medium (Pardy, 2015b) is the original problem.

The calculation of the velocity of sound in the relic photon sea which is the relic astrophysical black-body (Pardy, 2013a,b) is of the astrophysical meaning.

Zuev (2009) considers the Vavilov-Cherenkov phenomenon in nanofilms from Au, Ag, Cu, where the Vavilov-Cherenkov phenomenon is realized only as the surface plasmons which cannot escape from the 2D medium. The fundamental importance of the Vavilov-Cherenkov radiation is in its use for the modern detectors of very speed charged particles in the high energy physics. The detection of the Vavilov-Cherenkov radiation enables to detect not only the existence of the particle, however, also its direction of motion and its velocity and also its charge. The two-dimensional Vavilov-Cherenkov radiation is the promising application in LED, the light-emitting diode.

The light-emitting diode, LED, consists of several layers (sheets) of semiconducting materials. The Nobel prize laureates, Isamu Akasaki (Nagoya University, Japan), Hiroshi Amano (Nagoya University, Japan), Huji Nakamura (American citizen, University of California, Santa Barbara, USA) succeeded in increasing the lamps efficiency (Royal Swedish Academy of Sciences, 2014); . White LEDs currently reach more than 300 lm/W, representing more than 50% wallplug efficiency.

The relation of the Vavilov-Cherenkov effect to LED is evident. Namely, when LED (with additional dielectric sheet) is irradiated by high-energy electrons with velocity greater than the velocity of light in the sheet, then LED produces the 2D Vavilov-Cherenkov radiation if and only if the electrons move within the dielectric sheet inside the LED. The set of small grain-sand LED (fixed in adequate viscous gel emulsion) forms then the new detector of elementary particle physics. The two-dimensional Vavilov-Cherenkov radiation was still not applied, nevertheless, it is not excluded that it is the crucial effect in LED.

REFERENCES

Al-Jaber, Sami M. (2003). Planck's spectral distribution law in N dimensions, Int. Journal of Theor. Phys. 42, No. 1, 111.

Akhiezer, A.I. and Berestetzky, V.B. *Quantum Electrodynamics*; GITTL, Moscow, (1953).

Amusia, M. Ya. *Atomic photoeffect*; Nauka, Moscow, 1987, (in Russian).

Bass, F. G. and Yakovenko, V. M. (1965). Theory of radiation from a charge passing through an electrically inhomogeneous medium, Physics-Uspekhi **8**(2), 420-444.

Berestetzky, V. B., Lifshitz, E. M. and Pitaevskii, L. P. *Quantum electrodynamics*; Moscow, NAUKA, 1989. (in Russian).

Bernard. C. W. (1974). Feynman rules for gauge theories at finite temperature, Phys. Rev. D **9**, 3312.

Bethe, H. A. (1947). The electromagnetic shift of energy levels, Phys. Rev. **72**, 339.

Bludov, Yu. V., Peres, N. M. R. and Vasilevskiy, M. I. (2012). Graphene-based polaritonic crystal, arXiv:1204.3900v1,[cond-mat.mes-hall].

Bolotovsky, B. M. (2009). Vavilov-Cherenkov radiation: its discovery and application, Physics-Uspekhi **52**(11), 1099-1110.

Cherenkov, P. A. (1934). The visible radiation of pure liquids caused by X-rays, Comptes Rendus Hebdomaclaires des Seances de l' Academic des Sciences USSR **2**, 451.

Corda, Ch. (2015a). Precise model of Hawking radiation from the tunneling mechanism, Class. and Quantum Gravity **32**, 195007.

Corda, Ch. (2015b). Quasi-normal modes: the "electrons" of Blak holes as "gravitational atoms"? Implications for the black hole information puzzle. Advances in High Energy Physics, 867601.

Corda, Ch. (2015c). Time dependent Schrödinger equation for black hole evaporation: no information loss, Annals of Physics **353**, 71.

Cox, P. H., Hellman, W. S. and Yildiz, A. (1984). Finite temperature corrections to field theory: electron mass, magnetic moment, and vacuum energy, Ann. Phys. (N.Y.) **154**, 211.

Davydov, A. S. *Quantum mechanics*; 2-nd ed., Pergamon Press, Oxford, New York, 1976.

Dittrich, W. (1978). Source methods in quantum field theory, *Fortschr. Phys.* **26**, 289.

Dolan, L. and Jackiw, R. (1974). Symmetry behavior at finite temperature, Phys. Rev. D **9**, 3320.

Donoghue, J. F., Holstein, B. R. and Robinett, R. W. (1985). Quantum electrodynamics at finite temperature, Ann. Phys. (NY) **164**, No. 2, 233.

Drukarev, G. F. *Quantum mechanics*; St. Petersburgh University, 1988, (in Russian).

Dwight, H. B. *Tables of integrals; New York*, The Macmilan Company, 1961.

Einstein, A., (1905). Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristichen Gesichtspunkt, Annalen der Physik, **17**, 132.

Einstein, A., (1965). On the Heuristic Viewpoint Concerning the Production and Transformation of Light, AJP, **33**, No. 5, May (1965). (The English translation of (Einstein, 1965)).

Einstein, A. (1917). Zur quantentheorie der Strahlung, Physikalische Zeitschrift, 18, 121.

Feynman, R. P. *Statistical mechanics*; W. A. Benjamin, Inc., Reading, Massachusetts, 1972.

Garbuny, M. *Optical Physics*; Academic Press, New York and London, 1965.

Ginzburg, V. N. and Tsytovich, V. L. *The transition radiation and the transition scattering*; Moscow, 1984. (in Russian).

Gradshteyn, J. S. and Ryzhik, I. M. *Tables of integrals, sums, series and products*; Moscow, (1962). (in Russian).

Heaviside, O. (1889). On the electromagnetic effects due to the motion of electrification through a dielectric, Philos. Mag., S. 5, **27**, 324 339.

Holstein, B. R. *Topics in advanced quantum mechanics*; Addison-Wesley Publishing Company, Redwood City, CA, USA, 1992.

Isihara, A. *Statistical mechanics*; Academic Press, New York,London, 1971.

Jackson, J.D. *Classical Electrodynamics*; 3-rd ed., John Wiley & Sons, Inc., New York, 1999.

Johansson, A. E., Peressutti, G. and Skagerstam, B. S. (1986). Quantum field theory at finite temperature: renormalization and radiative corrections, Nucl. Phys. B **278**, 324.

Kelvin, L. (1901). Nineteenth century clouds over the dynamical theory of heat and light, Philos. Mag., S. **6**, 2, 140.

Kuznetsov, D. S. *The special functions*; Moscow, State Publishing House - High School, 1962. (in Russian).

Landau, L. D. and Lifshitz, E. M. *Quantum Mechanics - Non-relativistic Theory*; Pergamon Press, Oxford, 1991.

Landau, L. D. and Pitaevskii, L. P. *Statistical Physics*; Volume 5,
Third edition, revised and enlarged, Pergamon Press, Oxford, New York,.., 1980.

Levinger, J. S. *Nuclear photo-desintegration*; Oxford University Press, 1960, (in Russian).

Lide, D. R. *CRC Handbook of Chemistry and Physics*; CRC Press/Taylor and Francis, Boca Raton, FL, 2008.

Mallet, L. (1926). Spectral research of luminescence of water and other media with gamma radiation, Comptes Rendus, **183**, 274.; ibid. (1929a)., Comptes Rendus, **187**, 222.; ibid. (1929b)., Comptes Rendus, 188, 445.

Novoselov, K.S., Geim, A.K., Morozov, S.V., et al. (2005). Two-dimensional gas of mass-less Dirac fermions in graphene, Nature, 438, pp. 197-200,

Pardy, M. (1989a). Finite-temperature Cherenkov radiation, Phys. Lett. A, **134**, No. 6, 357.

Pardy, M. (1989b). Finite-temperature gravitational Cherenkov radiation, International Journal of Theor. Physics, **34**, No. 6, 951.

Pardy, M. (1994a). The Cherenkov effect with radiative corrections, Physics Letters B **325**, 517.

Pardy, M. (1994b). The two-body potential at finite temperature, CERN.TH.7397/94.

Pardy, M. (1997). Cherenkov effect and the Lorentz contraction, Phys. Rev. A **55**, No. 3, 1647.

Pardy, M. (2004). Massive photons and the Volkov solution, International Journal of Theoretical Physics, 43(1), 127.

Pardy, M. (2007a). *The synchrotron radiation from the Volkov solution of the Dirac equation*, arXiv: hep-ph/0703102v1.

Pardy, M., (2007b). ıThe photoeffect at the low temperature graphene in the strong magnetic field, hep-ph/0707.2668v2.

Pardy, M. (2009a). *The polychromatic form of the Einstein photoelectric equation*: arXiv:0904.1283v1 [physics. gen-ph]

Pardy, M. The photoelectric effect on graphene; Scientific Research and Essays Vol. 5(12), pp. 1571-1575, 18 June, 2010 Available online at http: www.academicjournals.org/SRE ISSN 1992-2248, 2010 Academic Journals

Pardy, M. Photoeffect in graphene and axion detection by graphene; In: *Graphene Simulation*; Edited by Jian Ru Gong, Published by InTech , Janeza Trdine 9, 51000 Rijeka, Croatia, ISBN 78-953-307-556-3, 2011.

Pardy, M. (2013a). ıVelocity of sound in the relic photon sea, arXiv: General Physics (physics.gen-ph)/1303.3201.

Pardy, M. (2013b). Velocity of sound in the blackbody photon gas, Results in Physics **3**, 70.

Pardy, M. (2015a). *Dielectric crystal in the Planck black-body*, arXiv:1505.02756v1 [physics.gen-ph] 5 May 2015

Pardy, M. (2015b). The two-dimensional Vavilov-Cherenkov radiation in LED, Results in Physics 3, 6971.

Pardy, M. (2016a). Energy Shift of H-Atom Electrons Due to Gibbons- Hawking Thermal Bath, Journal of High Energy Physics, Gravitation and Cosmology, 2016, 2, 472-477: http://www.scirp.org/journal/JHEPGC ISSN Online: 2380-4335 ISSN Print: 2380-4327.

Pardy, M. (2016b). *The thermal Casimir effect due to the black-body photons*, Intell. Arch., ID 1740, JULY, 7, 2016.

Partovi, H. M. (1994). QED corrections to Planck radiation law and photon thermodynamics, Phys. Rev. D **50**, 1118.

Planck, M. (1900). Zur Theorie des Gesetzes der Energieverteilung im Normalspektrum, Verhandlungen deutsch phys. Ges. **2**, 237.; ibid: (1901). Ann. Phys. **4**, 553.

Prudnikov, A. P., Brychkov, Yu. A. and Marichev, O. I. *Integrals and Series, Elementary functions*; Moscow, Nauka, 1984. (In Russian).

Rohlf, J. W. *Modern Physics from α to Z^0*; John Willey & Sons, Inc., New York, 1994.

Royal Swedish Academy of Sciences; *The Nobel Prize in Physics*, 2014.

Rumer Yu. B. and Ryvkin, M. Sch. *Thermodynamics, statistical physics, kinetics*; Nauka, Moscow, 1977, (in Russian).

Sauter, F., (1931). Über atomeren Photoeffekt bei Grosser Härte der Anvegenden Strahlung, Ann. der Phys. **9**, 217.

Schöpf, H-G. Theorie der Wärmestrahlung in historisch-kritischer Darstellung, (Alademie/Verlag, Berlin, 1978).

Schwinger, J., Tsai, W. Y. and Erber, T. (1976). Classical and quantum theory of synergic synchrotron Cherenkov radiation, Annals of Physics (NY) **96**, 303.

Schwinger, J. *Particles, Sources and Fields*; Vol. I Addison-Wesley, Reading, 1970.

Schwinger, J. (1973). *Particles, Sources and Fields*; Addison-Wesley Publ. Comp., Reading, Mass., Vol. **2**.

Sokolov, A. A., Loskutov, Yu. M. and Ternov, I. M. *Quantum mechanics*; State Pedagogical Edition, Moscow, 1962. (in Russian).

Sokolov, A. A., Ternov, I. M., Zhukovsky, V. Tch. and Borisov, A. V. *Quantum electrodynamics*; The Moscow University Press, 1983. (in Russian).

Sommerfeld, A. (1904). Zur Elektronentheorie: II. Grundlagen für eine allgemeine Dynamik des Elektrons, Göttingen Nachr., 99, 363-439.

Sparnay, M. J. (1958). Measurement of attractive forces between flat plates, Physica, **24**, 751.

Stobbe, M., (1930). Zur Quantenmechanik photoelektrischer Prozesse, Ann. der Phys. **7**, 661.

Tamm, I. E. and Frank, I. M. (1937). The coherent radiation of a fast electron in a medium, Dokl. Akad. Nauk SSSR 14, 109.

Uhlenbeck, G. E. and Ford, G. W. (1963). *Lectures in statistical physics*; (American mathematical society, Providence, Rhode Island).

Volkov, D. M. (1935). Über eine Klasse von Lösungen der Diracschen Gleichung, Zeitschrift für Physik **94**, 250.

Weinberg, S. (1974). Gauge and global symmetries at high temperature, Phys. Rev. D **9**, 3357.

Welton, Th. (1948). Some observable effects of the quantum-mechanical fluctuations of the electromagnetic field, Phys. Rev. **74**, 1157.

Zuev, V. S. (2009). *Vavilov-Cherenkov phenomenon in metal nanofilms*, arXiv: 0907.1145, [Optics (physics.optics)].