Stock Price Fluctuations in an Agent-Based Model with Market Liquidity

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Abstract

We study an agent-based stock market model with heterogeneous agents and friction. Our model is based on that of [9]: the process of a stock price in a discrete-time framework is determined by temporary equilibria via agents' excess demand functions, and the diffusion approximation approach is applied to characterize the continuous-time limit (as transaction intervals shorten) as a solution of the corresponding stochastic differential equation (SDE). In this paper we further make the assumption that some of the agents are bound by either short sale constraints or budget constraints. Then we show that the continuous-time process of the stock price can be derived from a certain SDE with oblique reflection. Moreover we find that the short sale (respectively, budget) constraint causes overpricing (respectively, underpricing).

Keywords: Agent-based models, Liquidity problems, Short sale/budget constraints, Stochastic differential equations with oblique reflection, The Skorokhod problem.

1 Introduction

It is usual in mathematical finance to describe the price evolution of a risky asset such as a stock by a diffusion process. Geometric Brownian motion (GBM) is one of the most standard such processes for price fluctuation. Because of its simplicity and convenience, the GBM model is widely used in the context of option pricing/hedging, optimal investment, and many other financial problems. An important theme is to justify GBM from the economic viewpoint. For instance, a heuristic equilibrium argument for GBM is discussed in [20]. The justification of GBM as the rational expectations equilibrium is discussed in [2] and [15].

Recently there have been various studies of agent-based market models to explain the fluctuation of a price process. One representative study is the microeconomic approach of [8] and [9]: The process of the stock price is first given as a sequence of temporary price equilibria in a discrete-time market model with heterogeneous agents and then the price process in a continuous-time model is derived as the limit as the transaction time intervals shorten. Let us

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Mathematical Subject Classification (2010) 91B24, 91B69, 60F17

JEL Classification (2010) D53, C69

introduce the outline of the model of [9]. Let I be the set of agents in the market and $\hat{e}_k^{n,i}(p,\omega)$ be trader i's excess demand function for a proposed stock price p at the time k/n, where $n \in \mathbb{N}$ and trades are executed at times $t = 0, 1/n, 2/n, \ldots$. The parameter ω is a sample point in the underlying probability space $(\Omega, \mathcal{F}, P)$. The stock price process $(S_{k/n}^n)_{k=0,1,2,\ldots}$ is given as follows: At the initial time $t = 0$, the stock price is given by $S_0^n = s_0$. Then agents exhibit their excess demand $\hat{e}_{i,0}^n(\cdot,\omega)$ and the next price is determined as a temporary equilibrium, that is, the solution p^* of

$$\sum_{i\in I} \hat{e}_{i,0}^n(p^*(\omega),\omega) = 0.$$

After transactions at $t = 0$, the stock price changes to $S_{1/n}^n = p^*$. Similarly, during the trading period k/n, the stock price before transactions is given by $S_{k/n}^n$ and agents' excess demands $(\hat{e}_k^{n,i}(\cdot,\omega))_{i\in I}$ make the stock price change to $S_{(k+1)/n}^n$. Finally, the process $(S_{k/n}^n)_k$ is given as the solution of

$$\sum_{i\in I} \hat{e}_k^{n,i}(S_{(k+1)/n}^n(\omega),\omega) = 0, \quad k = 0,1,2,\ldots, \quad S_0^n = s_0. \tag{1.1}$$

For mathematical convenience we rewrite (1.1) by using the log price $X_k^n = \log S_k^n$ of the stock, obtaining

$$\sum_{i\in I} e_k^{n,i}(X_{(k+1)/n}^n(\omega),\omega) = 0, \quad k = 1,2,\ldots, \quad X_0^n = x_0, \tag{1.2}$$

where $e_k^{n,i}(x,\omega) = \hat{e}_k^{n,i}(e^x,\omega)$ and $x_0 = \log s_0$. In [8] and [9], the individual excess demand function $e_k^{n,i}$ is assumed to be given by

$$\begin{aligned} e_k^{n,i}(x,\omega) &= \alpha_k^{n,i}(f_k^{n,i}(x,\omega) - x) + \delta_k^{n,i}(\omega), \\ f_k^{n,i}(x,\omega) &= X_{k/n}^n + \beta_k^{n,i}(X_{k/n}^n - F_i) + \gamma_k^{n,i}(X_{k/n}^n - x). \end{aligned} \tag{1.3}$$

The parameter $\delta_k^{n,i}(\omega)$ denotes the liquidity demand, $f_k^{n,i}(x)$ is the reference level of agent i, and F_i is agent i's individual perception of the fundamental (log-)value. For a more precise economic interpretation of (1.3), see Example 3) in Section 2 of [9]. In this case, (1.2) can be rewritten using the stochastic difference equation

$$X_{(k+1)/n}^n - X_{k/n}^n = \bar{\beta}_k^n(X_{k/n}^n - \bar{F}) + \bar{\delta}_k^n, \quad k = 0,1,2,\ldots, \tag{1.4}$$

where

$$\bar{\beta}_k^n = \frac{\sum_{i\in I}\alpha_k^{n,i}\beta_k^{n,i}}{\bar{\alpha}_k^n}, \quad \bar{F}_k^n = \frac{\sum_{i\in I}\alpha_k^{n,i}\beta_k^{n,i}F_i}{\bar{\alpha}_k^n}, \quad \bar{\delta}_k^n = \frac{\sum_{i\in I}\delta_k^{n,i}}{\bar{\alpha}_k^n}, \quad \bar{\alpha}_k^n = \sum_{i\in I}\alpha_k^{n,i}(1+\gamma_k^{n,i}).$$

Let $(X_t^n)_{t\geq 0}$ be an interpolated process of $(X_{k/n}^n)_k$ such as either

$$X_t^n = X_{k/n}^n \quad t \in (k/n, (k+1)/n) \tag{1.5}$$

or

$$X_t^n = (nt-k)X_{(k+1)/n}^n + (k+1-nt)X_{k/n}^n \quad t \in (k/n,(k+1)/n). \tag{1.6}$$

Then, under some mathematical assumptions, the process $(X_t^n)_t$ converges to an Ornstein–Uhlenbeck (OU) process of the form

$$dX_t = \bar{\beta}(X_t - \bar{F})dt + \bar{\sigma}dB_t, \quad X_0 = x_0.$$

(This is a simplified version of the result of [9]: They also treat an OU process in a random environment.) This implies that the continuous-time stock price $S_t = \exp(X_t)$ is a geometric OU process.

It is meaningful to consider the above diffusion approximation approach to derive the continuous-time process in a more general framework. A diffusion approximation for solutions of stochastic difference equation in the following form

$$X^n_{(k+1)/n} - X^n_{k/n} = \frac{1}{\sqrt{n}} F^n_k(X^n_{k/n}, \omega) + \frac{1}{n} G^n_k(X^n_{k/n}, \omega), \tag{1.7}$$

with $\mathrm{E}[F^n_k(x)] = 0$, is studied in [17] and [23]–[25] under some mixing conditions. In [13], a case of the functional difference equation

$$X^n_{(k+1)/n} - X^n_{k/n} = \frac{1}{\sqrt{n}} F^n_k((X^n_r)_{r \le k/n}, \omega) + \frac{1}{n} G^n_k((X^n_r)_{r \le k/n}, \omega) \tag{1.8}$$

is studied under strong mixing conditions and a certain additional dimensional condition. By using these results, we can apply the diffusion approximation approach to the agent-based market model for more general excess demand functions and derive several stock price models based on the framework of [9].

The aim of this paper is to construct an agent-based model of stock prices with market liquidity problems. In the real market, although there are agents who can buy and sell the stock freely to some extent, there also exist agents who cannot trade to their own satisfaction because of a shortage of cash, being prohibited from short selling, and so on. To consider how such a liquidity problem affects things, we construct a market model based on [9] under the following constraints:

(I) Some of the agents cannot sell the more of the stock than the number of shares held (a short sale constraint),

(II) Some of the agents cannot buy more of the stock higher than allowed by their budget. (a budget constraint).

In each case (I)–(II), we will show that the continuous-time process of the stock price which is derived by shortening the transaction intervals is the solution of a certain stochastic differential equation with oblique reflection (SDER). Moreover, the value of the stock price under the short sale constraint is larger than it would be without such a constraint. The effect of the short sale constraint is discussed in [1], [5], [10] and the references therein, and our result is consistent with a common expectation, viz., that the short sale constraint causes overpricing (nonetheless, [1] pointed out that whether short sale constraints will always lead to overpricing is far from certain). On the other hand, we will also show that a budget constraint drives the stock price down.

We now fix some notation. For an interval $A \subset [0, \infty)$, we denote by $C(A; \mathbb{R}^d)$ the set of continuous functions from A to $\mathbb{R}^d$ and we use the abbreviations $\mathcal{C}^d_T = C([0,T]; \mathbb{R}^d)$ and $\mathcal{C}^d =$

$C([0,\infty);\mathbb{R}^d)$ (when $d=1$, we simply write $\mathcal{C}_T$ and $\mathcal{C}$.) For $w \in \mathcal{C}^d$, we set $|w|_T = \sup_{0\le t\le T} |w(t)|$ and $|w|_\infty = \sup_{t\ge 0} |w(t)|$, where $|\cdot|$ is the Euclidean norm. We also define the following subspaces of $\mathcal{C}^d_T$.

$$\begin{aligned} \mathcal{C}^d_{T,+} &= \{w = (w^i)^d_{i=1} \in \mathcal{C}^d_T \ ; \ w^i(t) \ge 0, \quad i = 1,\dots,d\}, \\ \mathcal{C}^d_{T,\uparrow 0} &= \{w = (w^i)^d_{i=1} \in \mathcal{C}^d_T \ ; \ w^i(0) = 0, w^i(t) \text{ is non-decreasing in } t, \quad i = 1,\dots,d\}, \end{aligned}$$

and similarly $\mathcal{C}^d_+$ and $\mathcal{C}^d_{\uparrow 0}$ for $T=\infty$. We introduce the canonical σ-algebra $\mathcal{B}_t = \sigma(w(s) \ ; \ s \le t)$ of $\mathcal{C}$. We often consider the space of $\mathbb{R}^{1+N}$-valued functions and then we start the index at zero, i.e., $w(t) = (w^i(t))^N_{i=0}$. The notation $(x)_+$ stands for the positive part of x, that is, $(x)_+ = \max\{x, 0\}$.

2 Main Results

2.1 Model I: Stock Price Model with Short Sale Constraints

Let $I = \{1,\dots,N\}$ be a finite set of agents who are active in the market which consists of a single stock. We assume that each agent $i \in I$ always stays in the market and no new agents enter. First we consider the discrete trading case, with market clearing times $t = 1/n, 2/n, \dots$ for $n \in \mathbb{N} = \{1,2,3,\dots\}$. We denote by $X^n = (X^n_t)_{t\ge 0}$ the log-price process of a stock, and by $\varphi^{n,i} = (\varphi^{n,i}_t)_{t\ge 0}$ the number of shares of the stock held by agent $i \in I$. The fluctuations of the processes X^n and $\varphi^n = (\varphi^{n,i})^N_{i=1}$ are found as follows. At the initial time $t=0$, every agent $i \in I$ has $\varphi^{n,i}_0 = \Phi^i \ge 0$ shares of a stock and the initial log-price of the stock is $X^n_0 = x_0 \in \mathbb{R}$. After trading is finished for the period $t = k/n$, the log-price X^n_t, $t \in (k/n, (k+1)/n]$ is determined by the market clearing condition

$$\sum_{i\in I} e^{n,i}_k (X^n_{(k+1)/n}, X^n, \omega; \varphi^{n,i}_{k/n}) = 0 \tag{2.1}$$

and linear interpolation (1.6), where $e^{n,i}_k(x,w,\omega;\varphi) : \mathbb{R}\times\mathcal{C}\times\Omega^n\times\mathbb{R} \longrightarrow \mathbb{R}$ is the excess demand function of $i \in I$ at $t = k/n$ on the underlying probability space $(\Omega^n, \mathcal{F}^n, P^n)$, and $\varphi^{n,i}_k$ is the quantity of stock held by i. Here $e^{n,i}_k(x,w,\omega;\varphi)$ is assumed to be $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}_{k/n} \otimes \mathcal{F}^n \otimes \mathbb{R}$-measurable, so that

$$e^{n,i}_k(x,w,\omega;\varphi) = e^{n,i}_k(x, w(\cdot \wedge (k/n)), \omega; \varphi).$$

After determining the log-price up to $t = (k+1)/n$ by (2.1), the agents' holdings of the stock are set by

$$\varphi^{n,i}_{(k+1)/n} = \varphi^{n,i}_{k/n} + e^{n,i}_k (X^n_{(k+1)/n}, X^n, \omega; \varphi^{n,i}_{k/n}) \tag{2.2}$$

and

$$\varphi^{n,i}_t = (nt-k)\varphi^{n,i}_{(k+1)/n} + (k+1-nt)\varphi^{n,i}_{k/n}, \quad t \in (k/n, (k+1)/n) \tag{2.3}$$

for $i = 1, \dots, N$. We remark that (2.1) implies that the total number of shares of the stock in the market is constant, i.e., $\sum_{i=1}^{N} \varphi_t^{n,i} = \sum_{i=1}^{N} \Phi^i$ for any n and t.

All agents have their individual excess demand $\tilde{e}_k^{n,i}(x, w, \omega)$ before considering friction, but they do not always exhibit $\tilde{e}_k^{n,i}(x, w, \omega)$ itself to the market. Agents are divided into two groups, $I = I_1 \cup I_2$, where $I_1 = \{1, \dots, N_1\}$ and $I_2 = \{N_1 + 1, \dots, N\}$ for some $1 \leq N_1 \leq N$. Agents in the first group are prohibited from selling short, thus their excess demand at $t = k/n$ is not lower than $\varphi_{k/n}^{n,i}$ and the process of their stock holdings is always non-negative. Agents in the second group I_2 are allowed to sell short, so they can exhibit the value $\tilde{e}_k^{n,i}(\cdot, X^n, \omega)$ itself as their excess demand. Then the excess demand function is finally defined as

$$e_k^{n,i}(x, w, \omega; \varphi) = \begin{cases} \max\{\tilde{e}_k^{n,i}(x, w, \omega), -\varphi\}, & i \in I_1 \\ \tilde{e}_k^{n,i}(x, w, \omega), & i \in I_2. \end{cases} \tag{2.4}$$

We divide $\tilde{e}_k^{n,i}(x, w)$ into two parts.

$$\tilde{e}_k^{n,i}(x, w) = f_k^{n,i}(x, w) + g_k^{n,i}(w). \tag{2.5}$$

Here,

$$f_k^{n,i}(x, w) = \tilde{e}_k^{n,i}(x, w) - \tilde{e}_k^{n,i}(w(k/n), w), \quad g_k^{n,i}(w) = \tilde{e}_k^{n,i}(w(k/n), w).$$

The function $g_k^{n,i}(w)$ is the excess demand when the price of the stock does not change. We further divide $g_k^{n,i}(w)$ into

$$g_k^{n,i}(w) = \frac{1}{n}\bar{g}_k^{n,i}(w) + \frac{1}{\sqrt{n}}\tilde{g}_k^{n,i}(w),$$

where $\tilde{g}_k^{n,i}(w)$ is a mean zero random function. We remark that $g_k^{n,i}$, $\bar{g}_k^{n,i}$ and $\tilde{g}_k^{n,i}$ are all $\mathcal{B}_{k/n} \otimes \mathcal{F}^n$-measurable. The function $f_k^{n,i}(x, w)$ is an additional excess demand associated with the change of the price. Usually the higher the stock price becomes, the lower is an agent's demand for the stock. Thus it is natural that $f_k^{n,i}(x, w)$ is supposed to be decreasing in x. Furthermore, we assume

[A1] The function $f_k^{n,i}(x, w)$ is deterministic (i.e., independent of $\omega \in \Omega^n$), continuous in (x, w), and three times continuously differentiable in x. Moreover, there exist positive constants K_0 and δ_0 such that

$$-K_0 \leq \frac{\partial}{\partial x} f_k^{n,i}(x, w) \leq -\delta_0. \tag{2.6}$$

Furthermore, $\bar{g}_k^{n,i}(w)$ and $\tilde{g}_k^{n,i}(w)$ are continuous in w almost surely.

Condition [A1] implies that when I_2 is not empty (that is, when $N_1 < N$), the equation

$$\sum_{i \in I} e_k^{n,i}(x, w; \varphi^i) = 0 \tag{2.7}$$

has a unique solution x for any fixed w and $(\varphi^i)_i$, since the left-hand side of (2.7) is strictly decreasing in x. In fact, the existence of a unique solution of (2.7) is also guaranteed even if $I = I_1$, provided $\sum_{i=1}^{N} \Phi^i > 0$. However, hereafter we always assume $N_1 < N$ for some technical reasons which we will explain later.

By (2.1) and (2.2), we can construct the processes $(X_t^n)_t$ and $(\varphi_t^n)_t$. We are interested in the limit of the $(1+N)$-dimensional process $(\Xi_t^n)_t = (X_t^n, \varphi_t^{n,1}, \dots, \varphi_t^{n,N})_t$ as $n \to \infty$. More precisely, we consider the weak limit of the distribution $\mu^n = P(\Xi^n \in \cdot)$ on $\mathcal{C}^{1+N}$.

We will define more conditions.

[A2] For every $M > 0$, there exists a positive constant $C_M > 0$ such that

$$\sum_{l=0}^{3} \sup_{|x|,|w|_\infty \le M} \left| \frac{\partial^l}{\partial x^l} f_k^{n,i}(x,w) \right| + \mathrm{E}^n[\sup_{|w|_\infty \le M} |\bar{g}_k^{n,i}(w)|^{24}] + \mathrm{E}^n[\sup_{|w|_\infty \le M} |\tilde{g}_k^{n,i}(w)|^{24}] \le C_M$$

for any $n, k \in \mathbb{N}$ and $i = 1, \dots, N$, where E^n is the expectation with respect to P^n (we simply denote this by E when there is no possibility of confusion).

[A3] The σ-algebras $\sigma(\bar{g}_k^{n,i}, \tilde{g}_k^{n,i} \; ; \; i = 1, \dots, N)$, $k = 1, 2, \dots$, are independent.

[A4] Let

$$\begin{aligned} \alpha_k^{n,i}(w) &= -\frac{\partial}{\partial x} f_k^{n,i}(w(k/n), w), \quad \beta_k^{n,i}(w) = \mathrm{E}^n[\bar{g}_k^{n,i}(w)], \\ \gamma_k^{n,i}(w) &= \frac{\partial^2}{\partial x^2} f_k^{n,i}(w(k/n), w), \quad a_k^{n,ij}(w) = \mathrm{E}^n[\tilde{g}_k^{n,i}(w)\tilde{g}_k^{n,j}(w)]. \end{aligned}$$

For each $i, j \in I$ the following limits exist

$$\begin{aligned} \beta^i(t,w) &= \lim_{r\to\infty} \beta_{[nt]}^{n,i}(w), \quad \gamma^i(t,w) = \lim_{r\to\infty} \gamma_{[nt]}^{n,i}(w), \\ a^{ij}(t,w) &= \lim_{r\to\infty} a_{[nt]}^{n,ij}(w) \end{aligned}$$

uniformly on any compact subset of $\mathcal{C}$ and for any $t \ge 0$, and

$$\alpha^i(t,w) = \lim_{r\to\infty} \alpha_{[nt]}^{n,i}(w)$$

uniformly on any compact subset of $[0,\infty) \times \mathcal{C}$.

[A5] Define $Q_k^n(w) = (Q_k^{n,ij}(w))_{1\le i\le N, 1\le j\le N_1}$ as

$$Q_k^{n,ij}(w) = (1-\delta_{ij})\alpha_k^{n,i}(w)\tilde{\alpha}_k^{n,j}(w),$$

where δ_{ij} is the Kronecker delta and

$$\bar{\alpha}_k^n(w) = \sum_{i=1}^{N} \alpha_k^{n,i}(w), \quad \tilde{\alpha}_k^{n,j}(w) = 1/(\bar{\alpha}_k^n(w) - \alpha_k^{n,j}(w)).$$

There exists $V = (V^{ij})_{i,j=1}^{N_1} \in \mathbb{R}^{N_1} \otimes \mathbb{R}^{N_1}$ such that $V^{ii} = 0$, $Q_k^{n,ij}(w) \le V^{ij}$ for each $i, j = 1, \dots, N_1$ and the spectral radius of V is less than 1.

Note that the inequality $N_1 < N$ is essential for condition [A5]. Indeed, if $N_1 = N$, then the calculation

$$\begin{aligned}\sum_{i=1}^{N}((Q_k^n(w))^m)^{ij} &= \sum_{l_{m-1}=1}^{N}\cdots\sum_{l_1=1}^{N}\sum_{i=1}^{N} Q_k^{n,il_1}(w)Q_k^{n,l_1l_2}(w)\cdots Q_k^{n,l_{n-1}j}(w) \\ &= \sum_{l_{m-1}=1}^{N}\cdots\sum_{l_1=1}^{N} Q_k^{n,l_1l_2}(w)\cdots Q_k^{n,l_{n-1}j}(w) = \cdots = 1, \quad m \in \mathbb{N}\end{aligned}$$

indicates the convergence $\lim_{m\to\infty} ||(Q_k^n(w))^m||_1 = 1$, where $((Q_k^n(w))^m)^{ij}$ is the i,jth element of the mth power of the matrix $Q_k^n(w)$ and the norm $||\cdot||_1$ stands for $||A||_1 = \max_j \sum_{i=1}^{N} |A^{ij}|$ for $A = (A^{ij})_{ij} \in \mathbb{R}^N \otimes \mathbb{R}^N$, thus we cannot find such a matrix V in [A5].

[A6] Let $\sigma(t,w) = (\sigma^{ij}(t,w))_{i,j=1}^N$ be an N-dimensional matrix-valued function such that $a^{ij}(t,w) = \sum_{m=1}^{N} \sigma^{im}(t,w)\sigma^{jm}(t,w)$. For any $T > 0$, there exists a positive constant C_T such that

$$|\tilde{\beta}^i(t,w)| + |\tilde{\gamma}^i(t,w)| + |\sigma^{ij}(t,w)| \le C_T(1 + |w|_t)$$

for each $i,j = 1,\dots,N$, $0 \le t \le T$ and $w \in \mathbb{R}$, where

$$\tilde{\beta}^i(t,w) = \beta^i(t,w) + \tilde{\gamma}^i(t,w), \quad \tilde{\gamma}^i(t,w) = \frac{\gamma^i(t,w)}{2\bar{\alpha}(t,w)^2} \sum_{k,l=1}^{N} a^{kl}(t,w).$$

We now introduce an SDER.

$$\begin{aligned} dX_t &= \hat{b}^0(t,X)dt + \sum_{j=1}^{N} \hat{\sigma}^{0j}(t,X)dB_t^j + \sum_{j=1}^{N_1} \tilde{\alpha}^j(t,X)dL_t^j, \quad X_0 = x_0, \qquad (2.8) \\ d\varphi_t^i &= \hat{b}^i(t,X)dt + \sum_{j=1}^{N} \hat{\sigma}^{ij}(t,X)dB_t^j + 1_{I_1}(i)dL_t^i - \sum_{j=1}^{N_1} Q^{ij}(t,X)dL_t^j, \quad \varphi_0^i = \Phi^i, \quad i = 1,\dots,N, \end{aligned}$$

where

$$\tilde{\alpha}^i(t,w) = 1/(\bar{\alpha}(t,w) - \alpha^i(t,w)), \quad Q^{ij}(t,w) = (1 - \delta_{ij})\alpha^i(t,w)\tilde{\alpha}^j(t,w)$$

and $\hat{b}^i(t,w)$, $\hat{\sigma}^{ij}(t,w)$, $0 \le i \le N, 1 \le j \le N$ are given by

$$\hat{b}^i(t,w) = \begin{cases} \sum_{j=1}^N \tilde{\beta}^j(t,w)/\bar{\alpha}(t,w) & (i=0) \\ \tilde{\beta}^i(t,w) - \alpha^i(t,w)\sum_{j=1}^N \tilde{\beta}^j(t,w)/\bar{\alpha}(t,w) & (i \ge 1), \end{cases} \tag{2.9}$$

$$\hat{\sigma}^{ij}(t,w) = \begin{cases} \sum_{k=1}^N \sigma^{kj}(t,w)/\bar{\alpha}(t,w) & (i=0) \\ \sigma^{ij}(t,w) - \alpha^i(t,w)\sum_{k=1}^N \sigma^{kj}(t,w)/\bar{\alpha}(t,w) & (i \ge 1) \end{cases} \tag{2.10}$$

with $\bar{\alpha}(t,w) = \sum_{i=1}^N \alpha^i(t,w)$. We say that an $(1+N+N_1)$-dimensional continuous adapted stochastic process $(X_t, \varphi_t, L_t)_t = (X_t, (\varphi_t^i)_{i=1}^N, (L_t^i)_{i=1}^{N_1})$ is a solution of (2.8) on a given filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P)$ equipped with an N-dimensional $(\mathcal{F}_t)_t$-Brownian motion $B_t = (B_t^i)_{i=1}^n$ if

- $P(\varphi \in \mathcal{C}_+^N) = P(L \in \mathcal{C}_{\uparrow 0}^{N_1}) = 1$,
- $\int_0^\infty 1_{\{\varphi_r^i > 0\}} dL_r^i = 0$ for $i = 1, \dots, N_1$ almost surely,
- The processes $(X_t)_t$, $(\varphi_t)_t$, $(L_t)_t$ and $(B_t)_t$ satisfy

$$\begin{aligned} X_t &= x_0 + \int_0^t \hat{b}^0(r,X)dr + \sum_{j=1}^N \int_0^t \hat{\sigma}^{0j}(r,X)dB_r^j + \sum_{j=1}^{N_1}\int_0^t \tilde{\alpha}^j(r,X)dL_r^j, \qquad (2.11) \\ \varphi_t^i &= \Phi^i + \int_0^t \hat{b}^i(r,X)dr - \sum_{j=1}^N \int_0^t \hat{\sigma}^{ij}(r,X)dB_r^j + 1_{I_1}(i)L_t^i - \sum_{j=1}^{N_1}\int_0^t Q^{ij}(r,X)dL_r^j \end{aligned}$$

  for $t \ge 0$ and $i = 1, \dots, N$ almost surely.

We call the process $L = (L_t^i)_{i \in I_1, t \ge 0}$ a regulator associated with $\Xi = (X, \varphi)$. Now we present our final assumption.

[A7] A solution of (2.8) is unique in law.

For instance, if $\partial f_k^n/\partial x$, $i = 1, \dots, N$, is constant (and so is the matrix Q), condition [A7] holds under a Lipschitz condition on the coefficients $\hat{b}^i$ and $\hat{\sigma}^{ij}$ (see [4] and [19] for instance: although the form of our SDE (2.8) is a little special, the arguments in these papers also works.) For other sufficient conditions for [A7], see [7] and [18].

By [A7], the distribution $\mu = P(\Xi \in \cdot)$ is uniquely determined. We are now prepared to state our main result.

Theorem 1. *Assume [A1]–[A7]. Then the distribution μ^n converges weakly to μ on $\mathcal{C}^{1+N}$ as $n \to \infty$.*

The proof is in Section 3. Theorem 1 implies that the limit of $\Xi^n = (X^n, \varphi^{n,1}, \dots, \varphi^{n,N})$ is characterized as the solution of an SDER, viz., (2.8). The regulator process L prevents the shares of stock of an agent i from taking a negative value. The infinitesimal term $1_{I_1}(i)dL_t^i$ in (2.8) works only when the agent $i \in I_1$ hopes to sell more of the stock than they hold.

Here we consider the case where all agents in the market can sell short (i.e., $N_1 = 0$). This is a special case of Theorem 1 of [13] and X^n converges weakly to the unique solution $\hat{X}$ of

$$d\hat{X}_t = \hat{b}^0(t, \hat{X})dt + \sum_{j=1}^{N} \hat{\sigma}^{0j}(t, \hat{X})dB_t^j, \quad \hat{X}_0 = x_0. \tag{2.12}$$

In this case, no agents are bound by the short sale prohibition, and the process $(\hat{X}_t)_t$ represents the log-price of the stock without friction.

On the other hand, when I_1 is not empty, agents in I_1 may not be able to exhibit their primary excess demand (which is described as $\tilde{e}_k^{n,i}$ in the discrete-time model.) The (log-)price X_t is pushed up by the gap between the actual excess demand with the primary excess demand, so X_t is larger than $\hat{X}_t$. The following theorem describes such a phenomenon.

Theorem 2. *Assume [A1]–[A7]. Let $\Xi = (X, \varphi^1, \dots, \varphi^N)$ (resp., $\hat{X}$) be a solution of (2.8) (resp., (2.12)) on a given filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P)$ equipped with an N-dimensional Brownian motion B. Then $X_t \geq \hat{X}_t$ for any $t \geq 0$ almost surely.*

Theorem 2 is easily obtained by similar arguments to the proof of Proposition 5.2.18 in [12]. This suggests the assertion that the short sale constraint causes the overpricing in our model.

2.2 Model II: Stock Price Model under Budget Constraint

We also consider the case where some of the agents, $I_1 = \{1, \dots, N_1\}$, cannot borrow cash. In the previous section, an excess demand function with no friction $\tilde{e}_k^{n,i}(x, w)$ is understood as shares of the stock which an agent wants to buy. In this section, we interpret $\tilde{e}_k^{n,i}(x, w)$ to mean an excess demand in terms of dollars. We also define

$$e_k^{n,i}(x, w; y) = \begin{cases} \min\{\tilde{e}_k^{n,i}(x, w), y\}, & i \in I_1 \\ \tilde{e}_k^{n,i}(x, w), & i \in I_2. \end{cases}$$

The market clearing condition is now expressed by

$$\sum_{i=1}^{N} e_k^{n,i}(X_{(k+1)/n}^n, X^n; W_{k/n}^n) = 0, \tag{2.13}$$

where W_t^n is the agent's amount of cash held at time t. Then the process of the log-price of the stock $(X_t^n)_t$ and the cash holdings $(W_t^{n,i})_t$ of an agent i are given by (1.6), (2.13),

$$W_{(k+1)/n}^{n,i} = W_{k/n}^{n,i} - e_k^{n,i}(X_{(k+1)/n}^n, X^n; W_{k/n}^{n,i}), \tag{2.14}$$

and

$$W_t^{n,i} = (nt - k)W_{(k+1)/n}^{n,i} + (k + 1 - nt)W_{k/n}^{n,i}, \quad t \in (k/n, (k + 1)/n). \tag{2.15}$$

We assume that $\tilde{e}_k^{n,i}$ has the same form as (2.5). We also assume [A1]–[A7], replacing $\hat{b}$, $\hat{\sigma}$, and (2.8) with

$$\hat{b}^i(t,w) = \begin{cases} \sum_{j=1}^N \tilde{\beta}^j(t,w)/\bar{\alpha}(t,w) & (i=0) \\ -\tilde{\beta}^i(t,w) + \alpha^i(t,w)\sum_{j=1}^N \tilde{\beta}^j(t,w)/\bar{\alpha}(t,w) & (i \geq 1), \end{cases}$$

$$\hat{\sigma}^{ij}(t,w) = \begin{cases} \sum_{k=1}^N \sigma^{kj}(t,w)/\bar{\alpha}(t,w) & (i=0) \\ -\sigma^{ij}(t,w) + \alpha^i(t,w)\sum_{k=1}^N \sigma^{kj}(t,w)/\bar{\alpha}(t,w) & (i \geq 1) \end{cases},$$

and

$$\begin{aligned} dX_t &= \hat{b}^0(t,X)dt + \sum_{j=1}^N \hat{\sigma}^{0j}(t,X)dB_t^j - \sum_{j=1}^{N_1} \tilde{\alpha}^j(t,X)dL_t^j, \quad X_0 = x_0, \qquad (2.16) \\ dW_t^i &= \hat{b}^i(t,X)dt + \sum_{j=1}^N \hat{\sigma}^{ij}(t,X)dB_t^j + 1_{I_1}(i)dL_t^i + \sum_{j=1}^{N_1} Q^{ij}(t,X)dL_t^j, \quad W_0^i = c^i, \quad i=1,\ldots,N, \end{aligned}$$

where $c^i \geq 0$ is the initial cash holdings of agent i. Then we have the following theorem.

Theorem 3. *The distribution of $\tilde{\Xi}^n = (X^n, W^{n,1}, \ldots, W^{n,N})$ converges weakly to a solution of* (2.16) *on $\mathcal{C}^{1+N}$ as $n \to \infty$.*

Theorem 4. *Let $\tilde{\Xi} = (X, W^1, \ldots, W^N)$ (resp., $\hat{X}$) be a solution of* (2.16) *(resp., (2.12)) on a given filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P)$ equipped with an N-dimensional Brownian motion B. Then $X_t \leq \hat{X}_t$ for any $t \geq 0$ almost surely.*

We omit the proofs of Theorems 3–4 since they are almost the same as those of Theorems 1–2.

3 Proof of Theorem 1

Take any $M > |x_0|$ and let $\psi_M \in C^\infty(\mathbb{R};[0,1])$ be such that $\psi_M(y) = 1$ on $|y| \leq M/2$ and $\psi_M(y) = 0$ on $|y| \geq M$. We set

$$\tilde{e}_k^{n,M,i}(x,w) = -(1-\psi_M(w(k/n)))\alpha_k^{n,i}(w)(x - w(k/n)) + \psi_M(w(k/n))\tilde{e}_k^{n,i}(x,w).$$

We define $X^{n,M}$, $\varphi^{n,M,i}$ and $e_k^{n,M,i}(x,w;\varphi)$ similarly to (2.1)–(2.3), replacing $\tilde{e}_k^{n,i}$ with $\tilde{e}_k^{n,M,i}$. First we consider the convergence of the truncated processes $\Xi^{n,M} = (X^{n,M}, \varphi^{n,M,1}, \ldots, \varphi^{n,M,N})$, $n \in \mathbb{N}$ for fixed M. We can easily see the following proposition (Proposition 1 in [13]).

Proposition 1. *For any ω, if $|X_t^{n,M}(\omega)| \leq M$, then $|X_r^{n,M}(\omega)| \leq M$ for all $r \in [0,t]$.*

We rearrange our market clearing equation into the form of a difference equation. Since it follows that

$$\max\{\tilde{e}_k^{n,M,i}(x,w), -\varphi\} = \tilde{e}_k^{n,M,i}(x,w) + (-\tilde{e}_k^{n,M,i}(x,w) - \varphi)_+,$$

we get

$$\sum_{i=1}^{N} e_k^{n,M,i}(X_{(k+1)/n}^{n,M}, X^{n,M}; \varphi_{k/n}^{n,M,i}) = \sum_{i=1}^{N} \tilde{e}_k^{n,M,i}(X_{(k+1)/n}^{n,M}, X^{n,M}) + \sum_{i=1}^{N_1} \hat{\eta}_k^{n,M,i} = 0,$$

where $\hat{\eta}_k^{n,M,i} = (-\tilde{e}_k^{n,M,i}(X_{(k+1)/n}^{n,M}, X^{n,M}) - \varphi_{k/n}^{n,M,i})_+$. Using Taylor's theorem, we have

$$\begin{aligned} & X_{(k+1)/n}^{n,M} - X_{k/n}^{n,M} \\ = \; & \frac{1}{\bar{\alpha}_k^n(X^{n,M})}\Big\{\psi_M(X_{k/n}^{n,M}) \sum_{i=1}^{N} \left(g_k^{n,i}(X^{n,M}) + \frac{1}{2}\gamma_k^{n,i}(X^{n,M})(X_{(k+1)/n}^{n,M} - X_{k/n}^{n,M})^2 + \varepsilon_k^{n,M,i} \right) \\ & \qquad + \sum_{i=1}^{N_1} \hat{\eta}_k^{n,M,i} \Big\} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} & \varphi_{(k+1)/n}^{n,M,i} - \varphi_{k/n}^{n,M,i} \\ = \; & \psi_M(X_{k/n}^{n,M}) \left(g_k^{n,i}(X^{n,M}) + \frac{1}{2}\gamma_k^{n,i}(X^{n,M})(X_{(k+1)/n}^{n,M} - X_{k/n}^{n,M})^2 + \varepsilon_k^{n,M,i} \right) \\ & -\alpha_k^{n,i}(X^{n,M})(X_{(k+1)/n}^{n,M} - X_{k/n}^{n,M}) + 1_{I_1}(i)\hat{\eta}_k^{n,M,i}, \end{aligned} \tag{3.2}$$

where

$$\varepsilon_k^{n,M,i} = \frac{1}{2}\int_0^1 (1-u)^2 \frac{\partial^3}{\partial x^3} f_k^{n,i}(uX_{(k+1)/n}^{n,M} + (1-u)X_{k/n}^{n,M}, X^{n,M}) du (X_{(k+1)/n}^{n,M} - X_{k/n}^{n,M})^3.$$

Substituting (3.1) into itself and into (3.2), we get

$$\begin{aligned} X_{(k+1)/n}^{n,M} - X_{k/n}^{n,M} &= \frac{1}{\bar{\alpha}_k^n(X^{n,M})}\left\{ \sum_{i=1}^{N} H_k^{n,M,i}(X^{n,M}) + \sum_{i=1}^{N_1} \hat{\eta}_k^{n,M,i} \right\}, \\ \varphi_{(k+1)/n}^{n,M,i} - \varphi_{k/n}^{n,M,i} &= H_k^{n,M,i}(X^{n,M}) - \frac{\alpha_k^{n,i}(X^{n,M})}{\bar{\alpha}_k^n(X^{n,M})}\left\{ \sum_{j=1}^{N} H_k^{n,M,j}(X^{n,M}) + \sum_{j=1}^{N_1} \hat{\eta}_k^{n,M,j} \right\} \\ & \quad +1_{I_1}(i)\hat{\eta}_k^{n,M,i}, \end{aligned}$$

where

$$
\begin{aligned}
H_k^{n,M,i}(w) &= \psi_M(w(k/n))\left\{\frac{1}{\sqrt{n}}\bar{g}_k^{n,i}(w)+\frac{1}{n}h_k^{n,i}(w)\right\}+\tilde{\varepsilon}_k^{n,M,i},\\
h_k^{n,i}(w) &= \bar{g}_k^{n,i}(w)+\psi_M(w(k/n))^2\frac{\gamma_k^{n,i}(w)}{2\bar{\alpha}_k^n(w)}\sum_{j,m=1}^N \tilde{g}_k^{n,j}(w)\tilde{g}_k^{n,m}(w),\\
\tilde{\varepsilon}_k^{n,M,i} &= \psi_M(X_{k/n}^{n,M})\left\{\varepsilon_k^{n,M,i}+\frac{\gamma_k^{n,i}(X^{n,M})}{2\bar{\alpha}_k^n(X^{n,M})^2}\hat{\varepsilon}_k^{n,M}\right\},\\
\hat{\varepsilon}_k^{n,M} &= \left(\sum_{i=1}^{N_1}\hat{\eta}_k^{n,M,i}\right)\left\{\sum_{i=1}^{N_1}\hat{\eta}_k^{n,M,i}+2\psi_M(X_{k/n}^{n,M})\sum_{i=1}^{N}\left(\frac{1}{\sqrt{n}}\tilde{g}_k^{n,i}(X^{n,M})+\rho_k^{n,M,i}\right)\right\}\\
&\quad+\psi_M(X_{k/n}^{n,M})^2\left\{\frac{2}{\sqrt{n}}\sum_{i,j=1}^N \tilde{g}_k^i(X^{n,M})\rho_k^{n,M,j}+\left(\sum_{i=1}^N\rho_k^{n,M,i}\right)^2\right\},\\
\rho_k^{n,M,i} &= \frac{1}{n}\bar{g}_k^{n,i}(X^{n,M})+\frac{1}{2}\gamma_k^{n,i}(X^{n,M})(X_{(k+1)/n}^{n,M}-X_{k/n}^{n,M})^2+\varepsilon_k^{n,M,i}.
\end{aligned}
$$

Thus, if we set $\eta_k^{n,M,i}=(1-\alpha_k^{n,i}(w)/\bar{\alpha}_k^n(w))\hat{\eta}_k^{n,M,i}$,

$$
Z_t^{n,M,i}=\sum_{k=0}^{[nt]-1}H_k^{n,M,i}(X^{n,M})+(nt-[nt])H_{[nt]}^{n,M,i}(X^{n,M}),\quad L_t^{n,M,i}=\sum_{k=0}^{[nt]-1}\eta_k^{n,M,i}+(nt-[nt])\eta_{[nt]}^{n,M,i}
$$

and

$$
Y_t^{n,M,0} = x_0+\sum_{i=1}^N\int_0^t\frac{1}{\bar{\alpha}_{[nr]}^n(X^{n,M})}dZ_r^{n,M,i}, \tag{3.3}
$$

$$
Y_t^{n,M,i} = \Phi^i+Z_t^{n,M,i}-\sum_{j=1}^N\int_0^t\frac{\alpha_{[nr]}^{n,i}(X^{n,M})}{\bar{\alpha}_{[nr]}^n(X^{n,M})}dZ_r^{n,M,j},\quad i=1,\dots,N, \tag{3.4}
$$

then

$$
X_t^{n,M} = Y_t^{n,M,0}+\sum_{i=1}^{N_1}\int_0^t\tilde{\alpha}_{[nr]}^{n,i}(X^{n,M})dL_r^{n,M,i}, \tag{3.5}
$$

$$
\varphi_t^{n,M,i} = Y_t^{n,M,i}+1_{I_1}(i)L_t^{n,M,i}-\sum_{j=1}^{N_1}\int_0^t Q_{[nr]}^{n,ij}(X^{n,M})dL_t^{n,M,j}. \tag{3.6}
$$

The equality (3.6) seems to imply that $(\varphi^{n,M,i},L^{n,M,i})_{i=1}^{N_1}$ is a solution of the Skorokhod problem with oblique reflection in the non-negative orthant associated with $(Y^{n,M,i})_{i=1}^{N_1}$ (see [4], [11], [18], [19] and [21]). However this is not strictly true, since the equality $\int_0^\infty \varphi_r^{n,M,i}dL_r^{n,i}=0$ does not hold by virtue of the linear interpolation (2.3). The following proposition tells us that $(\varphi_{k/n}^{n,M,i},L_{k/n}^{n,M,i})_{i=1}^{N_1}$, $k\in\mathbb{Z}_+$, is, as it were, a solution of the corresponding Skorokhod problem in discrete-time.

Proposition 2. *For every $k \in \mathbb{Z}_+$,*

$$L^{n,M,i}_{k/n} = \max_{0\le l\le k}\left(\sum_{j=1}^{N_1}\int_0^{l/n} Q^{n,ij}_{[nr]}(X^{n,M})dL^{n,M,j}_r - Y^{n,M,i}_{l/n}\right)_+ . \tag{3.7}$$

Proof. It is obvious that the left-hand side of (3.7) is not less than the right-hand side. We suppose

$$L^{n,M,i}_{k/n} > \left(\sum_{j=1}^{N_1}\int_0^{l/n} Q^{n,ij}_{[nr]}(X^{n,M})dL^{n,M,j}_r - Y^{n,M,i}_{l/n}\right)_+ \tag{3.8}$$

for $l = 0.\dots,k$. By (3.8) with $l = k$, we have

$$Y^{n,M,i}_{k/n} + L^{n,M,i}_{k/n} - \int_0^{k/n} Q^{n,ij}_{[nr]}(X^{n,M})dL^{n,M,j}_r = \varphi^{n,M,i}_{k/n} > 0.$$

This inequality gives $\eta^{n,M,i}_{k-1} = 0$ (by the definition of $\hat\eta^{n,M,i}_{k-1}$) and thus $L^{n,M,i}_{(k-1)/n} = L^{n,M,i}_{k/n}$. Using (3.8) again with $l = k-1$, we similarly get $\eta^{n,M,i}_{k-2} = 0$. Inductively we see that $L^{n,M,i}_t = 0$ for $t \in [0, k/n]$ and this contradicts (3.8). Then we obtain the assertion. ■

Using the above proposition and the same arguments as Theorem 2 in [21], we get the following proposition.

Proposition 3. *For every $0 \le l \le k$,*

$$\sum_{i=1}^{N_1}|L^{n,M,i}_{k/n} - L^{n,M,i}_{l/n}|^2 \le \hat K \max_{l\le m\le k}\sum_{i=1}^{N_1}|Y^{n,M,i}_{m/n} - Y^{n,M,i}_{l/n}|^2$$

for some $\hat K > 0$ depending only on V.

The equality (3.2) also indicates

$$\varphi^{n,M,i}_{k/n} = \hat Y^{n,M,i}_k + \hat L^{n,M,i}_{k/n}, i = 1,\dots,N_1, \tag{3.9}$$

where

$$\begin{aligned}\hat Y^{n,M,i}_k &= \sum_{l=0}^{k-1}\Bigg\{\psi_M(X^{n,M}_{l/n})\left(g^{n,i}_l(X^{n,M}) + \frac{1}{2}\gamma^{n,i}_l(X^{n,M})(X^{n,M}_{(l+1)/n} - X^{n,M}_{l/n})^2 + \varepsilon^{n,M,i}_l\right)\\ &\qquad -\alpha^{n,i}_l(X^{n,M})(X^{n,M}_{(l+1)/n} - X^{n,M}_{l/n})\Bigg\},\\ \hat L^{n,M,i}_t &= \sum_{k=0}^{[nt]-1}\hat\eta^{n,M,i}_k + (nt-[nt])\hat\eta^{n,M,i}_{[nt]}.\end{aligned}$$

The equality (3.9) corresponds to the classical Skorokhod problem for each $i = 1,\dots,N_1$. Similarly to Proposition 2, we obtain the following.

Proposition 4. *For every* $k \in \mathbb{Z}_+$, $\hat{L}^{n,M,i}_{k/n} = \max_{0 \le l \le k} (-\hat{Y}^{n,M,i}_{l/n})_+$.

Next we evaluate the moment of the process.

Proposition 5. $\mathrm{E}[|X^{n,M}_{(k+1)/n} - X^{n,M}_{k/n}|^{24}] \le C_M/n^{12}$, $k \in \mathbb{Z}_+$ *for some* $C_M > 0$.

Proof. Set

$$f(t) = \tilde{f}(t(X^{n,M}_{(k+1)/n} - X^{n,M}_{k/n}) + X^{n,M}_{k/n}), \quad \tilde{f}(x) = \sum_{i=1}^{N} e^{n,M,i}_k(x, X^{n,M}; \varphi^{n,M,i}_{k/n}).$$

Then we have $f(1) = 0$ and

$$f(0) = \sum_{i=1}^{N_1} \max\{\psi_M(X^{n,M}_{k/n}) g^{n,i}_k(X^{n,M}), -\varphi^{n,M,i}_{k-1}\} + \sum_{i=N_1+1}^{N} \psi_M(X^{n,M}_{k/n}) g^{n,i}_k(X^{n,M}).$$

Using the fundamental theorem of calculus, we get

$$f(0) = f(0) - f(1) = -\int_0^1 \tilde{f}'(t(X^{n,M}_{(k+1)/n} - X^{n,M}_{k/n}) + X^{n,M}_{k/n}) dt (X^{n,M}_{(k+1)/n} - X^{n,M}_{k/n}). \quad (3.10)$$

Since $\varphi^{n,M,i}_t \ge 0$ for $i = 1, \ldots, N_1$ and $\tilde{f}'(x) \le -N_2\delta_0$, we get

$$|X^{n,M}_{(k+1)/n} - X^{n,M}_{k/n}| \le \frac{|f(0)|}{N_2\delta_0} \le \frac{1}{N_2\delta_0} \sum_{i=1}^{N} |\psi_M(X^{n,M}_{k/n}) g^{n,i}_k(X^{n,M})|. \quad (3.11)$$

By [A2] and Proposition 1, we have

$$\mathrm{E}[\psi_M(X^{n,M}_{k/n}) |g^{n,i}_k(X^{n,M})|^{24}] = \mathrm{E}[\sup_{|w|_\infty \le M} |g^{n,i}_k(y)|^{24}] \le \frac{C_M}{n^{12}}. \quad (3.12)$$

Our assertion follows immediately by (3.11) and (3.12). ■

The above proposition and [A1]–[A2] lead us to the following.

Proposition 6. $\mathrm{E}[|\tilde{e}^{n,M,i}_k(X^{n,M}_{(k+1)/n}, X^{n,M})|^{24}] + \mathrm{E}[|\eta^{n,M,i}_k|^{24}] + \mathrm{E}[|\hat{\eta}^{n,M,i}_k|^{24}] + \mathrm{E}[|\varepsilon^{n,M,i}_k|^8] \le C_M/n^{12}$, $k \in \mathbb{Z}_+$, *for some* $C_M > 0$.

Proposition 7. $\mathrm{E}[|L^{n,M}_t|^8] + \mathrm{E}[|\hat{L}^{n,M}_t|^8] \le C_{M,t}$ *for some* $C_{M,t} > 0$.

Proof. It suffices to estimate $\mathrm{E}[|\sum_{i=1}^{N_1} \hat{L}^{n,M,i}_{[nt]/n}|^8]$. By [A1], Proposition 4, and the equality

$$X^{n,M}_{(k+1)/n} - X^{n,M}_{k/n} = (X^{n,M}_{(k+1)/n} - X^{n,M}_{k/n}) 1_{\{|X^{n,M}_{k/n}| \le M\}} \quad (3.13)$$

(obtained by the definitions of $\tilde{e}_k^{n,M,i}$, $\hat{\eta}_k^{n,M,i}$, and $X_t^{n,M}$), we have

$$\begin{aligned}
\sum_{i=1}^{N_1} \hat{L}_{[nt]/n}^{n,M,i} &\leq \sum_{i=1}^{N_1} \max_{0\leq k\leq [nt]} \left| G_k^{n,M,i} - \tilde{G}_k^{n,M,i} + \frac{1}{n}\sum_{l=0}^{k-1}\left(\hat{h}_l^{n,M,i} - \frac{\alpha_l^{n,i}(X^{n,M})}{\bar{\alpha}_l^n(X^{n,M})}\sum_{j=1}^{N}\hat{h}_l^{n,M,j}\right)\right. \\
&\qquad \left. - \sum_{l=0}^{k-1}\frac{\alpha_l^{n,i}(X^{n,M})}{\bar{\alpha}_l^n(X^{n,M})}\sum_{j=1}^{N_1}\hat{\eta}_l^{n,M,j}\right| \\
&\leq \sum_{i=1}^{N_1}\left(\max_{0\leq k\leq [nt]}|G_k^{n,M,i}| + \max_{0\leq k\leq [nt]}|\tilde{G}_k^{n,M,i}|\right) + \frac{2}{n}\sum_{i=1}^{N}\sum_{k=0}^{[nt]-1}|\hat{h}_k^{n,M,i}| \\
&\qquad + \sum_{k=0}^{[nt]-1}\frac{\sum_{i=1}^{N_1}\alpha_k^{n,i}(X^{n,M})}{\bar{\alpha}_k^n(X^{n,M})}\sum_{j=1}^{N_1}\hat{\eta}_k^{n,M,j},
\end{aligned}$$

where

$$\begin{aligned}
G_k^{n,M,i} &= \frac{1}{\sqrt{n}}\sum_{l=0}^{k-1}\psi_M(X_{l/n}^{n,M})\tilde{g}_l^{n,i}(X^{n,M}), \\
\tilde{G}_k^{n,M,i} &= \frac{1}{\sqrt{n}}\sum_{j=1}^{N}\sum_{l=0}^{k-1}\frac{\alpha_l^{n,i}(X^{n,M})}{\bar{\alpha}_l^n(X^{n,M})}\psi_M(X_{l/n}^{n,M})\tilde{g}_l^{n,j}(X^{n,M}), \\
\hat{h}_k^{n,M,i} &= \psi_M(X_{k/n}^{n,M})\left(\bar{g}_k^{n,i} + \frac{n}{2}\gamma_k^{n,i}(X^{n,M})(X_{(k+1)/n}^{n,M} - X_{k/n}^{n,M})^2 + n\varepsilon_k^{n,M,i}\right).
\end{aligned}$$

Since

$$\frac{\sum_{i=1}^{N_1}\alpha_k^{n,i}(X^{n,M})}{\bar{\alpha}_k^n(X^{n,M})} \leq \frac{1}{1+N_2\delta_0/(N_1K_0)}$$

by virtue of [A1], we get

$$\begin{aligned}
\sum_{i=1}^{N_1} \hat{L}_{[nt]/n}^{n,M,i} &\leq \sum_{i=1}^{N_1}\left(\max_{0\leq k\leq [nt]}|G_k^{n,M,i}| + \max_{0\leq k\leq [nt]}|\tilde{G}_k^{n,M,i}|\right) + \frac{2}{n}\sum_{i=1}^{N}\sum_{k=0}^{[nt]-1}|\hat{h}_k^{n,M,i}| \\
&\qquad + \frac{1}{1+N_2\delta_0/(N_1K_0)}\sum_{i=1}^{N_1}\hat{L}_{[nt]/n}^{n,M,i},
\end{aligned}$$

and thus

$$\begin{aligned}
&\sum_{i=1}^{N_1} \hat{L}_{[nt]/n}^{n,M,i} \\
\leq\ & \left(1+\frac{N_1K_0}{N_2\delta_0}\right)\left\{\sum_{i=1}^{N_1}\left(\max_{0\leq k\leq [nt]}|G_k^{n,M,i}| + \max_{0\leq k\leq [nt]}|\tilde{G}_k^{n,M,i}|\right) + \frac{2}{n}\sum_{i=1}^{N}\sum_{k=0}^{[nt]-1}|\hat{h}_k^{n,M,i}|\right\}. \quad (3.14)
\end{aligned}$$

By [A4] and Propositions 5–6, we have

$$\sum_{i=1}^{N}\mathrm{E}\left[\left(\sum_{k=0}^{[nt]-1}|\hat{h}_k^{n,M,i}|\right)^8\right] \leq C'_{M,t} \quad (3.15)$$

for some $C'_{M,t} > 0$. Moreover, since $(G_k^{n,M,i})_k$ and $(\tilde{G}_k^{n,M,i})_k$ are both $(\mathcal{G}_k^n)_k$-martingales, where $\mathcal{G}_k^n$ is the σ-algebra generated by $\bar{g}_l^{n,i}$ and $\tilde{g}_l^{n,i}$ for $i = 1, \dots, N$, $l = 0, \dots, k-1$ (note that $\mathcal{G}_0^n$ is a trivial σ-algebra), the Doob inequality implies

$$
\begin{aligned}
&\mathrm{E}[\max_{0\le k\le [nt]} |G_k^{n,M,i}|^8 + \max_{0\le k\le [nt]} |\tilde{G}_k^{n,M,i}|^8] \le \left(\frac{8}{7}\right)^8 \mathrm{E}[|G_{[nt]}^{n,M,i}|^8 + |\tilde{G}_{[nt]}^{n,M,i}|^8] \\
&\le \frac{2}{n^2}\left(\frac{8}{7}\right)^8 \mathrm{E}\left[\left(\sum_{k=0}^{[nt]-1} \psi_M(X_{k/n}^{n,M}) \tilde{g}_k^{n,i}(X^{n,M})\right)^8\right] \le C''_{M,t} \qquad (3.16)
\end{aligned}
$$

for some $C''_{M,t} > 0$. Now by (3.14)–(3.16), we obtain the assertion. ■

Proposition 8. *For every $t > 0$ there exists a constant $C_{M,t} > 0$ such that*

$$
\mathrm{E}\Big[\Big(\sum_{k=0}^{[nt]-1} |\tilde{\varepsilon}_k^{n,M,i}|\Big)^4\Big] \le \frac{C_{M,t}}{n}.
$$

Proof. It suffices to show this for $\hat{\varepsilon}_k^{n,M}$ instead of $\tilde{\varepsilon}_k^{n,M,i}$. A straightforward calculation gives

$$
\begin{aligned}
\mathrm{E}\Big[\Big(\sum_{k=0}^{[nt]-1} |\hat{\varepsilon}_k^{n,M}|\Big)^4\Big] &\le \mathrm{E}\Big[\Big\{\Big(\sum_{i=1}^{N_1} \hat{L}_{[nt]}^{n,M,i}\Big) \max_{0\le k\le [nt]-1} |\hat{\psi}_k^{n,M}| + \sum_{k=0}^{[nt]-1} |\pi_k^{n,M}|\Big\}^4\Big] \\
&\le 8\,\mathrm{E}\Big[\Big(\sum_{i=1}^{N_1} \hat{L}_{[nt]}^{n,M,i}\Big)^4 \max_{0\le k\le [nt]-1} |\hat{\psi}_k^{n,M}|^4\Big] + 8N^3 \sum_{k=0}^{[nt]-1} \mathrm{E}[|\pi_k^{n,M}|^4],
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\psi}_k^{n,M} &= \sum_{i=1}^{N_1} \hat{\eta}_k^{n,M,i} + 2\psi_M(X_{k/n}^{n,M}) \sum_{j=1}^{N} \left(\frac{1}{\sqrt{n}} \tilde{g}_k^{n,i}(X^{n,M}) + \rho_k^{n,M,i}\right), \\
\pi_k^{n,M} &= \psi_M(X_{k/n}^{n,M})^2 \left\{\frac{2}{\sqrt{n}} \sum_{i,j=1}^{N} \tilde{g}_k^i(X^{n,M}) \rho_k^{n,M,j} + \left(\sum_{i=1}^{N} \rho_k^{n,M,i}\right)^2\right\}.
\end{aligned}
$$

Since [A2] and Proposition 5 imply $\mathrm{E}[\psi_M(X^{n,M})(\rho_k^{n,M,i})^8] \le C'_M/n^4$ and thus $\mathrm{E}[(\hat{\psi}_k^{n,M})^8] \le C''_M/n^4$, $\mathrm{E}[(\pi_k^{n,M})^4] \le C''_M/n^2$ for some $C'_M, C''_M > 0$ (by virtue of Proposition 6), using Proposition 7, we get

$$
\begin{aligned}
\mathrm{E}\Big[\Big(\sum_{k=0}^{[nt]-1} |\hat{\varepsilon}_k^{n,M}|\Big)^4\Big] &\le C'''_{M,t} \left\{\mathrm{E}\Big[\Big(\sum_{i=1}^{N_1} \hat{L}_{[nt]}^{n,M,i}\Big)^8\Big]^{1/4} \left(\sum_{k=0}^{[nt]-1} \mathrm{E}[|\hat{\psi}_k^{n,M}|^8]\right)^{1/4} + \frac{1}{n^2}\right\} \\
&\le C''''_{M,t}\left(\frac{1}{\sqrt{n}} + \frac{1}{n}\right)
\end{aligned}
$$

for some $C'''_{M,t}, C''''_{M,t} > 0$. This implies the assertion. ■

Proposition 9. $\mathrm{E}[\sup_{0\le t\le T}|Z_t^{n,M}|^4]\le C_{M,T}$ *for some* $C_{M,T}>0$.

Proof. Our assertion is obtained from [A2], Proposition 8, (3.16), by calculating

$$\begin{aligned}
&\mathrm{E}[\sup_{0\le t\le T}|Z_t^{n,M}|^4]\\
\le\ & \mathrm{E}\left[\max_{0\le k\le[nT]}\left|\sum_{l=0}^{k}\left\{\psi_M(X_{k/n}^{n,M})\left(\frac{1}{\sqrt{n}}\tilde{g}_k^{n,i}(X^{n,M})+\frac{1}{n}h_k^{n,i}(X^{n,M})\right)+\tilde{\varepsilon}_k^{n,M,i}\right\}\right|^4\right]\\
\le\ & C'_{M,T}\left\{\mathrm{E}[\max_{0\le k\le[nT]}|G_k^{n,M,i}|^4]+\frac{1}{n}\sum_{k=0}^{[nT]}\mathrm{E}[\sup_{|w|_\infty\le M}|h_k^{n,M,i}|^4]+\mathrm{E}[|\sum_{k=0}^{[nT]}\tilde{\varepsilon}_k^{n,M,i}|^4]\right\}
\end{aligned}$$

for some $C'_{M,T}>0$. ■

Proposition 10. *Let* $(\xi_k^n)_k$ *be uniformly bounded random variables such that* ξ_k^n *is* $\mathcal{G}_{k-1}^n$*-adapted. (Here,* $(\mathcal{G}_k^n)_k$ *is defined as in the proof of Proposition 7.) Put* $\hat{Z}_t^{n,M,i}=\int_0^t\xi_{[nr]}^n dZ_r^{n,M,i}$*. Then* $(\hat{Z}_t^{n,M})_t$ *is tight on* $\mathcal{C}$*.*

Proof. It suffices to show

$$\mathrm{E}[|\hat{Z}_u^{n,M,i}-\hat{Z}_t^{n,M,i}|^2|\hat{Z}_t^{n,M,i}-\hat{Z}_s^{n,M,i}|]\le C_{M,T}(u-s)^{3/2},\quad n\in\mathbb{N} \tag{3.17}$$

and

$$\limsup_{n\to\infty}\mathrm{E}[|\hat{Z}_u^{n,M,i}-\hat{Z}_t^{n,M,i}|^2]\le C_{M,T}(u-t) \tag{3.18}$$

for any $0\le s\le t\le u\le T$ and for some $C_{M,T}>0$ (cf. [14]). Set $\Phi=|\hat{Z}_t^{n,M,i}-\hat{Z}_s^{n,M,i}|$ for brevity. Using the inequality

$$\left(\sum_{l=1}^{k}x_l\right)^2=\sum_{l=1}^{k}x_l^2+2\sum_{l=1}^{k}x_l(x_1+\cdots+x_l),\quad x_1,\dots,x_k\in\mathbb{R}$$

and the uniform boundedness of $(\xi_k^n)_k$, we get

$$\begin{aligned}
&\mathrm{E}[|\hat{Z}_u^{n,M,i}-\hat{Z}_t^{n,M,i}|^2\Phi|]\\
\le\ & 3\,\mathrm{E}\left[\left\{|\xi_{[nu]}^nH_{[nu]}^{n,M,i}|^2+|\xi_{[nt]}^nH_{[nt]}^{n,M,i}|^2+\left(\sum_{k=[nt]}^{[nu]-1}\xi_k^nH_k^{n,M,i}\right)^2\right\}\Phi\right] \qquad (3.19)\\
\le\ & C'\left\{\left(\mathrm{E}[(H_{[nu]}^{n,M,i})^4]^{1/2}+\mathrm{E}[(H_{[nt]}^{n,M,i})^4]^{1/2}+\sum_{k=[nt]}^{[nu]-1}\mathrm{E}[(H_k^{n,M,i})^4]^{1/2}\right)\mathrm{E}[\Phi^2]^{1/2}\right.\\
&\left.+\sum_{k=[nt]}^{[nu]-1}\left|\mathrm{E}[\xi_k^nH_k^{n,M,i}(X^{n,M})(\hat{Z}_{k/n}^{n,M,i}-\hat{Z}_{[nt]/n}^{n,M,i})\Phi]\right|\right\} \qquad (3.20)
\end{aligned}$$

for some positive constant C'. By [A2] and Proposition 8, we see that

$$\mathrm{E}[(H_k^{n,M,i})^4] \le \frac{C''_{M,T}}{n^2}, \quad k \le [nu]-1 \tag{3.21}$$

for some $C''_{M,T} > 0$. Moreover, [A3] implies

$$\mathrm{E}[\xi_k^n \tilde{g}_k^{n,i}(X^{n,M})(\hat{Z}_{k/n}^{n,M,i} - \hat{Z}_{[nt]/n}^{n,M,i})\Phi] = 0,$$

and hence

$$\begin{aligned}
&\left|\mathrm{E}[\xi_k^n H_k^{n,M,i}(X^{n,M})(\hat{Z}_{k/n}^{n,M,i} - \hat{Z}_{[nt]/n}^{n,M,i})\Phi]\right| \\
\le\;& \mathrm{E}\left[\left|\frac{1}{n}\psi_M(X_{k/n}^{n,M})h_k^{n,i}(X^{n,M}) + \tilde{\varepsilon}_k^{n,M,i}\right|^4\right]^{1/4} \mathrm{E}[\sup_{0\le k\le [nu]-1}(\hat{Z}_{k/n}^{n,M,i} - \hat{Z}_{[nt]/n}^{n,M,i})^4]^{1/4}\,\mathrm{E}[\Phi^2]^{1/2} \\
\le\;& C'''_{M,T}\left(\frac{1}{n} + \mathrm{E}[(\tilde{\varepsilon}_k^{n,M,i})^4]^{1/4}\right)\mathrm{E}[\sup_{0\le r\le T}|Z_r^{n,M,i}|^4]^{1/4}\,\mathrm{E}[\Phi^2]^{1/2}, \quad k \le [nu]-1
\end{aligned} \tag{3.22}$$

for some $C'''_{M,T} > 0$. The inequalities (3.20)–(3.22) and Propositions 8–9 imply

$$\mathrm{E}[|\hat{Z}_u^{n,M,i} - \hat{Z}_t^{n,M,i}|^2\Phi] \le C''''_{M,T} \times \frac{[nu]-[nt]+1}{n}\,\mathrm{E}[\Phi^2]^{1/2}$$

for some $C''''_{M,T} > 0$. Replacing Φ with 1 and performing the same calculation, we have

$$\mathrm{E}[|\hat{Z}_t^{n,M,i} - \hat{Z}_s^{n,M,i}|^2] \le C''''_{M,T} \times \frac{[nt]-[ns]+1}{n}, \tag{3.23}$$

hence

$$\mathrm{E}[|\hat{Z}_u^{n,M,i} - \hat{Z}_t^{n,M,i}|^2|\hat{Z}_t^{n,M,i} - \hat{Z}_s^{n,M,i}|] \le (C''''_{M,T})^{3/2}\left(\frac{[nu]-[ns]+1}{n}\right)^{3/2}. \tag{3.24}$$

The inequality (3.23) immediately leads to (3.18). The inequality (3.17) is now obtained by (3.24) and the same argument as in the proof of Theorem 14.1 in [3]. ■

As a consequence of Propositions 3 and 10, we can see the tightness of the processes with fixed M.

Proposition 11. *A family of processes* $(X^{n,M}, Y^{n,M}, Z^{n,M}, L^{n,M}, \hat{L}^{n,M})_n$ *is tight on* $\mathcal{C}^{2(1+N+N_1)}$.

Proof. The tightness of $(Y^{n,M}, Z^{n,M})$ is obtained directly from Proposition 10. Then Theorem 7.3 in [3] implies

$$\lim_{\delta\to 0}\limsup_{n\to\infty} P(w_T(\delta; Y^{n,M,i}) \ge \varepsilon') = 0, \quad i = 0,\dots,N \tag{3.25}$$

for every $\varepsilon' > 0$ and $T > 0$, where $w_T(\delta; x)$ stands for a modulus of continuity, i.e., $w_T(\delta;x) = \sup_{0\le s<t\le T, |t-s|\le\delta}|x(t)-x(s)|$.

For a while, we set n sufficiently large so that $1/n < \delta$. Let $0 \le s < t \le T$ be such that $|t-s| \le \delta$. By [A1], we get

$$\begin{aligned} |\hat{L}^{n,M,i}_t - \hat{L}^{n,M,i}_s| &\le \hat{\eta}^{n,M,i}_{[nt]} + \hat{\eta}^{n,M,i}_{[ns]} + \sum_{k=[ns]+1}^{[nt]-1} \left(1 + \frac{\alpha^{n,i}_k(X^{n,M})}{\bar{\alpha}^n_k(X^{n,M}) - \alpha^{n,i}_k(X^{n,M})} \right) \eta^{n,M,i}_k \\ &\le 2 \max_{0 \le k \le [nT]} \hat{\eta}^{n,M,i}_k + \left(1 + \frac{K_0}{(N-1)\delta_0} \right) w_T(\delta; L^{n,M,i}). \end{aligned} \tag{3.26}$$

Similarly, Proposition 3 implies

$$|L^{n,M,i}_t - L^{n,M,i}_s| \le 2 \max_{0 \le k \le [nT]} \eta^{n,M,i}_k + \sqrt{\hat{K}} \sum_{j=1}^{N_1} w_T(\delta; Y^{n,M,j}). \tag{3.27}$$

By (3.27), Proposition 6, and the Chebyshev inequality, it follows that

$$\begin{aligned} P(w_T(\delta; L^{n,M,i}) \ge \varepsilon) &\le P\left(\sqrt{K_0} \sum_{j=1}^{N_1} w_T(\delta; Y^{n,M,j}) \ge \varepsilon/2 \right) + P(2 \max_{0 \le k \le [nT]} \eta^{n,M,i}_k \ge \varepsilon/2) \\ &\le \sum_{j=1}^{N_1} P\left(w_T(\delta; Y^{n,M,j}) \ge \frac{\varepsilon}{2\sqrt{K_0} N_1} \right) + \frac{256}{\varepsilon^4} \sum_{k=0}^{[nT]} \mathrm{E}[(\eta^{n,M,i}_k)^4] \\ &\le \sum_{j=1}^{N_1} P\left(w_T(\delta; Y^{n,M,j}) \ge \frac{\varepsilon}{2\sqrt{K_0} N_1} \right) + \frac{256 C_M([nT]+1)}{\varepsilon^4 n^2} \end{aligned}$$

for any $\varepsilon > 0$. Taking $\limsup_n$, letting $\delta \to 0$, and applying (3.25), we get

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P(w_T(\delta; \hat{L}^{n,M,i}) \ge \varepsilon) = 0. \tag{3.28}$$

Similarly, (3.27), (3.28), and Proposition 6 imply

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P(w_T(\delta; L^{n,M,i}) \ge \varepsilon) = 0 \tag{3.29}$$

for any $\varepsilon > 0$. Furthermore, the inequality

$$|X^{n,M}_t - X^{n,M}_s| \le |Y^{n,M,0}_t - Y^{n,M,0}_s| + \frac{1}{(N-1)\delta_0} \sum_{i=1}^{N_1} |L^{n,M,i}_t - L^{n,M,i}_s|$$

gives

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P(w_T(\delta; X^{n,M}) \ge \varepsilon) = 0. \tag{3.30}$$

Our assertion is obtained from (3.28)–(3.30) and the fact that the initial values $X^{n,M}_0$, $L^{n,M,i}_0$, $\hat{L}^{n,M,i}_0$ are all constants. ∎

Proposition 11 tells us that for any sequence $(n_k)_k \subset \mathbb{N}$ there is a subsequence $(n_{k_l})_l \subset (n_k)_k$ such that $(X^{n_{k_l},M}, Y^{n_{k_l},M}, Z^{n_{k_l},M}, L^{n_{k_l},M}, \hat{L}^{n_{k_l},M})$ converges weakly to a certain continuous process $(X^M, Y^M, Z^M, L^M, \hat{L}^M)$ defined on some probability space $(\Omega^M, \mathcal{F}^M, P^M)$ on $\mathcal{C}^{2(1+N+N_1)}$ as $l \to \infty$. Furthermore, since [A1] and [A4] imply that the convergences

$$\tilde{\alpha}^{n,i}_{[nt]}(X^{n,M}) \longrightarrow \tilde{\alpha}^i(t, X^M), \quad Q^{n,ij}_{[nt]}(X^{n,M}) \longrightarrow Q^{ij}(t, X^M), \quad n \to \infty$$

are uniform in $t \in [0,T]$ for all $T > 0$, using Proposition 7, Theorem 2.2 in [16], and the continuous mapping theorem (Theorem 2.7 in [3]), we obtain the weak convergence of

$$\begin{aligned} &\left(X^{n_{k_l},M}, \varphi^{n_{k_l},M}, Y^{n_{k_l},M}, Z^{n_{k_l},M}, L^{n_{k_l},M}, \tilde{I}^{n_{k_l},M}, \tilde{J}^{n_{k_l},M}\right) \\ &\longrightarrow \left(X^M, \varphi^M, Y^M, Z^M, L^M, \tilde{I}^M, \tilde{J}^M\right), \quad l \to \infty, \end{aligned} \tag{3.31}$$

where

$$\begin{aligned} \tilde{I}^{n,M,i}_t &= \int_0^t \tilde{\alpha}^{n,i}_{[nr]}(X^{n,M}) dL^{n,M,i}_r, \quad \tilde{J}^{n,M,i}_t = \int_0^t Q^{n,ij}_{[nt]}(X^{n,M}) dL^{n,M,i}_r, \\ \tilde{I}^{M,i}_t &= \int_0^t \tilde{\alpha}^i(r, X^M) dL^{M,i}_r, \quad \tilde{J}^{M,i}_t = \int_0^t Q^{ij}(t, X^M) dL^{M,i}_r \end{aligned}$$

and

$$\varphi^{M,i}_t = Y^{M,i}_t + 1_{I_1}(i) L^{M,i}_t - \sum_{j=1}^{N_1} \tilde{J}^{M,j}_t. \tag{3.32}$$

Note that (3.5) and (3.31) tell us

$$X^M_t = Y^{M,0}_t + \tilde{I}^{M,i}_t. \tag{3.33}$$

Let us introduce a filtration on $(\Omega^M, \mathcal{F}^M, P^M)$. We define $\mathcal{G}^M_t = \sigma(X^M_r, Z^M_r, L^M_r; r \le t)$ and let $(\mathcal{F}^M_t)_t$ be an enlarged filtration of $(\mathcal{G}^M_t)_t$ such that $(\Omega^M, \mathcal{F}^M, (\mathcal{F}^M_t)_t, P^M)$ satisfies the usual condition. We notice that the processes φ^M, Y^M and $\hat{L}^M$ are also $(\mathcal{F}^M_t)_t$-adapted. Now let us define

$$\begin{aligned} N^{M,i}_t &= Z^{M,i}_t - \int_0^t \tilde{\beta}^{M,i}(r, X^M) dr, \\ \tilde{N}^{M,ij}_t &= N^{M,i}_t N^{M,j}_t - \int_0^t \psi_M(X^M_r)^2 a^{ij}(r, X^M) dr, \end{aligned} \tag{3.34}$$

where $\tilde{\beta}^{M,i}(t,w) = \psi_M(w(t))\beta^i(t,w) + \psi_M(w(t))^3 \tilde{\gamma}^i(t,w)$.

Proposition 12. *For each $i, j = 1, \dots, N$, the processes $(N^{M,i}_t)_t$ and $(\tilde{N}^{M,ij}_t)_t$ are both $(\mathcal{F}^M_t)_t$-martingales.*

Proof. It suffices to show that $(N^{M,i}_t)_t$ and $(\tilde{N}^{M,ij}_t)_t$ are $(\mathcal{G}^M_t)_t$-martingales. Set

$$\begin{aligned} N^{n,M,i}_t &= \frac{1}{\sqrt{n}} \sum_{k=0}^{[nt]-1} \tilde{g}^{n,i}_k(X^{n,M}) + \frac{nt - [nt]}{\sqrt{n}} \psi_M(X^{n,M}_{k/n}) \tilde{g}^{n,i}_{[nt]}(X^{n,M}), \\ \bar{h}^{n,M,i}_k(w) &= \mathrm{E}[h^{n,M,i}_k(w)], \quad \tilde{\beta}^{n,M,i}_k(w) = \psi_M(w(k/n)) \bar{h}^{n,M,i}_k(w). \end{aligned}$$

Then we have

$$
\begin{aligned}
&\mathrm{E}\left[\sup_{0\le t\le T}\left|Z_t^{n,M,i}-N_t^{n,M,i}-\int_0^t \tilde\beta_{[nr]}^{n,M,i}(X^{n,M})dr\right|^2\right]\\
&\le \frac{2}{n^2}\,\mathrm{E}\left[\max_{0\le k\le [nT]+1}\left(\tilde H_k^{n,M,i}\right)^2\right]+2\,\mathrm{E}[|\sum_{k=0}^{[nt]}\tilde\varepsilon_k^{n,M,i}|^2],
\end{aligned}
$$

where

$$
\tilde H_k^{n,M,i}=\sum_{l=0}^{k-1}\psi_M(X_{k/n}^{n,M})(h_k^{n,M,i}(X^{n,M})-\bar h_k^{n,M,i}(X^{n,M})).
$$

Then the same calculation as (3.16) and Proposition 8 leads us to

$$
\mathrm{E}\left[\sup_{0\le t\le T}\left|Z_t^{n,M,i}-N_t^{n,M,i}-\int_0^t \tilde\beta_{[nr]}^{n,M,i}(X^{n,M})dr\right|^2\right]\longrightarrow 0,\quad n\to\infty. \tag{3.35}
$$

Moreover [A4] implies $\lim_{n\to\infty}\tilde\beta_{[nt]}^{n,M,i}(w)=\tilde\beta^{M,i}(t,w)$ uniformly on any compact subset of $\mathcal{C}$ for all $t\ge 0$. Thus, using (3.31), we get

$$
\begin{aligned}
&\left(X^{n_{k_l},M},Z^{n_{k_l},M},L^{n_{k_l},M},\left(\int_0^{\cdot}\tilde\beta_{[nr]}^{n,M,i}(X^{n,M})dr\right)_i\right)\\
&\longrightarrow\left(X^M,Z^M,L^M,\left(\int_0^{\cdot}\tilde\beta^{M,i}(r,X^M)dr\right)_i\right),\quad l\to\infty
\end{aligned}
\tag{3.36}
$$

weakly on $\mathcal{C}$. The convergences (3.35)–(3.36) and the continuous mapping theorem imply that

$$
(X^{n_{k_l},M},Z^{n_{k_l},M},L^{n_{k_l},M},N^{n_{k_l},M})\ \longrightarrow\ (X^M,Z^M,L^M,N^M),\quad l\to\infty\ \text{ weakly on } \mathcal{C}^{2N}. \tag{3.37}
$$

On the other hand, by [A3]–[A4], we have

$$
\mathrm{E}[(N_t^{n,M,i}-N_s^{n,M,i})\Phi((X_{s_l}^{n,M})_{l=1}^m,(Z_{s_l}^{n,M})_{l=1}^m,(L_{s_l}^{n,M})_{l=1}^m)]=0
$$

and

$$
\begin{aligned}
&\mathrm{E}[(N_t^{n,M,i}N_t^{n,M,j}-N_s^{n,M,i}N_s^{n,M,j})\Phi((X_{s_l}^{n,M})_{l=1}^m,(Z_{s_l}^{n,M})_{l=1}^m,(L_{s_l}^{n,M})_{l=1}^m)]\\
=\;&\mathrm{E}[(N_t^{n,M,i}-N_s^{n,M,i})(N_t^{n,M,j}-N_s^{n,M,j})\Phi]\\
=\;&\int_s^t \mathrm{E}[\psi_M(X_r^{n,M})^2 a_{[nr]}^{n,ij}(X^{n,M})\Phi]dr\\
&-\frac{(nt-[nt])([nt]+1-nt)}{n}\,\mathrm{E}[\psi_M(X_{[nt]/n}^{n,M})a_{[nt]}^{n,ij}(X^{n,M})\Phi]\\
&-\frac{(ns-[ns])([ns]+1-ns)}{n}\,\mathrm{E}[\psi_M(X_{[ns]/n}^{n,M})a_{[ns]}^{n,ij}(X^{n,M})\Phi]
\end{aligned}
$$

for any $m \in \mathbb{N}$, $0 \le s_1 \le \cdots \le s < t$ and any bounded continous function $\Phi : \mathbb{R}^{(1+N+N_1)m} \longrightarrow \mathbb{R}$. Thus, using [A2], (3.37), and the dominated convergence theorem, we obtain

$$\mathrm{E}^M[(N_t^{M,i} - N_s^{M,i})\Phi((X_{s_l}^M)_{l=1}^m, (Z_{s_l}^M)_{l=1}^m, (L_{s_l}^M)_{l=1}^m)] = 0, \tag{3.38}$$

$$\begin{aligned} &\mathrm{E}^M[(N_t^{M,i}N_t^{M,j} - N_s^{M,i}N_s^{M,j})\Phi((X_{s_l}^M)_{l=1}^m, (Z_{s_l}^M)_{l=1}^m, (L_{s_l}^M)_{l=1}^m)] \\ &= \mathrm{E}^M[\int_s^t \psi_M(X_r^M)^2 a^{ij}(r, X^M)dr\Phi((X_{s_l}^M)_{l=1}^m, (Z_{s_l}^M)_{l=1}^m, (L_{s_l}^M)_{l=1}^m)] \end{aligned} \tag{3.39}$$

(where E^M stands for the expectation under P^M), which imply our assertion. ■

By Proposition 12 and the martingale representation theorem (Theorem 3.4.2 in [12]), we can construct an enlarged filtered space $(\hat{\Omega}^M, \hat{\mathcal{F}}^M, (\hat{\mathcal{F}}_t^M)_t, \hat{P}^M)$ of $(\Omega^M, \mathcal{F}^M, (\mathcal{F}_t^M)_t, P^M)$ and find an N-dimensional $(\hat{\mathcal{F}}_t^M)_t$-Brownian motion $(B_t^M)_t$ such that

$$N_t^{M,i} = \sum_{j=1}^N \int_0^t \psi_M(X_r^M)\sigma^{ij}(r, X^M)dB_r^{M,j}, \tag{3.40}$$

where the stochastic processes on $(\Omega^M, \mathcal{F}^M, (\mathcal{F}_t^M)_t, P^M)$ are regarded as defined on $(\hat{\Omega}^M, \hat{\mathcal{F}}^M, (\hat{\mathcal{F}}_t^M)_t, \hat{P}^M)$ canonically. Moreover the process $(Z_t^{M,i})_t$ becomes an $(\hat{\mathcal{F}}_t^M)_t$-semimartingale and we can define the stochastic integral

$$\int_0^\cdot \xi_t dZ_t^{M,i} = \int_0^\cdot \xi_t \tilde{\beta}^{M,i}(t, X^M)dt + \sum_{j=1}^N \int_0^\cdot \xi_t \psi_M(X_t^M)\sigma^{ij}(t, X^M)dB_t^{M,j}$$

for an $(\hat{\mathcal{F}}_t^M)_t$-progressively measurable process $(\xi_t)_t$ (under suitable moment conditions).

Proposition 13. *The following equalities hold.*

$$\begin{aligned} Y_t^{M,0} &= x_0 + \sum_{j=1}^N \int_0^t \frac{1}{\bar{\alpha}(r, X^M)} dZ_r^{M,j}, \\ Y_t^{M,i} &= \Phi^i + Z_t^{M,i} - \int_0^t \frac{\alpha^i(r, X^M)}{\bar{\alpha}(r, X^M)} dZ_r^{M,j}, \quad i = 1, \ldots, N \end{aligned}$$

Proof. By Proposition 8 and

$$\frac{1}{n} \sum_{i=1}^N \sum_{k=0}^{[nt]-1} \mathrm{E}[|\psi_M(X_{k/n}^{n,M})\tilde{g}_k^{n,i}(X^{n,M})|^2 + |\psi_M(X_{k/n}^{n,M})h_k^{n,M,i}(X^{n,M})|] \le C_{M,t}$$

for some $C_{M,t} > 0$, we can apply Theorem 2.2 in [16] to arrive at the weak convergence of

$$\begin{aligned} &\left(Y^{n_{k_l},M}, Z^{n_{k_l},M}, \left(\int_0^\cdot \frac{1}{\bar{\alpha}_{[n_{k_l}r]}^{n_{k_l}}(X^{n_{k_l},M})} dZ_r^{n_{k_l},M,i}\right)_i, \left(\int_0^\cdot \frac{\alpha^{n_{k_l},i}(X^{n_{k_l},M})}{\bar{\alpha}_{[n_{k_l}r]}^{n_{k_l}}(X^{n_{k_l},M})} dZ_r^{n_{k_l},M,i}\right)_i\right) \\ &\longrightarrow \left(Y^M, Z^M, \left(\int_0^\cdot \frac{1}{\bar{\alpha}(r, X^M)} dZ_r^{M,i}\right)_i, \left(\int_0^\cdot \frac{\alpha^i(r, X^M)}{\bar{\alpha}(r, X^M)} dZ_r^{M,i}\right)_i\right), \quad l \to \infty. \end{aligned} \tag{3.41}$$

Our assertion is now obtained using (3.3)–(3.4), (3.41), and the continuous mapping theorem. ■

By (3.31)–(3.34), (3.40), and Proposition 12–13, we obtain the following proposition.

Proposition 14. *The pair* (X^M, φ^M) *(and the regulator process* L^M*) is the solution of*

$$\begin{aligned} dX_t^M &= \hat{b}^{M,0}(t,X^M)dt + \sum_{j=1}^{N} \hat{\sigma}^{M,0j}(t,X^M)dB_t^{M,j} + \sum_{j=1}^{N_1} \tilde{\alpha}^j(t,X^M)dL_t^{M,j}, \quad X_0^M = x_0, \\ d\varphi_t^{M,i} &= \hat{b}^{M,i}(t,X^M)dt + \sum_{j=1}^{N} \hat{\sigma}^{M,ij}(t,X^M)dB_t^{M,j} + 1_{I_1}(i)dL_t^{M,i} \\ &\qquad - \sum_{j=1}^{N_1} Q^{ij}(t,X^M)dL_t^{M,j}, \quad \varphi_0^i = \Phi^i, \quad i = 1,\dots,N, \end{aligned} \tag{3.42}$$

where $\hat{b}^M$ *and* $\hat{\sigma}^M$ *are given by (2.9)–(2.10) upon replacing* $\tilde{\beta}^i$ *and* σ^{ij} *by* $\tilde{\beta}^{M,i}$ *and* $\sigma^{M,ij}$*.*

Proof. It is obvious that $(\varphi_t^{M,i})_t$ is non-negative, $(L_t^{M,j})_t$ is non-decreasing, and $L_0^{M,j} = 0$ for $i = 1,\dots,N$ and $j = 1,\dots,N_1$. The rest of the proof is to show

$$\int_0^\infty \varphi_r^{M,i} dL_r^{M,i} = 0, \quad i = 1,\dots,N_1 \ \text{ almost surely.} \tag{3.43}$$

By the definition of $L^{n,M}$, we have

$$\int_0^T \varphi^{n,M,i}_{([nr]+1)/n} dL_r^{n,M,i} = \sum_{l=0}^{[nT]-1} \varphi^{n,M,i}_{(l+1)/n}\eta_l^{n,M,i} + (nT - [nT])\varphi^{n,M,i}_{([nT]+1)/n}\eta^{n,M,i}_{[nT]} = 0, \quad T \geq 0. \tag{3.44}$$

Propositions 6–7 imply

$$\begin{aligned} & \mathrm{E}[\sup_{0\leq t\leq T} |\int_0^t (\varphi^{n,M,i}_{([nr]+1)/n} - \varphi_r^{n,M,i})dL_r^{n,M,i}|] \\ \leq\ & \mathrm{E}[(L_T^{n,M,i})^2]^{1/2} \left(\sum_{j=1}^{N} \mathrm{E}[\max_{0\leq k\leq [nT]} (H_k^{n,M,j})^2]^{1/2} + \sum_{j=1}^{N_1} \mathrm{E}[\max_{0\leq k\leq [nT]} (\hat{\eta}_k^{n,M,j})^2]^{1/2} \right) \\ \leq\ & \mathrm{E}[(L_T^{n,M,i})^2]^{1/2} \left\{ \sum_{j=1}^{N} \left(\sum_{k=0}^{[nT]} \mathrm{E}[(H_k^{n,M,j})^4] \right)^{1/4} + \sum_{j=1}^{N_1} \left(\sum_{k=0}^{[nT]} \mathrm{E}[(\hat{\eta}_k^{n,M,j})^4] \right)^{1/4} \right\} \\ \leq\ & \frac{C_{M,T}}{n^{1/4}} \longrightarrow 0, \quad n \to \infty, \quad T > 0 \end{aligned} \tag{3.45}$$

for some $C_{M,T} > 0$. Using (3.44)–(3.45) and Theorem 2.2 in [16], we obtain (3.43). ∎

Here, (3.32) and (3.43) imply that $((\varphi^{M,i})_{i=1}^{N_1}, (L^{M,i})_{i=1}^{N_1})$ is a solution of the Skorokhod problem associated with $(Y^{M,i})_{i=1}^{N_1}$ (for given X^M). Then, applying the standard argument of the Skorokhod problem, we get

$$L_t^{M,i} = \sup_{0\leq r\leq t} \left(\sum_{j=1}^{N_1} \int_0^t Q^{ij}(r,X^M)dL_r^{M,j} - Y_t^{M,i} \right)_+, \quad i = 1,\dots,N_1.$$

Hence, similar to Proposition 3, the same arguments as in the proof of Theorem 2 of [21] leads us to

$$\sum_{i=1}^{N_1} |L_t^{M,i} - L_s^{M,i}|^2 \leq \hat{K} \sup_{s\leq r\leq t} \sum_{i=1}^{N_1} |Y_r^{M,i} - Y_s^{M,i}|^2, \quad 0 \leq s < t \tag{3.46}$$

for some $\hat{K} > 0$ which depends only on V.

Proposition 15. $\sup_M \mathrm{E}^M[\sup_{0\leq t\leq T} |X_t^M|^4] < \infty$ *for all* $T > 0$.

Proof. Take any $R > 0$ and set $\tau_R = \inf\{t \geq 0\,;\, |X_t^M| \geq R\}$ and $m_t^R = \mathrm{E}^M[\sup_{0\leq r\leq \min\{t,\tau_R\}} |X_r^M|^4]$. From (3.46), we see that

$$\begin{aligned}
&\sum_{i=1}^{N_1} \sup_{s\leq r\leq t} \left| \int_s^r \tilde{\alpha}^i(u, X^M) dL_u^{M,i} \right|^2 \leq \frac{1}{(N-1)^2\delta_0^2} \sum_{i=1}^{N_1} |L_t^{M,i} - L_s^{M,i}|^2 \\
&\leq \frac{\hat{K}}{(N-1)^2\delta_0^2} \sum_{i=1}^{N_1} \sup_{s\leq r\leq t} \left| \int_s^r \hat{b}^{M,i}(v, X^M) dv + \sum_{j=1}^{N} \int_0^r 1_{[s,\infty)}(v) \hat{\sigma}^{M,ij}(v, X^M) dB_v^{M,j} \right|^2,
\end{aligned} \tag{3.47}$$

hence [A6], Proposition 14, the Hölder inequality, and the Burkholder–Davis–Gundy inequality imply

$$\begin{aligned}
m_t^R &\leq C\Bigg\{1 + T^3 \sum_{i=0}^{N} \int_0^t \mathrm{E}^M[1_{\{\tau_R\geq r\}} \sup_{0\leq s\leq r} |\hat{b}^{M,i}(s, X^M)|^4] dr \\
&\qquad + T \sum_{i=0}^{N} \sum_{j=1}^{N} \int_0^t \mathrm{E}^M[1_{\{\tau_R\geq r\}} \sup_{0\leq s\leq r} |\hat{\sigma}^{M,ij}(s, X^M)|^4] dr \Bigg\} \\
&\leq C_T' \left\{1 + \int_0^t m_r^R dr\right\}, \quad t \leq T
\end{aligned}$$

for some $C > 0$ and $C_T' > 0$. Then we apply the Gronwall inequality to get $m_T^R = 0$. Our assertion is now obtained by letting $R \to \infty$. ■

The inequality (3.46) and Proposition 15 immediately give the following proposition.

Proposition 16. $\sup_M \mathrm{E}^M[(L_T^M)^4] < \infty$ *for all* $T > 0$.

Proposition 17. *The family of processes* $(X^M, \varphi^M, Y^M, Z^M, L^M)_M$ *is tight on* $\mathcal{C}^{2+2N+N_1}$.

Proof. By (3.46), (3.47), and a calculation similar to that in the proof of the above proposition, we have

$$\begin{aligned}
&\mathrm{E}^M[|X_t^M - X_s^M|^4] + \sum_{i=1}^{N} E^M[|\varphi_t^{M,i} - \varphi_s^{M,i}|^4] + \sum_{i=1}^{N_1} \mathrm{E}^M[|L_t^{M,i} - L_s^{M,i}|^4] \\
&\leq C \sum_{i=0}^{N} \mathrm{E}^M[\sup_{s\leq r\leq t} |Y_t^{M,i} - Y_s^{M,i}|^4] \leq C_T'(t-s) \int_s^t (1 + \mathrm{E}^M[\sup_{0\leq v\leq r} |X_v^M|^4]) dr \\
&\leq C_T'(1 + \sup_M \mathrm{E}^M[\sup_{0\leq r\leq T} |X_r^M|^4])(t-s)^2, \quad 0 \leq s \leq t \leq T
\end{aligned} \tag{3.48}$$

for some $C, C'_T > 0$. Similarly,

$$\mathrm{E}^M[|Z_t^M - Z_s^M|^4] \le C''_T(1 + \sup_M \mathrm{E}^M[\sup_{0\le r\le T} |X_r^M|^4])(t-s)^2 \tag{3.49}$$

for some $C''_T > 0$. The inequalities (3.48)–(3.49), Proposition 15, and Theorem 2.3 in [22] then give the assertion. ∎

Proof of Theorem 1. By Proposition 17, we see that for any non-decreasing sequence $(M_k)_k$ there is a subsequence $(M_{k_l})_l \subset (M_k)_k$ and continuous processes (X, φ, Y, Z, L) on a certain probability space $(\Omega, \mathcal{F}, P)$ such that

$$(X^{M_{k_l}}, \varphi^{M_{k_l}}, Y^{M_{k_l}}, Z^{M_{k_l}}, L^{M_{k_l}}) \;\longrightarrow\; (X, \varphi, Y, Z, L), \quad l \to \infty \;\text{ weakly.} \tag{3.50}$$

We define $(N_t^i)_t$ by

$$N_t^i = Z_t^i - \int_0^t \tilde{\beta}^i(r, X)dr. \tag{3.51}$$

As in Proposition 12, we get the weak convergence $(X^{M_{k_l}}, Z^{M_{k_l}}, L^{M_{k_l}}, N^{M_{k_l}}) \longrightarrow (X, Z, L, N)$ by Proposition 17. Thus, using (3.38)–(3.39) and the martingale representation theorem, we can find an N-dimensional $(\mathcal{F}_t)_t$-Brownian motion $(B_t)_t$ on a certain filtered space $(\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t)_t, \hat{P})$, which contains the original probability space $(\Omega, \mathcal{F}, P)$, such that

$$N_t^i = \sum_{j=1}^N \int_0^t \sigma^{ij}(r, X)dB_r^j. \tag{3.52}$$

for $i = 1, \dots, N$. As in Proposition 13, we get

$$Y_t^0 = x_0 + \sum_{j=1}^N \int_0^t \frac{1}{\bar{\alpha}(r, X)} dZ^j, \quad Y_t^i = \Phi^i + Z_t^i - \int_0^t \frac{\alpha^i(r, X)}{\bar{\alpha}(r, X)} dZ^j, \quad i = 1, \dots, N. \tag{3.53}$$

Moreover, by (3.43), Proposition 16, and Theorem 2.2 of [16], we get

$$\int_0^\infty \varphi_t^i dL_t^i = 0, \quad i = 1, \dots, N_1. \tag{3.54}$$

By (3.50)–(3.54) and Proposition 14, we see that (X, φ, L) is a solution of our SDER (2.8). Since [A7] implies that the distribution of (X, φ, L) is uniquely determined, we get the weak convergence $(X^M, \varphi^M, L^M) \longrightarrow (X, \varphi, L)$ as $M \to \infty$. The proof of Theorem 1 can be completed using the arguments in step (vi) of the proof of Theorem 3 in [14]. ∎

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