

Five Exponential Diophantine Equations and Mayhem Problem M429

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1 Introduction

In the March 2010 issue of the journal *Crux Mathematicorum with Mathematical Mayhem*, mayhem problem M429 was proposed (see reference [1]):

Determine all positive integers a, b, c that satisfy,

$$a^{(b^c)} = (a^b)^c; \text{ or equivalently}$$

$$a^{b^c} = a^{bc}.$$

A solution, by this author, was published in the December 2010 issue of *Cruz Mathematicorum with Mathematical Mayhem* (see [2]). According to this solution, the following ordered triples of positive integers a, b, c are precisely those that satisfy the above exponential equation:

The triples of the form $(1, b, c)$, with b, c being any positive integers;
the triples of the form $(a, b, 1)$, with a, b positive integers and with $a \geq 2$;
and the triples of the form $(a, 2, 2)$ with $a \in \mathbb{Z}^+$, and $a \geq 2$.

In the language of diophantine equations, we are dealing with the three-variable diophantine equation

$$x^{(y^z)} = x^{yz}. \quad (1)$$

Accordingly, the above results can be expressed in Theorem 1 as follows.

Theorem 1. *Consider the three-variable diophantine equation, $x^{(y^z)} = x^{yz}$, over the set of positive integers \mathbb{Z}^+ . If S is the solution set of the above diophantine equation, then $S = S_1 \cup S_2 \cup S_3$, where S_1, S_2, S_3 are the pairwise disjoint sets,*

$$S_1 = \{(1, b, c) | b, c \in \mathbb{Z}^+\};$$

$$S_2 = \{(a, b, 1) | a \geq 2, a, b \in \mathbb{Z}^+\};$$

$$S_3 = \{(a, 2, 2) | a \geq 2 \text{ and } a \in \mathbb{Z}^+\}.$$

Motivated by mayhem problem M429, in this work we tackle another four exponential, three-variable diophantine equations. These are:

$$x^{(y^z)} = x^{(z^y)}, \quad (2)$$

$$x^{(y^z)} = y^{xz}, \quad (3)$$

$$x^{yz} = y^{xz}, \quad (4)$$

and

$$x^{(y^z)} = z^{xy} \quad (5)$$

In Section 2, we state Theorems 2, 3, 4, and 5. In Theorems 2, 3 and 4, the solutions sets of the diophantine equations (2), (3), and (4) are stated.

These three solution sets are determined with the aid of the two-variable exponential diophantine equation found in Section 3, whose solution set is given in Result 2.

The proofs of Theorems 2,3, and 4, are given in Section 4. The proof of Theorem 5 is presented in Section 5. In Theorem 5, some solutions to equation (5) are given.

2 The four theorems

Theorem 2. *Consider the three-variable diophantine equation (over \mathbb{Z}^+),*

$$x^{(y^z)} = x^{z^y}.$$

Let S be the solution set of this equation.

Then, $S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$, where

$$S_1 = \{(1, b, c) \mid b, c \in \mathbb{Z}^+\} \quad \text{where}$$

$$S_2 = \{(a, 1, 1) \mid a \geq 2, a \in \mathbb{Z}^+\}$$

$$S_3 = \{(a, b, b) \mid a \geq 2, b \geq 2, a, b \in \mathbb{Z}^+\}$$

$$S_4 = \{(a, 4, 2) \mid a \geq 2, a \in \mathbb{Z}^+\}$$

$$S_5 = \{(a, 2, 4) \mid a \geq 2, a \in \mathbb{Z}^+\}$$

Theorem 3. *Consider the three-variable diophantine equation (over \mathbb{Z}^+),*

$$x^{(y^z)} = y^{xz}$$

Let S be the solution set of this equation. Then, $S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$ where

$$S_1 = \{(1, 1, c) \mid c \in \mathbb{Z}^+\},$$

$$S_2 = \{(a, a, 1) \mid a \geq 2, a \in \mathbb{Z}^+\}$$

$$\text{(singleton set)} \quad S_3 = \{(4, 2, 1)\},$$

$$\text{(singleton set)} \quad S_4 = \{(2, 4, 1)\}$$

$$S_5 = \{(b^c, b, c) \mid b \geq 2, c \geq 2, b, c \in \mathbb{Z}^+\}$$

Theorem 4. Consider the three-variable diophantine equation (over \mathbb{Z}^+)

$$x^{yz} = y^{xz}.$$

Let S be its solution set. Then,

$$S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7,$$

where

$$S_1 = \{(1, 1, c) \mid c \in \mathbb{Z}^+\}$$

$$S_2 = \{(a, a, 1) \mid a \geq 2, a \in \mathbb{Z}^+\}$$

$$\text{(singleton set)} \quad S_3 = \{(4, 2, 1)\},$$

$$\text{(singleton set)} \quad S_4 = \{(2, 4, 1)\}$$

$$S_5 = \{(a, a, c) \mid a \geq 2, c \geq 2, a, c \in \mathbb{Z}^+\}$$

$$S_6 = \{(4, 2, c) \mid c \geq 2, c \in \mathbb{Z}^+\}$$

$$S_7 = \{(2, 4, c) \mid c \geq 2, c \in \mathbb{Z}^+\}$$

Theorem 5. Consider the three-variable equation (over \mathbb{Z}^+)

$$x^{(y^z)} = z^{xy}$$

(i) Let S be the set of those solutions, (x, y, z) such that at least one of x, y , or z is equal to 1. Then

$$S = \{(1, b, 1) \mid b \in \mathbb{Z}^+\}$$

(ii) The only solution (x, y, z) to the above equation, such that $x \geq 2$, $y \geq 2$, $z \geq 2$, and with $x = z$, is the triple $(2, 2, 2)$

(iii) Let F be the family of solutions (x, y, z) such that $x \geq 2$, $y \geq 2$, $z \geq 2$ and with $y = z \neq x$. Then

$$F = \{(b^b, b, b) \mid b \geq 2, b \in \mathbb{Z}^+\}$$

3 A key exponential diophantine equation

The diophantine equation, $x^y = y^x$, over the positive integers, is instrumental in determining the solution sets of the diophantine equations (2), (3), and (4). The following, Result 1, can be found in W. Sierpinski's book, "Elementary Theory of Numbers", (see reference [3]). The proof is about half a page long.

Result 1. Consider the two-variable equation, $x^y = y^x$, over the set of positive rational numbers, \mathbb{Q}^+ . Then all the solutions to this equation, with x and y being positive rationals, and with $y > x$, are given by

$$x = \left(1 + \frac{1}{n}\right)^n, \quad y = \left(1 + \frac{1}{n}\right)^{n+1},$$

where n is a positive integer: $n = 1, 2, 3, \dots$.

A simple or cursory examination of the formulas in Result 1 easily leads to Result 2. Observe that these formulas can be written in the form,

$$x = \left(\frac{n+1}{n}\right)^n, \quad y = \left(\frac{n+1}{n}\right)^{n+1}.$$

For $n = 1$, we obtain the integer solution $x = 2$ and $y = 4$. However, for $n \geq 2$, the number $\frac{n+1}{n}$ is a proper rational, i.e., a rational which is not an integer.

This is clear since n and $n + 1$ are relatively prime, and $n \geq 2$. Thus, since for $n \geq 2$, $\frac{n+1}{n}$ is a proper rational, so must be any positive integer power of $\frac{n+1}{n}$. This observation takes us immediately to Result 2 below.

Result 2. *Consider the two-variable diophantine equation (over \mathbb{Z}^+)*

$$x^y = y^x.$$

Let S be its solution set. Then, $S = S_1 \cup S_2 \cup S_3$. Where

$$S_1 = \{(a, a) \mid a \in \mathbb{Z}^+\},$$

(singleton set) $S_2 = \{(4, 2)\},$

and (singleton set) $S_3 = \{(2, 4)\}$

Result 2 is used in the proofs of Theorems 2, 3, and 4 below.

4 Proofs of Theorems 2, 3, and 4

- (1) *Proof. Theorem 2* Suppose that (a, b, c) is a solution to equation (2). We have

$$a^{(b^c)} = a^{(c^b)} \tag{6}$$

If $a = 1$, then b and c can be arbitrary positive integers; and (6) is satisfied.

If $b = 1$ and $a \geq 2$, then by (6) we get

$$a = a^c. \tag{6a}$$

Since $a \geq 2$, by inspection, we see that (6a) is satisfied only when $c = 1$.

So, we obtain the solutions of the form $(a, 1, 1)$ with $a \geq 2$. If $a \geq 2$, $b \geq 2$, and $c = 1$, equation (6) yields

$$a^b = a,$$

which is impossible with $a \geq 2$ and $b \geq 2$.

Finally, assume that $a \geq 2$, $b \geq 2$, and $c \geq 2$ in (6). Then (6) \Leftrightarrow (since $a \geq 2$) $b^c = c^b$; and by Result 2, it follows that either $b = 4$ and $c = 2$; or $b = 2$ and $c = 4$; or $b = c$. We have shown that if (a, b, c) is a positive integer solution of equation (2), then (a, b, c) must belong to one of the sets S_1, S_2, S_3, S_4 , or S_5 . Conversely, a routine calculation shows that any member of these five sets is a solution to (2). \square

- (2) *Proof. Theorem 3.* Let (a, b, c) be a solution to equation (3). We then have,

$$a^{(b^c)} = b^{ac} \tag{7}$$

If $a = 1$, then by (7), $1 = b^c$, which in turn implies $b = 1$; and c an arbitrary positive integer.

If $a \geq 2$ and $b = 1$, (7) becomes impossible for any value of c . If $a \geq 2$, $b \geq 2$, and $c = 1$, (7) yields $a^b = b^a$; and by Result 2 we must have either $a = 4$ and $b = 2$, or $a = 2$ and $b = 4$; or $a = b$. If $a \geq 2$, $b \geq 2$, $c \geq 2$. Then by (7),

$$a^{(b^c)} = (b^c)^a \tag{7a}$$

Combining (7a) with Result 2 implies that either $a = 4$ and $b^c = 2$, which is impossible since $b \geq 2$ and $c \geq 2$, or that $a = 2$ and $b^c = 4$, which gives $a = 2 = b = c$. Or, the third possibility, $a = b^c$. We have shown that if (a, b, c) is a positive integer solution of equation (3), then it must belong to one of the sets S_1, S_2, S_3, S_4 or S_5 . Conversely, a routine calculation shows that any member of these five sets is a solution to (3). \square

- (3) *Proof. Theorem 4.* Let (a, b, c) be a positive integer solution to equation (4)

$$a^{bc} = b^{ac} \tag{8}$$

If $a = 1$, we obtain $1 = b^c$; and so $b = 1$, with c being an arbitrary positive integer.

If $a \geq 2$ and $b = 1$, (8) gives $a^c = 1$, which is impossible since $a \geq 2$.

If $a \geq 2$, $b \geq 2$, and $c = 1$, we obtain from (8)

$$a^b = b^a \tag{8a}$$

Equation (8a), combined with Result 2, implies that either $a = 4$ and $b = 2$; or $a = 2$ and $b = 4$; or $a = b$. If $a \geq 2$, $b \geq 2$, and $c \geq 2$, we have from (8)

$$a^{bc} = b^{ac} \Leftrightarrow (a^b)^c = (b^a)^c \tag{8b}$$

Equation (8b) demonstrates that the c th powers of the positive integers a^b and b^a are equal. Since these two integers are greater than 1, equation (8b) implies

$$a^b = b^a$$

which once more, when combined with Result 2, implies either $a = 4$ and $b = 2$ or $a = 2$ and $b = 4$; or $a = b$.

We have shown that if (a, b, c) is a positive integer solution of equation (4), it must belong to one of the sets $S_1, S_2, S_3, S_4, S_5, S_6$, or S_7 . Conversely, a routine calculation establishes that any member of these seven sets is a solution to (4). \square

- (5) **Proof of Theorem 5** The following lemma can be easily proved by using mathematical induction. We omit the details. We will use the lemma in the proof of Theorem 5.

Lemma 1.

- (i) If $b \geq 3$, then $b^{n-1} > n$ for all positive integers $n \geq 2$.
- (ii) $2^{n-1} > n$, for all positive integers $n \geq 3$.
- (iii) If $c \geq 2$, then $c^n > n$, for all positive integers n .

Proof. **Theorem 5**

- (i) Let (a, b, c) be a solution to equation (5) with at least one of a, b, c being equal to 1.

If $a = 1$, (5) implies $1 = c^b$, and so $c = 1$ as well; and b is an arbitrary positive integer.

If $b = 1$ and $a \geq 2$ we get $a = c^a$ which is impossible if $c \geq 2$, by Lemma 1(iii); and clearly, $c \neq 1$ since $a \geq 2$.

Also, the case $b \geq 2$, $a \geq 2$, and $c = 1$ is ruled out by inspection. We conclude that if (a, b, c) is a solution to (5), with one of a, b, c being 1, then it must be of the form $(1, b, 1)$. Conversely, a straightforward calculation established that $(1, b, 1)$ is a solution of (5) for every positive integer b .

- (ii) Let (a, b, c) be a solution to (5) with $a \geq 2$, $b \geq 2$, $c \geq 2$, and $a = c$. We have, by (5), $a^{(b^a)} = a^{ab} \Leftrightarrow$ (since $a \geq 2$) $b^a = ab$, or equivalently, $b^{a-1} = a$, which, when combined with Lemma 1, parts (i) and (ii), implies that either $b \geq 3$ and $a = 1$; which is ruled out since $a \geq 2$; or alternatively, $b = 2$ and $a \leq 2$ which gives $a = 2$. We obtain $a = b = c = 2$. Conversely, $(2, 2, 2)$ is a solution to equation (5), $2^4 = 2^4$.
- (iii) Let (a, b, c) be a solution to (5) with $a \geq 2$, $b \geq 2$, $c \geq 2$, and with $b = c \neq a$. We have,

$$a^{(b^b)} = b^{ab};$$

or equivalently,

$$a^{(b^b)} = (b^b)^a \tag{9}$$

Equation (9) combined with Result 2 implies that, either $a = 4$ and $b^b = 2$ or $a = 2$ and $b^b = 4$; or $a = b^b$. The first possibility is ruled out since $b^b \geq 2^2 > 2$, because $b \geq 2$. The second possibility yields $b^b = 4$, $b = 2$, but then also have $a = 2$ and so $a = b = c = 2$, contrary to $b = c \neq a$. The third possibility establishes $(a, b, c) = (b^b, b, b)$. Conversely, (b^b, b, b) is a solution to (5) for any positive integer $b \geq 2$. Both sides of (5) are equal to $b^{(b^{b+1})}$.

□

References

- [1] *CruX Mathematicorum with Mathematical Mayhem*, **36**, No.2, March, 2010.
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