AN INVITATION TO THE THEORY OF GEOMETRIC FUNCTIONS

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Abstract. This compilation is an invitation to the theory of geometric functions. The foundation techniques in the field are explained and some developments mentioned. We will begin with the basic terminologies and concepts, then some subjects of inquiry in geometric functions theory. The main emphasis is on the important class of Caratheodory functions and their relations with various other classes of functions. The compilation will be accessible to a general mathematical audience.

1. Introduction

Let us begin by saying that: this talk is on complex-valued functions which are analytic in some *simply-connected* proper subdomain of the complex plane. They may be of several variables. However, our focus in this talk is mainly on those functions which are of one complex variable. Such a function (say g) is said to be analytic (regular or holomorphic) at a point z_0 in its domain if it is *complex-differentiable* there. Because these functions are analytic (and thus complex-differentiable of all orders), they have Taylor series developments about each point in their domain. They are thus expressible in certain series forms with centres at (say) z_0. Since by simple translation the nonzero centres z_0 may be *shifted* to zero, we may assume without loss of generality that the centres of the series developments of these functions are the origin. Thus an analytic function g may be expressed as:

$$g(z) = b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots .$$

By Cauchy integral formula, the coefficients $b_k = g^{(k)}(0)/k!$ where

$$g^{(k)}(0) = \frac{k!}{2\pi i} \int_\Gamma \frac{g(w)}{w^{k+1}} dw$$

and Γ is a rectifiable simple closed curve containing the origin, and g is analytic inside and on it. The function $g(z)$ is said to be univalent if it does not take the same value twice. That is if z_1, z_2 are points in the domain (say D) of g, then

$$g(z_1) = g(z_2) \Longrightarrow z_1 = z_2.$$

Put another way,

$$z_1 \neq z_2 \Longrightarrow g(z_1) \neq g(z_2), \quad \text{for all } \; z_1, z_2 \in D.$$

The unit disk: We would assume the domain of g to be the unit disk $D = \{z : |z| < 1\}$. Any justification for this? Yes. In 1851, Bernard Riemann showed that there always exists an analytic function mapping one simply connected proper subset of the complex plane to another of similar description (see [12], page 320).

> *Let $D_1 \subsetneq \mathbb{C}$ be simply connected in the z-plane and $D_2 \subsetneq \mathbb{C}$ also simply connected in the w-plane. There exists an analytic function $g(z)$ which maps D_1 onto D_2.*

This original version of the Riemann mapping theorem (RMT), though incomplete, gave rise to the theory of geometric functions. In 1907, Koebe discovered that analytic and univalent mappings have a nice property (conformality), which guarantees the Riemann assertion [1]. Assuming, without loss of generality, that $D_2 = D = \{z : |z| < 1\}$, we have the following complete version, which is the bedrock of the modern theory of geometric functions:

> *Let $D_1 \subsetneq \mathbb{C}$ be simply connected. If $z_0 \in D_1$, then there exists a unique function $g(z)$, analytic and univalent which maps D onto the the open unit disk D in such a way that $g(z_0) = 0$ and $g'(z_0) > 0$.*

Thus with the univalence (and thus conformality) of g, any simply connected proper subset of the complex plane can be mapped conformally onto the open unit disk. Furthermore, since inverse image of a conformal map is also conformal, it follows from the RMT that simply connected proper subsets of the complex plane are conformal equivalents. In particular, simply connected proper subsets of the complex plane are conformally equivalent to the unit disk, so that many varieties of problems about such domains are invariably reducible to the special case of the open unit disk.

Normalization: The function g shall be normalized such that:

(i) it takes the value zero at the origin (that is it takes the origin to the origin, $g(0) = 0$) and

(ii) its derivative takes the value 1 at the origin, that is $g'(0) = 1$.

Why? Observe from the RMT that, without loss of generality, we may, by translation, take $z_0 = 0$ so that the assertion becomes:

> *Let $D \subsetneq \mathbb{C}$ be simply connected. If D contains the origin, then there exists a unique function g, analytic and univalent which maps D_1 onto the the open unit disk D in such a way that $g(0) = 0$ and $g'(0) > 0$.*

The requirements that $g(0) = 0$ and $g'(0) > 0$ thus justify the normalization. Now, how is this to be achieved? Define

$$f(z) = \frac{g(z) - b_0}{b_1}$$

provided the coefficient $b_1 \neq 0$. Is this condition true of all analytic function g? Definitely not! The analytic function $g(z) = z^2$ is a counterexample. However, there are yet many others so normalizable. So, we know, sure, that the class of analytic functions so normalizable is nonempty. In fact, there exists a subset of them which have a nice underlying property. Alas, these are those that are univalent (already defined above). In geometric functions' parlance, such functions are variously called *simple*, *schlicht* (German) or *odnolistni* (Russian). It is not so difficult to see that:

> *If an analytic function, f, is univalent in a domain D, then $f'(z) \neq 0$ there.*

Proof. Suppose without loss of generality that D contains the origin and so $f(z) = b_0 + b_1 z + b_2 z^2 + \cdots$. Suppose also that $f'(0) = 0$ so that $f(z) = b_0 + b_2 z^2 + \cdots$. Then

$$\frac{f(z) - b_0}{b_2} = z^2 + \frac{b_3}{b_2} z^3 + \cdots .$$

For sufficiently small z (near the origin, say), z^2 dominates and thus by Rouche's theorem $\frac{f(z)-b_0}{b_2}$ has two roots near the origin, which contradicts that $f(z)$ is univalent in D. This proves that $f'(z) \neq 0$ in D if $f(z)$ is univalent there. □

Now, it follows that analytic and univalent functions do not have zero b_1, and we are thus guaranteed that with the univalence of g, the desired normalization can be effected. The class of normalizable (univalent) functions are thus isolated and denoted by S, say, and are represented by:

$$f(z) = z + a_2 z^2 + \cdots \tag{1}$$

where $a_k = b_k / b_1$, $k = 2, 3, \cdots$ and $b_1 \neq 0$.

The range of f: Is the nomenclature *geometric function theory* a misnomer or not for this field of study? No, it isn't. In the words of Macgregor [19]:

> *The significance of geometric ideas and problems in complex analysis is what is suggested by the term geometric function theory. These ideas also occur in real analysis, but geometry has had a much greater impact in complex analysis and it is a very fundamental aspect of its vitality.*

Duren [8] adds:

> *The interplay of geometry and analysis is perhaps the most fascinating aspect of complex function theory. The theory of univalent functions is concerned primarily with such relations between analytic structure and geometric behaviour.*

The ranges of these functions describe variously distinct geometries and classical characterizations. An example is: if f is a normalized analytic and univalent function in E, then its range contains some disk $|w| < \delta$. Furthermore, the ranges of some of them describe *star, close-to-star, convex, close-to-convex or linearly accessible, spiral* geometries: some in certain directions, some uniformly, some with respect to conjugate symmetric points and so on. These functions whose ranges describe certain geometries are thus known as geometric functions and their study became known as Geometric Function Theory.

In particular, a region of the complex plane is said to have star geometry with respect to a fixed point in it if every other point of it is visible from the fixed point. In other words, a ray or line segment issuing from the fixed point inside it to any other point of it lies entirely in it. If a region has star geometry with respect to every point in it, it is called convex. That is, the line segment joining any two points of this region lies entirely inside it.

Functions whose ranges have star geometry are known as starlike functions while those whose ranges have convex geometry are called convex. This same notion is expressed in many other classes of functions.

Between analysis and geometry: Any connections? Yes. Researchers have made groundbreaking discoveries between analysis and geometry. They have succeeded not only in describing those geometries in succinct mathematical terms, but also in establishing close links between certain prescribed properties of analytic functions and the geometries of their ranges. For example, if a function f maps the unit disk onto a star domain, then the real part of the quantity zf'/f is positive. The converse is also true. Similarly, if f maps the unit disk onto a convex domain, then the real part of the quantity zf''/f' is greater than -1. The converse is also true. Furthermore, with the truth also of the converse, if f maps the unit disk respectively onto a close-to-star, close-to-convex or linearly accessible, spiral domains, then the real part of the quantities f/g, g starlike; f'/g', g is convex; and $e^{i\theta}zf'/f$ is positive.

The above, perhaps, have led to the thinking that if the positivity condition of the real parts of many of these quantities is necessary as well as sufficient for univalence, then what can be said of other quantities such as f', those involving higher derivatives or defined by certain operators and more recently of linear combinations of two or more of such quantities?

Any examples?: Yes. The leading member of the large family of normalized univalent functions is the famous Koebe functions given by

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots .$$

The Koebe function maps the open unit disk onto the entire complex plane except a slit along the negative real axis from $-\frac{1}{4}$ through to $-\infty$. For many problems regarding the entire family of normalized univalent functions (and some subsets of it), the Koebe function assumes the best possible extremum.

We demonstrate this by examples in later sections. A trivial member of the family is the identity mapping $f(z) = z$. The identity mapping is ubiquitous; it can be found in any subclass of the class of univalent functions.

Best possible property: The class of univalent functions and many subclasses of it are being studied in the abstract sense. Many characterizations of them apply in the general sense to all members of the class under consideration as it is the case with many subjects of pure mathematics. Now, if a property T on a class of functions (or any set, J, for that matter) is such that there exists a member of the class J assuming the extremum, then such a property is said to be *best possible* on J. For example, in S, the coefficient characterization inequality $|a_2| \leq 2$ is best possible since the Koebe function, $k(z)$, which is a member of S, takes the equality. This is to say the property $|a_2| \leq 2$ cannot be made better as along as the Koebe function is a member of the set under consideration. The synonyms of "best possible" as can be found in usage by many workers in this field are "*sharp*" and "*cannot be improved*".

Investigations about best possible properties in geometric functions theory are generally referred to as extremal problems.

2. Some subjects of inquiry

A wide range of problems of mathematical analysis are being solved in the theory of geometric functions as many as their results are being applied in many branches of mathematics, physical sciences and engineering. Before long let us refer to the great compilation by S. D. Bernardi:

> *Bibliography of Schlicht functions*, Courant Institute of Mathematical Sciences, New York University, 1966; Part II, *ibid*, 1977. Reprinted with Part III added by Mariner Publishing Co. Tampa, Florida, 1983.,

which itemizes the many subject areas of Geometric Function Theory plus the list of the many research outputs in those areas.

We now begin a mention of some of them. First we note that these univalent functions exist infinitely in nature so much so that the simple definition, $f(z_1) = f(z_2) \Longrightarrow z_1 = z_2$ or its equivalent $z_1 \neq z_2 \Longrightarrow f(z_1) \neq f(z_2)$, cannot be used in general to identify, isolate or recognize many of them. This has given birth to several new methods of mathematical analysis with the sole aim of finding conditions sufficient for univalence of analytic functions in the unit disk. In particular these methods came under what treat as the first subject of inquiry, and are usually refered to as:

Sufficient conditions for univalence. Results in this direction are as many as there are researchers in the field. They continue to appear in print with no end in sight. Notable and simplest among them is the statement:

[Noshiro-Warschawski Theorem [9]] *If f is analytic in a convex domain D and Re $f'(z) > 0$ there, then f is univalent there.*

Proof. The proof depends on the fact that the function f is defined on a line segment joining any two distinct points of its domain, say, $L : tz_2+(1-t)z_1$, so that by the transformation $z = tz_2+(1-t)z_1$ ($dz = (z_2-z_1)dt$), the linear segment $z := tz_2+(1-t)z_1$ implies that when $z = z_2$ then $(1-t)z_2 = (1-t)z_1$. This holds if and only if $t = 1$ since $z_1 \neq z_2$. Similarly when $z = z_1$ we have $tz_2 = tz_1$, which holds also if and only if $t = 0$ since $z_1 \neq z_2$. Thus we have:

$$\begin{aligned} f(z_2) - f(z_1) &= \int_{z_1}^{z_2} f'(z)dz \\ &= (z_2 - z_1) \int_0^1 f'(tz_2 + (1-t)z_1)dt. \end{aligned}$$

The right hand of the above equation is nonvanishing since Re $f'(z) > 0$, which shows that the condition $z_1 \neq z_2$ implies $f(z_1) \neq f(z_2)$. Thus f is univalent. □

In fact, the assertion of the Noshiro-Warschawski theorem is contained in an equivalent but more general statement, which is:

[Close-to-convexity [9]] *If f is analytic in the unit disk E and if for some convex function g, Re $f'(z)/g'(z) > 0$ there, then f is univalent there.*

Proof. Let D be the range of g and consider $h(w) = f(z) = f(g^{-1}(w))$, $w \in D$. Then since $h(w) = f(z)$, then $h'(w)dw = f'(z)dz$. But $z = g^{-1}(w)$, that is $w = g(z)$ so that $dw = g'(z)dz$. Hence $h'(w)dw = f'(z)dw/g'(z)$, that is

$$h'(w) = \frac{f'(z)}{g'(z)}$$

so that Re $h'(w) > 0$ in D. Thus $h(w) = f(z)$ is univalent in D. □

Subclasses of S. Perhaps more than any other, the subject of finding necessary and sufficient conditions for univalence has led to identifying many more subfamilies of the class of univalent functions in the unit disk. Some of these subclasses are discussed in Section 3.

Transformations preserving the class S. Close also to the necessary and sufficient univalence conditions is the investigation about which transformations preserve univalence in the unit disk.

The most basic of these transformations are: conjugation, $\overline{f(\bar{z})}$; rotation, $e^{-i\theta}f(e^{i\theta}z)$; dilation, $f(rz)/r$ for $0 < r < 1$; disk automorphism, $[f((z+\sigma)/(1+\bar{\sigma}z)) - f(\sigma)]/[(1-|\sigma|^2)f'(\sigma)]$, $\sigma \in E$; omitted-value, $\xi f(z)/[\xi - f(z)]$, $f(z) \neq \xi$, $\xi \in E$; square root, $\sqrt{f(z^2)}$; and the composition/range transformations, $\varphi(f(z))$ where φ is similarly normalized analytic and univalent but in the range of f.

All these transformations are easily verified via the definition $f(z_1) = f(z_2) \Longrightarrow z_1 = z_2$, except the square root transformation, which requires a little explanation as follows: note that the function $g(z) = \sqrt{f(z^2)} = z + c_3z^3 + c_5z^5 + \cdots$ is an odd analytic function such that $g(-z) = -g(z)$. So if $g(z_1) = g(z_2)$, then $f(z_1^2) = f(z_2^2)$ and thus $z_1^2 = z_2^2$. That is $z_1 = \pm z_2$. But if $z_1 = -z_2$, then $g(z_1) = g(z_2) = g(-z_1) = -g(z_1)$, so that $2g(z_1) = 0 \Longrightarrow f(z_1) = 0$ and $z_1 = 0 (= z_2)$ since $f(z) = 0$ only at the origin. Thus we have $g(z_1) = g(z_2) \Longrightarrow z_1 = z_2$, which shows that g is univalent.

Advances in the subject have led to consideration of more difficult transformations preserving S, particularly those ones which are solutions of certain linear/nonlinear differential equations. The first form of this which was introduced and studied by Alexander (see [15]) is:

$$f(z) = \mathcal{J}(g) = \int_0^z \frac{g(t)}{t} dt$$

and it is the solution of the first-order linear differential equation: $zf'(z) = g(z)$. Perhaps the simplest form is what came to be known as the Libera integral transform defined as:

$$f(z) = \mathcal{J}(g) = \frac{2}{z} \int_0^z g(t)dt, \text{ (See [17]).}$$

The Libera integral is also the solution of the first-order linear differential equation: $zf'(z) + f(z) = 2g(z)$. Various other integrals have been considered, many being generalizations of the Libera integral. For example, the following:

$$f(z) = \mathcal{J}(g) = \frac{\alpha + c}{z^c} \int_0^z t^{c-1} g(t)^\alpha dt,$$

which is a generalization of the Libera integral also preserves the class S and it is a solution of the nonlinear differential equation, $\alpha z f(z)^{\alpha-1} f'(z) + cf(z)^\alpha = (\alpha + c)g(z)^\alpha$.

Transformations of this type examine the nature and propeties of the solutions of certain differential equations given that f has some known properties and or the extent of such properties being transferable to the solutions.

It is noteworthy that in spite of the fact that the class S is preserved by many 'big' transformations as above, simple addition does not preserve it. For instance $f(z) = z/(1 - z)$ and $g(z) = z/(1 + iz)$ both belong to S. However,

$$f'(z) + g'(z) = \frac{2 - 2(1 - i)z}{(1 - z)^2(1 + iz)^2}$$

which shows that $f'(z) + g'(z) = 0$ for $z = (1 + i)/2$, which is a location in the unit disk implies that the sum $f(z) + g(z)$ does not belong to S. This excellently simple counterexample is due to Kozdron[16].

Convolution or Hadamard product. It is interesting to note that many a transformation of f is expressible as 'convolution' of f with certain other analytic function with predetermined behaviour, which may pass on to the transformation. For example the Libera transform (2) is the convolution $\mathcal{J} = g * f$ where g is the analytic function

$$g(z) = z + \sum_{k=2}^{\infty} \frac{2}{k+1} z^k.$$

This function g has some nice geometric properties which may pass on to the Libera transform via the 'convolution' as would be found in literatures. The convolution therefore is defined thus: let $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ and $g(z) = b_0 + b_1 z + b_2 z^2 + \cdots$ be analytic funtions in the unit disk, then the convolution (or Hadamard product) of $f(z)$ and $g(z)$ (written as $(f * g)(z)$) is defined as

$$(f * g)(z) = z + \sum_{k=0}^{\infty} a_k b_k z^k.$$

The concept of convolution arose from the integral

$$h(r^2 e^{i\theta}) = (f * g)(r^2 e^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i(\theta - t)}) g(re^{it}) dt, \quad r < 1$$

and has proved very useful in dealing with certain problems of the theory of analytic and univalent functions, especially closure of families of functions under certain transformations as has preceeded and has also attracted much attention of resaerchers in GFT.

Quasi-convolution or quasi-Hadamard product. Built on the foundation convolution theory, more difficult transformation problems have given birth to the subject of quasi-convolution defined in various ways in the literature. In particular, we defined the quasi-convolution of f and g in [3] as:

$$\phi(z)^\alpha = (f^\alpha * g^\alpha)(z) = z^\alpha + \sum_{k=2}^{\infty} A_k(\alpha) B_k(\alpha) z^{\alpha+k-1}$$

where $\alpha > 0$ is real, $f(z)^\alpha = z^\alpha + A_2(\alpha) z^{\alpha+1} + ...$ and $g(z)^\alpha = z^\alpha + B_2(\alpha) z^{\alpha+1} + ...$ (where $A_k(\alpha)$, $B_k(\alpha)$ respectively depend on the coeficients a_k of $f(z)$ and b_k of $g(z)$, and α) and the quasi-convolution is denoted by $\phi(z)^\alpha = (f^\alpha * g^\alpha)(z)$. The justification for inquiry into the concept of quasi-convolution also lies in some very interesting applications of it as can be found in [3].

Radius problems. If we suppose that some transformations or geometric conditions fail to preserve univalence (for instance) in the unit disk, then it is natural to ask if such transformations (or conditions) could preserve it in any subdisk $E_0 = \{z : |z| < \rho < 1\} \subset E$. Problems of this sort became known as radius problems. More precisely, it is about finding the radius ρ of the largest subdisk E_0 in which certain transformations of a univalent function f or some geometric conditions guarantee univalence. This radius ρ is particularly known as the radius of univalence (for instance). By "for instance" we imply that this notion is not restricted to the subject of univalence only. In fact, and interestingly, this has raised many more questions like: the radius of starlikeness, convexity, close-to-convexity and many more. A basic result in this direction is:

> [Noshiro, Yamaguchi [30]] *If f is analytic and satisfies Re $f(z)/z > 0$ in E, then it is univalent in the subdisk $|z| < \sqrt{2} - 1$.*

Growth and Distortion of f. The idea of growth of analytic function f refers to the size of the image domain, that is $|f(z)|$. The term, distortion, arises from the geometric interpretation of $|f'(z)|$ as the infinitesimal magnification factor of the arclength under the mapping f, or from the Jacobian $|f'(z)|^2$ as the infinitesimal magnification factor of the area of the image domain. These concepts tell much about the boundedness of these functions and their derivatives. For the class S, we have:

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}$$

and

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}.$$

For the many subclasses of these functions, the precise bounds on them and their derivatives are still being investigated and improved.

Coefficient inequalities. A close look at the series development of f suggests that many properties of it like the growth, distortion and in fact univalence, may be affected (or be told) by the size of its coefficients. Duren says:

> *In most general form, the* coefficient problem *is to determine the region of $\mathbb{C}^{n-1}$ occupied by the points $(a_2, \cdots, a_n)$ for all $f \in S$. The deduction of such precise analytic information from the geometric hypothesis of univalence is exceedingly difficult.*

The most contained in this part of this article are sourced from the survey by Duren [8], which is ample for detailed issues regarding the coefficient problems in the field.

The coefficient problem has been reformulated in the more special manner of estimating $|a_n|$, the modulus of the nth coefficient. Perhaps, no problem of the field has challenged its people as much as the coefficient problem. As early as in 1916, Bieberbach conjectured that the nth coefficient of a univalent function is less or equal to that of the Koebe function. In mathematical language, he says:

> *For each function $f \in S$, $|a_n| \le n$ for $n = 2, 3, \cdots$. Strict inequality holds for all n unless f is the Koebe function or one of its rotations.*

The conjecturer, Bieberbach, himself proved that $|a_2| \le 2$ as a simple corollary to the *area theorem* (see page 29 of [9] for the *area theorem* and the proof of Bieberbach theorem), which is due to Gromwall. The third was settled in 1923 by Loewner. The fourth was solved in 1955 by Garabedian and Schiffer, while in 1960 Charzynski and Schiffer gave an elementary proof of same result. The proofs for the fifth and sixth came several years latter. Thereafter, the great puzle had remained unsolved until only recently when, precisely 1985, De Brange announced the final solution to the notorious conjecture. In total, the conjecture had stood for sixty-nine years unsolved! These long years were not unproductive however, as the conjecture had inspired the development of important new methods and techniques in the theory in particular and complex analysis in general.

Fekete-Szegö problem. Closely related to the Bieberbach conjecture is that of finding the sharp estimate for the coefficients of odd univalent functions, which has the most general form of the square root transformation of a function $f \in S$:

$$l(z) = \sqrt{f(z^2)} = z + c_3 z^3 + c_5 z^5 + \cdots .$$

For odd univalent functions, Littlewood and Parley in 1932 proved that for each n the modulus $|c_{2n+1}|$ is less than an absolute constant A, (which their method showed is less than 14) and they added the footnote "No doubt the true bound is given by $A = 1$" which became known as the Littlewood-Parley conjecture. The truth of this conjecture for certain subclasses of S enshrouded its falsity in general, though only for a short while. As early as in 1933 (about a year after the conjecture) it was settled in negation by what came to be known as the Fekete-Szegö problem.

For each $f \in S$, Fekete and Szegö obtained the sharp bound:

$$|a_3 - \alpha a_2^2| \le 1 + 2e^{-2\alpha/(1-\alpha)}, \;\; 0 \le \alpha \le 1.$$

This results gives $|c_5| < 1/2 + e^{-2/3} = 1.013\cdots$ since $c_5 = (a_3 - a_2^2/4)/2$. Thus the Fekete-Szegö problem has continued to recieve attention until even in the many subclasses of S. The functional $|a_3 - \alpha a_2^2|$ is well known as the Fekete-Szegö functional. Many other functionals have risen after it, each finding application in certain problems of the geometric functions. For $\alpha = 1$, it is important to mention a more general problem of this type, which is the Hankel determinant problem.

Hankel determinant problem. Let $n \geq 0$ and $q \geq 1$, the q-th Hankel determinant of the coefficients of $f \in \mathcal{S}$ is defined as:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

The determinant has been investigated by several authors with the subject of inquiry ranging from rate of growth of $H_q(n)$ as $n \to \infty$ to the determination of precise bounds on $H_q(n)$ for specific q and n for some favored classes of functions. It is interesting to note that $|H_2(1)| \equiv |a_3 - a_2^2|$, the Fekete-Szegö functional for $\alpha = 1$.

Other coefficient related problems. These include the determination of coefficient relationship: for instance, Robertson [23] conjectured that for positive integers m, n, the coefficients of close-to-convex functions satisfy the inequalities $|m|a_m| - n|a_n|| \leq |m^2 - n^2|$. Solving problems of this type could sometimes provide a relief from the search for bounds on a_n for many classes of functions as this works out as easy corollaries by the triangle inequality. In particular if $m = n+1$, this is problem is referred to as successive coefficient problem.

Covering. As a consequence of the Bieberbach Theorem on the second coefficient of functions in S and their omitted-value transformation, Koebe, in 1907, discovered that range of every function, $f \in S$, covers the disk $|\xi| < 1/4$. This assertion has thus been known as the Koebe One-Quarter Theorem and its proof is as follows:

Proof. If $f \in S$ omits $\xi \in \mathbb{C}$, then

$$g(z) = \frac{\xi f(z)}{\xi - f(z)} = z + \left(a_2 + \frac{1}{\xi}\right) z^2 \cdots$$

is analytic and univalent in E. So by Bieberbach theorem

$$\left|a_2 + \frac{1}{\xi}\right| \leq 2$$

so that

$$\left|\frac{1}{\xi}\right| - |a_2| \leq \left|a_2 + \frac{1}{\xi}\right| \leq 2.$$

Hence we have

$$\left|\frac{1}{\xi}\right| - |a_2| \leq 2.$$

That is, $|1/\xi| \leq 2 + |a_2|$. Then combined with the fact that $|a_2| \leq 2$, we have $|\xi| \geq 1/4$. This means that values omitted by f lie outside the disk $|\xi| < 1/4$, hence range of every $f \in S$ covers that disk. □

Investigating the largest area of the image domain covered by all members of a class of univalent functions has also become an interesting subject.

Partial sums. The inquistion regarding partial sums

$$s_n(z) = z + a_2z^2 + \cdots + a_nz^n$$

of the series development of f is about the extent to which known geometric properties of f are carried on to its partial sums. Another result of Yamaguchi[30] is suitable to mention here:

> [Yamaguchi [30]] If f satisfies Re $f(z)/z > 0$ in E, then the kth partial sums $s_k(z) = z + a_2z^2 + \cdots + a_kz^k$ is univalent in the subdisk $|z| < \frac{1}{4}$.

Linear sums or combinations. Since it is well known that the class of univalent functions is not preserved under addition, it has also become of interest to find out: if ϕ and φ are some geometric quantities describing certain subclasses of S, then under what conditions is a combination such as $\phi + \varphi$ or the linear sum $(1 - t)\phi + t\varphi$ preserving some known geometric properties based on ϕ and φ? Examples of this type of problems can be found in in the literatures [2, 4, 27, 28].

3. Some subclasses of S

Sequel to what has preceeded with regard to some of the subclasses of the class of univalent functions, we mention that the fundamental basis or justification for discussing new subclasses lies in the fact that through them certain classes of functions may be associated with some special properties, not commonly associable with certain other classes. Thus the many subjects of inquiry are being reinvestigated in several class of functions to sharpen, smoothen or better many known results particularly in the direction of a new subclass. Very many results concerning them abound like scattered pearls in literatures. Some of the well known subclasses of S (all univalent functions) are mentioned hereafter.

Starlike and convex functions: These are the earliest known subclasses of the classes of univalent functions. They were identified in 1936 by Robertson [24]. They consist of functions which map the open unit disk onto starlike and convex domains respectively. The starlike functions have analytic representation that the real part of the quantity zf'/f is positive, that is

$$\text{Re}\,\frac{zf'(z)}{f(z)} > 0,$$

while the convex functions have representation that the real part of the quantity $1 + zf''/f'$ is positive, that is

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0.$$

They have received so much attention of workers in the field. They are finding applications in many areas of applied mathematics (see [29] for example).

Close-to-convex functions: These functions were introduced in 1952 by Kaplan [14]. They are functions for which for each $r < 1$, the tangent to the curve $E(r) = \{f(re^{i\theta}) : 0 \leq \theta \leq 2\pi\}$ never turns back on itself as much as π radian. They have the analytic representation that the real part of the quantity f'/g', (g is a convex function) is positive, that is

$$\mathrm{Re}\,\frac{f'(z)}{g'(z)} > 0.$$

Functions of bounded turning: These are functions whose derivatives have positive real parts, that is

$$\mathrm{Re}\, f'(z) > 0.$$

They were first identified by Macgregor [19] in 1962, and are a special case of the class of close-to-convex functions.

Quasi-convex functions: They are functions for which the real part of the quantity $(zf')'/g'$, (g is a convex function) is positive, that is

$$\mathrm{Re}\,\frac{(zf'(z))'}{g'(z)} > 0.$$

They were first studied by Noor and Thomas [20] in 1980, and are subclasses of the class of close-to-convex functions.

Bazilevic functions. They are functions identified in 1955 by Bazilevic [6] consisting of normalized functions defined by the integral

$$f(z) = \left\{\frac{\alpha}{1+\xi^2}\int_0^z [h(t) - i\xi]t^{-\left(1+\frac{i\alpha\xi}{1+\xi^2}\right)} g(t)^{\left(\frac{\alpha}{1+\xi^2}\right)} dt\right\}^{\frac{1+i\xi}{\alpha}}$$

where h is an analytic function which has positive real part in E and normalized by $h(0) = 1$ and g is starlike in E. The numbers $\alpha > 0$ and ξ are real and all powers meaning principal determinations only. They contain many other class of function as special cases. The class of Bazilevic functions is probably the largest known subfamily of the family of univalent functions.

Other subclasses and generalizations. There are many other subclasses of the above classes of functions which have appeared in print. Many generalizations have also appeared via derivative as well as integral operators. These include the well known derivative operators of Salagean [26]

$$D^n f(z) = z[D^{n-1)} f(z)]' \text{ with } D^0 f(z) = f(z)$$

and of Ruscheweyh [25]

$$\mathcal{D}^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) = \frac{z(z^{n-1}f(z))^{(n)})}{n!}$$

and some further generalizations of them.

Inclusion relations. By inclusion relations between classes of functions, it can be easily proved that all properties of functions of the 'bigger' classes are shared by their subclasses. This way, univalence, in particular, of new classes of functions are established. It is important to mention in this note two well known inclusion relations between the above mentioned classes:

convexity $\Longrightarrow$ *quasi-convexity* $\Longrightarrow$ *close-to-convexity* $\Longrightarrow$ *univalence.*

convexity $\Longrightarrow$ *starlikeness* $\Longrightarrow$ *close-to-convexity* $\Longrightarrow$ *univalence.*

Studies in inclusion relations between classes of functions have become well known and have continued to receive attention so much so that of late, efforts have gone as far as studying *best* inclusion relations and radius of best inclusions.

4. Caratheodory, related functions and generalizations

A thorough look at the series development (1) for f and the various geometric quantities zf'/f, $1+zf''/f'$, f/g, f'/g', and many more, (which possess the property of positivity of real parts) suggests clearly the existence of a series form:

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots \,. \tag{2}$$

The form (2) satisfies $h(0) = 1$ and Re $h(z) > 0$ (positive real parts). The present author is not aware the discovery of which predates which of the two functions, f (normalized by $f(0) = 0$ and $f'(0) = 1$) and h (normalized by $h(0) = 1$). However, it is not out of place to insinuate that the prediscovery of f over h. For otherwise, the discovery of f certainly would have spurred inquisition into h. The study of h provides much insight into the natures of any f having goemetries described above. The function h is called the Caratheodory function (named after Caratheodory who not only noticed the obvious, but expended much energy in its characterizations).

Representions of h: The class of Caratheodory functions has two other representations, one due to subordination principle while the other is an integral known as Herglotz representation. By *subordination* of an analytic function f to another g, it is meant that there exist a function of unit bound, $\vartheta(z)$ ($|\vartheta(z)| < 1$, normalized by $\vartheta(0) = 0$) such that $f(z) = g(\vartheta(z))$. Precisely, in terms ϑ, the Caratheodory function h has the form:

$$h(z) = \frac{1+\vartheta(z)}{1-\vartheta(z)}, \quad z \in E.$$

That is, h is *subordinate* to the Möbius function

$$L_0(z) = \frac{1+z}{1-z}.$$

The Möbius function play a central role in the family of functions of the like of h. It assumes the extremum in the most extremal problem for such functions.

The Herglotz representation of any h is the integral form:

$$h(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t),$$

where $d\mu(t) \geq 0$ and $\int d\mu(t) = \mu(2\pi) - \mu(0) = 1$.

The various representations of h have very important applications as may be discovered through further studies.

Two inequalities for unit-bound functions: The unit bound functions, also known as Schwarz functions, satisfy two basic inequalities which are noteworthy. The first is due Schwarz (see [7]) and reads:

If $\vartheta(z)$ is a function of unit bound in E, then for each $0 < r < 1$, $|\vartheta'(0)| < 1$ and $|\vartheta(re^{i\theta})| < r$ unless $\vartheta(z) = e^{i\sigma}z$ for some real number σ.

The above result is commonly refered to as the Schwarz's Lemma. It has the implication that if $\vartheta(z)$ is a function of unit bound in E, so also is $u(z) = \vartheta(z)/z$, that is $|u(z)| < 1$, but not necessarily normalized by $u(0) = 0$. The second inequality, due to Caratheodory [7], is as follows:

If $\vartheta(z)$ is a function of unit bound (not necessarily normalized) in E, then

$$|\vartheta'(z)| \leq \frac{1 - |\vartheta(z)|^2}{1 - |z|^2}$$

with strict inequality holding unless $\vartheta(z) = e^{i\sigma}z$ for some real number σ.

Transformations preserving the class of Caratheodory functions: The Caratheodory functions are also preserved under a number of transformations: suppose g, h are Caratheodory, then so is p defined as (i) $p(z) = g(e^{it}z)$, t real; (ii) $p(z) = g(tz)$, $t \in [-1, 1]$; (iii) $p(z) = g[(z+t)/(1+\bar{t}z)]/g(t)$, $|t| < 1$; (iv) $p(z) = (g(z)+it)/(1+itg(z))$, t real; (v) $p(z) = [g(z)]^t$, $t \in [-1, 1]$ and (vi) $p(z) = [g(z)]^t[h(z)]^\tau$, $t, \tau, t + \tau \in [0, 1]$.

Proof. By simple computation it is easy to see that in all cases, $p(0) = 1$. Thus it only remains to show that the real parts of the transformations are positive.

For (i) - (iii), this follows from the fact that each of the points $e^{it}z$, tz and $(z + t)/(1 + \bar{t}z)$, (with associated conditions on t) are transformations of points in $|z| < 1$ to points in there [To show that $|(z + t)/(1 + \bar{t}z)| < 1$, assume the converse. That is $|z + t| \geq |1 + \bar{t}z|$. Then squaring both sides we obtain $|z|^2 + |t|^2 \geq 1 + |t|^2|z|^2$, wherefrom we obtain a contradiction that $|z| \geq 1$. This proves the point].

In fact (iv) is a linear transformation of the right half plane onto itself. To see this, note that $Re\left\{\frac{g+it}{1+itg}\right\} = Re\left\{\frac{(g+it)(1-it\bar{g})}{(1+itg)(1-it\bar{g})}\right\} = \frac{Re(g+t^2\bar{g})}{|1+itg|^2} > 0$ since the real parts of g and $\bar{g}$ is greater that zero.

As for (v) and (vi), they follow from the fact that $Re\ z^t \geq (Re\ z)^t$ when $t \in [0,1]$ and $Re\ z > 0$, which is due, by elementary calculus, to the fact that $y = \cos t\theta/(\cos\theta)^t$ attains its maximum value at $t_0 \in [0,1]$ (t_0 is given by $t_0 = \frac{\arctan\left(\frac{-\log\cos\theta}{\theta}\right)}{\theta}$, for all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\theta \neq 0$) and $y(t)$ is decreasing on $t \in [t_0, 1]$; and $y(t)$ is increasing on $t \in [0, t_0]$. In particular, $y(t) = \cos t\theta/(\cos\theta)^t \geq y(0) = y(1) = 1$. Then for each $t \in [-1,0]$ with respect to (v), the function p takes the reciprocal of its values for $t \in [0,1]$. This concludes the proof. □

The class is also preserved under convex combination, that is, if p_j, $j = 1, 2, \cdot, n$ are Caratheodory functions and λ_j are real numbers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$, then the convex combination $\lambda_1 p_1 + \lambda_2 p_2 + \cdots + \lambda_n p_n$ is also a Caratheodory function.

Two basic inequalities for h: We now mention two basic coefficient inequalities for h, the first based on its Herglotz representation while the other depends on its representation by functions of unit bound $\vartheta(z)$.

> [Caratheodory (See [9])] *If $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a Caratheodory function, then $|c_k| \leq 2$, $k = 1, 2, \cdots$. The Möbius function takes the equality.*

Proof. If we expand the Herglotz representation of h in series form. The Herglotz representation $h(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t)$ can be written as

$$\begin{aligned} h(z) &= \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t) \\ &= \int_0^{2\pi} \left(1 + 2ze^{-it} + 2z^2 e^{-2it} + 2z^3 e^{-3it} + \cdots\right) d\mu(t) \\ &= 1 + \sum_{k=1}^{\infty} \left(2 \int_0^{2\pi} e^{-ikt} d\mu(t)\right) z^k. \end{aligned}$$

So, when compared with $h(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$ gives

$$c_k = 2 \int_0^{2\pi} e^{-ikt} d\mu(t).$$

So, we have

$$\begin{aligned} |c_k| &\leq 2 \int_0^{2\pi} |e^{-ikt}| d\mu(t), \text{ since } d\mu(t) \text{ is nonnegative} \\ &= 2 \int_0^{2\pi} d\mu(t) = 2, \text{ since } \int_0^{2\pi} d\mu(t) = 1. \end{aligned}$$

□

[Pommerenke [22]] *If $h(z) = 1+c_1z+c_2z^2+\cdots$ is a Caratheodory function, then*

$$\left|c_2 - \frac{c_1^2}{2}\right| \le 2 - \frac{|c_1|^2}{2}.$$

Equality holds for the function:

$$h(z) = \frac{1+\frac{1}{2}(c_1+\varepsilon\overline{c_1})z+\varepsilon z^2}{1-\frac{1}{2}(c_1-\varepsilon\overline{c_1})z-\varepsilon z^2},\ |\varepsilon| = 1.$$

Proof. Suppose $\vartheta(z)$ is a function of unit bound in E, normalized by $\vartheta(0) = 0$. Then by Schwarz's Lemma there exists an analytic function $u(z)$ also of unit bound such that $\vartheta(z) = zu(z)$. Then

$$\begin{aligned} u(z) = \vartheta(z)/z &= \frac{1}{z}\frac{h(z)-1}{h(z)+1} \\ &= \frac{1}{2}c_1 + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right)z + \cdots \end{aligned}$$

satisfies $|u(z)| \le 1$ in E so that

$$|u'(z)| \le \frac{1-|u(z)|^2}{1-|z|^2}.$$

Thus

$$\begin{aligned} |u'(0)| &= \frac{1}{2}c_2 - \frac{1}{4}c_1^2 \\ &\le 1-|u(0)|^2 = 1-\frac{1}{4}|c_1|^2. \end{aligned}$$

□

The above two basic inequalities have great implications in geometric functions theory, especially with regard to coefficient problems.

Generalizations of h: Further advances have led to various generalizations of h. Janowski [13] redefined h in terms of ϑ, saying that: given fixed real numbers a, b such that $a \in (-1, 1]$ and $b \in [-1, a)$ (that is $-1 \le b < a \le 1$), then h is defined as:

$$h(z) = \frac{1+a\vartheta(z)}{1+b\vartheta(z)},$$

where the Caratheodory function corresponds to the extremes $b = -1$, $a = 1$. For various choice values of a, b, the function h also maps the unit disk to some portions of the right half plane.

Integral iterations of h: Perhaps, if any, the most significant of our contributions to this important field of study is the development of iterations for the very important families of functions: the Caratheodory and Janowski functions (See [5] and the previous works cited therein). These are:

$$h_n(z) = \frac{\alpha}{z^\alpha}\int_0^z t^{\alpha-1}h_{n-1}(t)dt,\ \ n \ge 1,$$

with $p_0(z) = p(z)$.

$$h_{\sigma,n}(z) = \frac{\sigma - (n-1)}{z^{\sigma-(n-1)}} \int_0^z t^{\sigma-n} h_{\sigma,n-1}(t)dt, \quad n \geq 1$$

with $p_{\sigma,0}(z) = p(z)$. These transformations preserve many geometric structures of the family of functions with positive real part normalized by $h(0) = 1$; particularly the positivity of the real parts, compactness, convexity and subordination. Another fascinating aspect of these transformations is that with them, investigations of the various classes of functions associated with them have become easy, short and elegant. They have been very helpful in dealing easily with certain problems of the theory of analytic and univalent function in the most intriguing simplicity.

Problem-solving technique based on h: Many techniques have been developed in the field. However, the most fundamental and beginner-friendly is one based on the close association that exists between the Caratheodory functions (together with its further developments) and many classes of functions. Many fundamental results have been established as regards this class of functions. Thus investigating various problems of geometric functions via an underlying h has been well accepted among researchers in this field as a princpal technique. In the next (and last) section are given a few examples and insight into the technique of constructing the extremal functions.

5. Illustrating examples

The examples in this section are simple. Our objective is to demonstrate, in repeated and beginner-friendly manner, how results can be obtained in certain classes of functions using an underlying Caratheodory function, h. All the results here are best possible. The construction of extremal functions are greatly simplified.

Theorem A. *If $f \in \mathcal{S}$ satisfy Re $f(z)/z > 0$, then its coefficients satisfy the inequality: $|a_k| \leq 2$. Equality is attained by $f(z) = z(1+z)/(1-z)$.*

Proof. Since Re $f(z)/z > 0$, then $f(z)/z$ is a function with positive real parts. Hence, $f(z)/z = h(z)$ for some $h(z)$ with positive real parts. Comparing coefficients of the series expansion of $f(z)/z$ and h, we have $a_k = c_{k-1}$, $k = 2, 3, \cdots$ so that $|a_k| \leq 2$ since $|c_k| \leq 2$, $k = 1, 2, \cdots$.

The construction of the extremal function is by setting the geometric quantity $f(z)/z$ equal to the extremal function for h, which is $L_0(z) = (1+z)/(1-z)$. This simply gives $f(z) = z(1+z)/(1-z)$. □

Theorem B. *The coefficients of functions of bounded turning (Re $f'(z) > 0$) satisfy the inequality: $|a_k| \leq 2/k$. Equality is attained by $f(z) = -2\ln(1-z) - z$.*

Proof. Since Re $f'(z) > 0$, then f' is a function with positive real parts. Hence, $f'(z) = h(z)$ for some $h(z)$ with positive real parts. Comparing coefficients of the series expansion of f' and h, we have $a_k = c_{k-1}/k$, $k = 2, 3, \cdots$ so that $|a_k| \leq 2/k$ since $|c_k| \leq 2$, $k = 1, 2, \cdots$.

Again the construction of the extremal function is by setting the geometric quantity $f'(z)$ equal to the extremal function for h, which is $L_0(z) = (1+z)/(1-z)$. Thus, we have

$$f'(z) = \frac{1+z}{1-z}.$$

Integrating both sides, we have

$$f(z) = \int_0^z \frac{1+t}{1-t} dt = -\ln(1-z) - z.$$

□

Theorem C. *If f is starlike ($Re\ zf'(z)/f(z) > 0$), then $|a_k| \leq k$. Equality is attained by the Koebe function $k(z) = z/(1-z)^2$.*

Proof. Given that $f(z) = z + a_2 z^2 + \cdots$. Since Re $zf'(z)/f(z) > 0$, then $zf'(z)/f(z)$ is a function with positive real parts. Hence, $zf'(z)/f(z) = h(z)$ for some $h(z) = 1 + c_1 z + \cdots$ with positive real parts. Equating $zf'(z)/f(z)$ and h we have $zf'(z) = h(z)f(z)$. Expanding both sides in series, we have

$$zf'(z) = z + 2a_2 z^2 + 3a_3 z^3 + \cdots$$

while

$$f(z)h(z) = z + (a_2 + c_1)z^2 + (a_3 + a_2 c_1 + c_2)z^3 + (a_4 + a_3 c_1 + a_2 c_2 + c_3)z^4 + \cdots$$

so that

$$ka_k = a_k + \sum_{j=1}^{k-1} a_j c_{k-j},\ a_1 = 1$$

and thus

$$(k-1)a_k = \sum_{j=1}^{k-1} a_j c_{k-j},\ a_1 = 1.$$

We now proceed by induction. For $k = 2$, we have $a_2 = a_1 c_1$ with $a_1 = 1$ so that $|a_2| = |c_1| \leq 2$ as required. Next we suppose the inequality is true for $k = n$, then for $k = n+1$ we have

$$na_{n+1} = \sum_{j=1}^{n} a_j c_{n+1-j}$$

so that

$$n|a_{n+1}| \leq \sum_{j=1}^{n} |a_j||c_{n+1-j}| \leq 2\sum_{j=1}^{n} j = n(n+1).$$

Thus we have $|a_{n+1}| \leq n+1$ and the inequality follows by induction.

As for the extremal function, the construction is by setting the geometric quantity $zf'(z)/f(z)$ equal to the extremal function for h, which is $L_0(z) = (1+z)/(1-z)$. Thus, we have

$$\frac{zf'(z)}{f(z)} = \frac{1+z}{1-z}$$

so that

$$\frac{f'(z)}{f(z)} = \frac{1+z}{z(1-z)} = \frac{1}{z} + \frac{2}{1-z}.$$

Now integrating both sides, we have $\ln f(z) = \ln z - 2\ln(1-z)$ which gives $f(z) = z/(1-z)^2$, which is the Koebe function. □

Theorem D. *If f is convex (Re $[1 + zf''(z)/f'(z)] > 0$), then $|a_k| \le 1$. Equality is attained by the function $f(z) = 1/(1-z)$.*

Proof. Observe that we can write the geometric quantity $1 + zf''(z)/f'(z)$ as $(zf'(z))'/f'(z)$ so that the convexity condition Re $[1 + zf''(z)/f'(z)] > 0$ now becomes Re $[z(zf'(z))']/[zf'(z)] > 0$. That is $zf'(z)$ is starlike [*In fact f is convex if and only if $zf'(z)$ is starlike is a result due to Alexander and is known as Alexander theorem*]. So by the result for starlike function, the coefficients of $zf'(z)$ satisfy $|a_k| \le k$. Hence we have $k|a_k| \le k$ which gives $|a_k| \le 1$ as required.

Now to the construction of the extremal function, set the geometric quantity $1 + zf''(z)/f'(z)$ equal to the extremal function for h, which is $L_0(z) = (1+z)/(1-z)$. Thus, we have

$$\frac{z(zf'(z))'}{zf'(z)} = \frac{1+z}{1-z}$$

so that as in the previous proof we have $zf'(z) = z/(1-z)^2$. Furthermore we have $f'(z) = 1/(1-z)^2$, which on integration gives $f(z) = 1/(1-z)$. □

Theorem E. *If f is close-to-convex (Re $f'(z)/g'(z) > 0$, g is convex), then $|a_k| \le k$. Equality is attained by the Koebe function $k(z) = z/(1-z)^2$.*

Proof. Given that $f(z) = z + a_2z^2 + \cdots$. Since Re $f'(z)/g'(z) > 0$, then $f'(z)/g'(z)$ is a function with positive real parts. Hence, $f'(z)/g'(z) = h(z)$ for some $h(z) = 1 + c_1z + \cdots$ with positive real parts. Equating $f'(z)/g'(z)$ and h we have $f'(z) = h(z)g'(z)$. Let $g(z) = z + b_2z^2 + \cdots$. Expanding both sides in series, we have

$$f'(z) = 1 + 2a_2z + 3a_3z^2 + \cdots$$

while

$$\begin{aligned} h(z)g'(z) = 1 + (2b_2 + c_1)z + (3b_3 + 2b_2c_1 + c_2)z^2 \\ + (4b_4 + 3b_3c_1 + 2b_2c_2 + c_3)z^3 + \cdots \end{aligned}$$

so that

$$ka_k = kb_k + \sum_{j=1}^{k-1} jb_jc_{k-j},\ b_1 = 1$$

and thus

$$k|a_k| \leq k|b_k| + \sum_{j=1}^{k-1} j|b_j||c_{k-j}|,\ a_1 = 1.$$

Since $|b_k| \leq 1,\ k = 2, 3, \cdots$ for convex functions and $|c_k| \leq 2,\ k = 1, 2, \cdots$ for h, it follows that $k|a_k| \leq k + 2\sum_{j=1}^{k-1} j = k + k(k-1)$ so that the desired inequality follows.

The extremal function is also to be constructed is by choosing $g = 1/(1-z)$ in the geometric quantity $f'(z)/g'(z)$ and setting this equal to the extremal function for h, which is $L_0(z) = (1+z)/(1-z)$. Thus, we have

$$\frac{f'(z)}{g'(z)} = \frac{f'(z)}{1/(1-z)^2} = \frac{1+z}{1-z}$$

so that

$$f'(z) = \frac{1+z}{(1-z)^3}.$$

Integrating both sides, we have $f(z) = z/(1-z)^2$, which is the Koebe function. □

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