

Triple linking numbers and triple point numbers of certain T^2 -links

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Abstract

The triple linking number of an oriented surface link was defined as an analogical notion of the linking number of a classical link. We consider a certain m -component T^2 -link ($m \geq 3$) determined from two commutative pure m -braids a and b . We present the triple linking number of such a T^2 -link, by using the linking numbers of the closures of a and b . This gives a lower bound of the triple point number. In some cases, we can determine the triple point numbers, each of which is a multiple of four.

1 Introduction

A *surface link* is a smooth embedding of a closed surface into the Euclidean 4-space \mathbb{R}^4 . A T^2 -*link* is a surface link each of whose components is of genus one. In this paper, we consider a certain m -component T^2 -link which is determined from two commutative pure m -braids a and b . The triple linking number of an oriented surface link is defined in [1] as an analogical notion of the linking number of a classical link. The aim of this paper is to present the triple linking number of such a T^2 -link, by using the linking numbers of the closures of a and b . Further, we study the triple point number. The triple linking numbers give a lower bound of the triple point number. In some cases, we can determine the triple point numbers, each of which is a multiple of four.

We review the linking number of an oriented classical link L as follows. For positive integers i and j with $i \neq j$, the *linking number* of the i th and j th components of L is the total number of positive crossings minus the

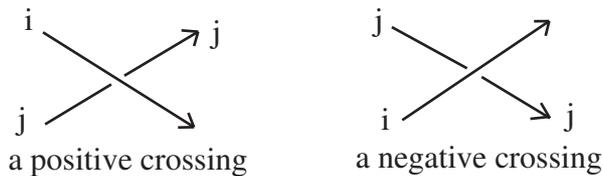


Figure 1.1: A positive crossing and a negative crossing.

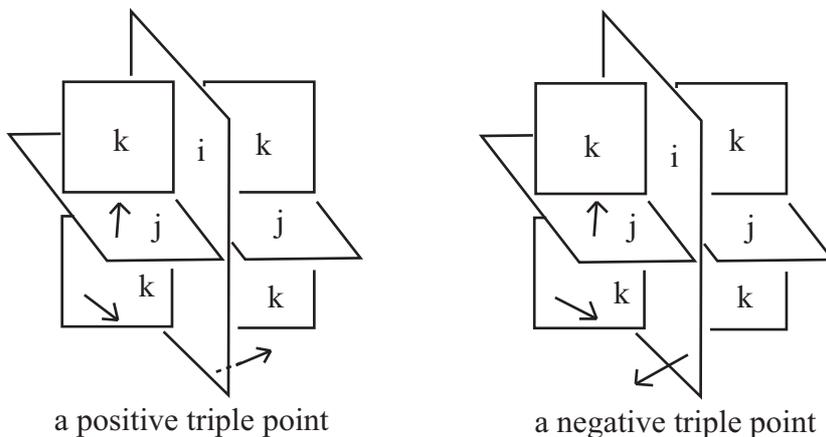


Figure 1.2: A positive triple point and a negative triple point, where we denote the orientations of sheets by normals.

total number of negative crossings of a diagram of L such that the over-arc (resp. under-arc) is from the i th (resp. j th) component; see Fig. 1.1. We denote it by $Lk_{i,j}(L)$. It is known [16] that $Lk_{j,i}(L) = Lk_{i,j}(L)$.

The triple linking number of an oriented surface link S is defined as follows (see [1, Definition 9.1], see also [3]). For positive integers i , j , and k with $i \neq j$ and $j \neq k$, the *triple linking number* of the i th, j th, and k th components of S is the total number of positive triple points minus the total number of negative triple points of a surface diagram of S such that the top, middle, and bottom sheet are from the i th, j th, and k th component of S respectively ([1]); see Fig. 1.2. We denote it by $Tlk_{i,j,k}(S)$. It is known [1] that $Tlk_{k,j,i}(S) = -Tlk_{i,j,k}(S)$ if i, j, k are mutually distinct, and otherwise $Tlk_{i,j,k}(S) = 0$.

We show the following theorem. We consider a torus-covering T^2 -link, which is a T^2 -link in the form of an unbranched covering over the standard torus (see Definition 2.1). It is known [14, Lemma 2.8] that a torus-covering T^2 -link is determined from two commutative classical m -braids, which we call basis m -braids (see Section 2.1). We denote by $\mathcal{S}_m(a, b)$ the torus-covering T^2 -link with basis m -braids a and b . In this paper, we consider the

case when the basis braids are pure m -braids for $m \geq 3$. For an m -braid c , let us denote by \hat{c} the closure of c .

Theorem 1.1. *Let a and b be commutative pure m -braids for $m \geq 3$. Then the triple linking number $\text{Tlk}_{i,j,k}(\mathcal{S}_m(a,b))$ ($i \neq j$ and $j \neq k$) is given by*

$$\text{Tlk}_{i,j,k}(\mathcal{S}_m(a,b)) = -\text{Lk}_{i,j}(\hat{a})\text{Lk}_{j,k}(\hat{b}) + \text{Lk}_{i,j}(\hat{b})\text{Lk}_{j,k}(\hat{a}),$$

where $\text{Lk}_{i,j}(\hat{a})$ (resp. $\text{Lk}_{i,j}(\hat{b})$) is the linking number of the i th and j th components of \hat{a} (resp. \hat{b}). Here we define the l th component of $\mathcal{S}_m(a,b)$ (resp. \hat{c} for $c = a$ or b) by the component containing the l th string of the basis braids (resp. c) for $l = 1, 2, \dots, m$.

The triple point number of a surface link S , denoted by $t(S)$, is the minimal number of triple points among all possible generic projections of S . For the results known about triple point numbers, see [3, Section 4.4.2]. By definition, we can see that $t(S) \geq \sum_{i \neq j, j \neq k} |\text{Tlk}_{i,j,k}(S)|$; thus the triple linking numbers of S give a lower bound of the triple point number of S . In particular, we show the following theorem.

Theorem 1.2. *Let $m \geq 3$. Let b be a pure m -braid, let Δ be a full twist of a bundle of m parallel strings, and let n be a non-negative integer. Put $\mu = \sum_{i < j} |\text{Lk}_{i,j}(\hat{b})|$, and put $\nu = \sum_{i < j < k} (\nu_{i,j,k} + \nu_{j,k,i} + \nu_{k,i,j})$, where $\nu_{i,j,k} = \min_{i,j,k} \{|\text{Lk}_{i,j}(\hat{b})|, |\text{Lk}_{j,k}(\hat{b})|\}$ if $\text{Lk}_{i,j}(\hat{b})\text{Lk}_{j,k}(\hat{b}) > 0$ and otherwise zero. Then*

$$t(\mathcal{S}_m(b, \Delta^n)) \geq 4n(\mu(m-2) - \nu).$$

In some cases, we can determine the triple point numbers. Let $\sigma_1, \sigma_2, \dots, \sigma_{m-1}$ be the standard generators of the m -braid group.

Theorem 1.3. *Let $m \geq 3$. Let b be an m -braid presented by a braid word which consists of $\sigma_i^{2(-1)^i}$ ($i = 1, 2, \dots, m-1$); note that b is a pure braid. Then*

$$t(\mathcal{S}_m(b, \Delta^n)) = 4n(m-2) \left(\sum_{i < j} |\text{Lk}_{i,j}(\hat{b})| \right).$$

Further the triple point number is realized by a surface diagram in the form of a covering over the torus.

It is known [4] (see also [5]) that any oriented surface link is bordant to the split union of oriented “necklaces”, and [5] any surface link is un-orientably bordant to the split union of necklaces and connected sums of standard projective planes; see also [17]. A necklace has the triple point number $4n$ (see Section 4). For other examples of surface links (not necessarily orientable) which realize large triple point numbers, see [6, 13, 15, 18]. In the papers [6, 13, 15] (resp. [18]), they use quandle cocycle invariants (resp. normal Euler numbers) to give lower bounds of triple point numbers.

Quandle cocycle invariants [1, 2, 3] can be regarded as an extended notion of triple linking numbers ([1, 4]), useful to give lower bounds of triple point numbers; see [6, 9, 10, 13, 15, 20, 21].

This paper is organized as follows. In Section 2, we give the definition of a torus-covering T^2 -link. In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorems 1.2 and 1.3.

2 A torus-covering T^2 -link

It is known [11, 12] that any oriented surface link can be presented by a branched covering over the standard 2-sphere. A torus-covering link was introduced in [14] as a new construction of a surface link, by considering the standard torus instead of the standard 2-sphere. In this section, we define a torus-covering T^2 -link (Definition 2.1, see [14]). The triple linking number of the i th, j th, and k th components of an oriented surface link is the sum of the signs of all the triple points of type (i, j, k) (see Section 2.2, see also [1]). We can obtain the sign and type of each triple point of a surface diagram of $\mathcal{S}_m(a, b)$, by studying the braid word transformation sequence from ab to ba (Lemma 2.2).

2.1 A torus-covering T^2 -link

Let T be the standard torus in \mathbb{R}^4 , i.e. the boundary of the standard solid torus in $\mathbb{R}^3 \times \{0\}$. Let us fix a point x_0 of T , and take a meridian \mathbf{m} and a longitude \mathbf{l} of T with the base point x_0 . A *meridian* is an oriented simple closed curve on T which bounds a 2-disk in the solid torus whose boundary is T and which is not null-homologous in T . A *longitude* is an oriented simple closed curve on T which is null-homologous in the complement of the solid torus in the three space $\mathbb{R}^3 \times \{0\}$ and which is not null-homologous in T . Let $N(T)$ be a tubular neighborhood of T in \mathbb{R}^4 .

Definition 2.1. A *torus-covering T^2 -link* is a surface link S in \mathbb{R}^4 such that S is embedded in $N(T)$ and $p|_S : S \rightarrow T$ is an unbranched covering map, where $p : N(T) \rightarrow T$ is the natural projection.

For the definition and properties of a torus-covering link whose component might be of genus more than one, see [14].

Let us consider a torus-covering T^2 -link S . The intersections $S \cap p^{-1}(\mathbf{m})$ and $S \cap p^{-1}(\mathbf{l})$ are closures of classical braids. Cutting open the solid tori $p^{-1}(\mathbf{m})$ and $p^{-1}(\mathbf{l})$ at the 2-disk $p^{-1}(x_0)$, we obtain a pair of classical braids. We call them *basis braids* ([14]). The basis braids of a torus-covering T^2 -link are commutative, and for any commutative m -braids a and b , there exists a unique torus-covering T^2 -link with basis braids a and b ([14, Lemma 2.8]).

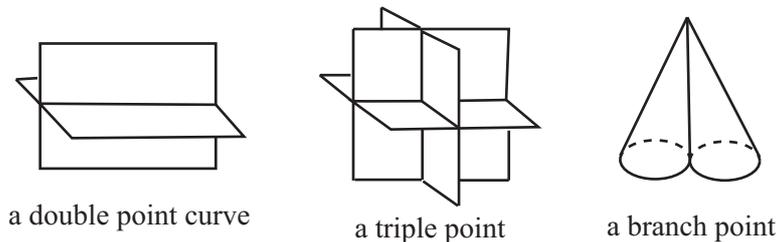


Figure 2.1: The singularity of a surface diagram.

For commutative m -braids a and b , we denote by $\mathcal{S}_m(a, b)$ the torus-covering T^2 -link with basis m -braids a and b ([14]).

In this paper, we consider the case when the basis braids are pure braids, and we define the l th component of $\mathcal{S}_m(a, b)$ by the component containing the l th string of the basis braids for $l = 1, 2, \dots, m$.

2.2 Triple points of $\mathcal{S}_m(a, b)$

Triple linking numbers are defined using triple points of a surface diagram. We will review a surface diagram of a surface link S (see [3]). For a projection $\pi : \mathbb{R}^4 \rightarrow \mathbb{R}^3$, the closure of the self-intersection set of $\pi(S)$ is called the singularity set. Let π be a generic projection, i.e. the singularity set of the image $\pi(S)$ consists of double points, isolated triple points, and isolated branch points; see Fig. 2.1. The closure of the singularity set forms a union of immersed arcs and loops, which we call double point curves. Triple points (resp. branch points) form the intersection points (resp. the end points) of the double point curves. A *surface diagram* of S is the image $\pi(S)$ equipped with over/under information along each double point curve with respect to the projection direction.

Throughout this paper, we consider the surface diagram of $\mathcal{S}_m(a, b)$ by the projection which projects $N(T) = I \times I \times T$ to $I \times T$ for an interval I , where we identify $N(T)$ with $I \times I \times T$ in such a way as follows. Since T is the boundary of the standard solid torus in $\mathbb{R}^3 \times \{0\}$, the normal bundle of T in $\mathbb{R}^3 \times \{0\}$ is a trivial bundle. We identify it with $I \times T$. Then we identify $N(T)$ with $I \times I \times T$, where the second I is an interval in the fourth axis of \mathbb{R}^4 . Perturbing $\mathcal{S}_m(a, b)$ if necessary, we can assume that this projection is generic. We call this surface diagram the surface diagram of $\mathcal{S}_m(a, b)$ in the form of a covering over the torus.

Around a triple point t in a surface diagram of an oriented surface link S , there are three types of sheets with respect to the projection direction. We call them the *top sheet*, *middle sheet* and *bottom sheet* from the higher one. We define that the *sign* of t is $+1$ (resp. -1) if t is a positive (resp. negative) triple point, and we call the triplet (i, j, k) the *type* of t , where the

top, middle, and bottom sheet are from the i th, j th, and k th component of S respectively ([1]); see Fig. 1.2. The triple linking number $\text{Trk}_{i,j,k}(S)$ is the sum of the signs of all the triple points of type (i, j, k) ([1]).

We can obtain the sign and type of each triple point of $\mathcal{S}_m(a, b)$ by studying the braid word transformation sequence from ab to ba , as follows. Let us consider the surface obtained from T by removing a sufficiently small open neighborhood of $\mathbf{m} \cup \mathbf{l}$, and denote it by X . The boundary of X is a circle. Let us take two points x and x' in ∂X which divide ∂X into two paths ρ_0 and ρ_1 from x to x' such that ρ_0 (resp. ρ_1) is presented by $\mathbf{m} \cdot \mathbf{l}$ (resp. $\mathbf{l} \cdot \mathbf{m}$). Take an isotopy $\{\rho_u\}_{u \in [0,1]}$ of paths in X which connects ρ_0 and ρ_1 such that $\rho_u \cap \rho_v = \{x, x'\}$ for $u \neq v$.

Put $S_u = \mathcal{S}_m(a, b) \cap p^{-1}(\rho_u)$ ($u \in [0, 1]$). Since $\mathcal{S}_m(a, b)$ is a covering over T , $\{S_u\}_u$ is an isotopy of classical m -braids such that $S_0 = ab$ and $S_1 = ba$. Thus $\{S_u\}_u$ is presented by a braid word transformation sequence (see [12]) from ab to ba related by a finite sequence of the following transformations:

- (1) Insertion of the word $\sigma_i \sigma_i^{-1}$ or $\sigma_i^{-1} \sigma_i$, or deletion of the word $\sigma_i \sigma_i^{-1}$ or $\sigma_i^{-1} \sigma_i$,
- (2) Substitution of $\sigma_i \sigma_j$ for $\sigma_j \sigma_i$ where $|i - j| > 1$,
- (3) Substitution of $\sigma_i \sigma_j \sigma_i$ for $\sigma_j \sigma_i \sigma_j$ where $|i - j| = 1$.

Thus we have a braid word transformation sequence from ab to ba . Conversely, when we have a braid word transformation sequence from ab to ba , we can construct $\mathcal{S}_m(a, b)$.

In the part of the surface diagram related by the transformation (1) or (2), the singularity set consists of disjoint double point curves; thus it does not contain triple points. For the transformation (3), we have the following lemma. The transformation (3) presents a Reidemeister move of type III. From now on we will call it a Reidemeister move of type III.

Lemma 2.2. *Let us consider a braid word transformation sequence from ab to ba for commutative pure m -braids a and b . We consider the surface diagram of $\mathcal{S}_m(a, b)$ in the form of a covering over T . Then the triple points are in the part of the surface diagram iff it is related by Reidemeister moves of type III (the transformation (3)), and the sign and type of each triple point are as follows. Let*

$$w \sigma_i \sigma_j \sigma_i w' \rightarrow w \sigma_j \sigma_i \sigma_j w' \quad (|i - j| = 1) \quad (2.1)$$

be a transformation in a braid word transformation sequence from ab to ba , where w and w' are some m -braids. The part of $\mathcal{S}_m(a, b)$ presented by this transformation has one triple point. The sign and type of the triple point is $+(\tau(j+1), \tau(j), \tau(i))$ if $i < j$, and $-(\tau(i+1), \tau(i), \tau(j))$ if $i > j$, where τ is the permutation associated with the m -braid w^{-1} .

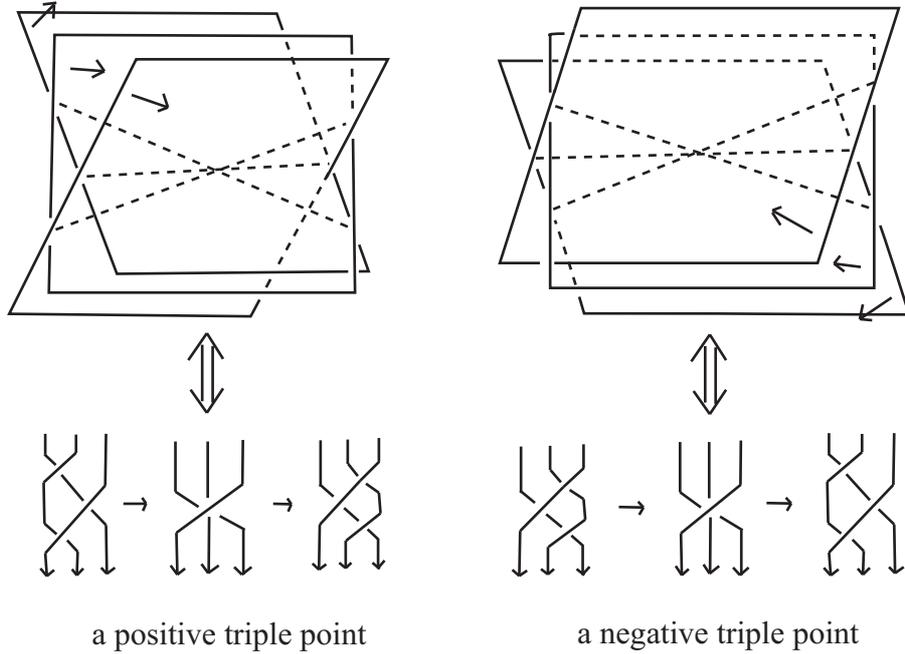


Figure 2.2: A positive triple point and a negative triple point.

Proof. It suffices to determine the sign and type of the triple point presented by the transformation (2.1). Let us denote the triple point by t .

First we determine the sign of t . A positive triple point and a negative triple point as in Fig. 1.2 can be deformed as in Fig. 2.2; thus a positive triple point is presented by a braid word transformation $\sigma_k \sigma_{k+1} \sigma_k \rightarrow \sigma_{k+1} \sigma_k \sigma_{k+1}$, and a negative triple point is presented by $\sigma_{k+1} \sigma_k \sigma_{k+1} \rightarrow \sigma_k \sigma_{k+1} \sigma_k$ (see also [7, Proposition 4.43 (3)]). Thus the sign of t is positive (resp. negative) if $i < j$ (resp. $i > j$).

Next we determine the type of t . The triple point t is constructed by the $\tau(k)$ th, $\tau(k+1)$ th, and $\tau(k+2)$ th strings of m -braids presented by (2.1). The top, middle, and bottom sheet are constructed by the $\tau(k+2)$ th, $\tau(k+1)$ th, and $\tau(k)$ th strings. Since the l th string of ab is from the l th component of $\mathcal{S}_m(a, b)$ for any l , so is the l th string of m -braids presented by (2.1). Thus the type of the triple point is $(\tau(k+2), \tau(k+1), \tau(k))$. Thus the conclusion follows. \square

3 Triple linking numbers

In this section we prove Theorem 1.1.

Proof of Theorem 1.1. Since triple linking numbers are determined from

three components of a surface link, it is sufficient to consider three components of $\mathcal{S}_m(a, b)$. Since $\mathcal{S}_m(a, b)$ is in the form of a covering over T , any three components of $\mathcal{S}_m(a, b)$ form a torus-covering T^2 -link. Thus it suffices to show for the case when $m = 3$. From now on we consider two commutative pure 3-braids a and b .

We determine the presentations of a and b as follows. It is known (see [8]) that the pure 3-braid group P_3 is presented by generators A_{12}, A_{23}, A_{13} , and the relations

$$\begin{aligned} A_{12}^{-1}A_{23}A_{12} &= A_{13}A_{23}A_{13}^{-1}, \\ A_{12}^{-1}A_{13}A_{12} &= A_{13}A_{23}A_{13}A_{23}^{-1}A_{13}^{-1}, \end{aligned}$$

where

$$A_{12} = \sigma_1^2, \quad A_{23} = \sigma_2^2, \quad A_{13} = \sigma_2\sigma_1^2\sigma_2^{-1}.$$

Put $A = A_{12}, B = A_{23}$ and $C = A_{13}$. Then

$$P_3 = \langle A, B, C \mid A^{-1}BA = CBC^{-1}, A^{-1}CA = CBCB^{-1}C^{-1} \rangle.$$

Put $\Delta = ACB$; note that Δ is a full twist. By eliminating C by $C = A^{-1}\Delta B^{-1}$, we have

$$\begin{aligned} P_3 &= \langle A, B, \Delta \mid A\Delta = \Delta A, B\Delta = \Delta B \rangle \\ &= Z \oplus U, \end{aligned}$$

where Z is an infinite cyclic group generated by Δ , and U is the free group with two generators A and B ; note that since the center of U is trivial, the center of P_3 is Z . Thus we can see that any element of P_3 is written uniquely by $\Delta^l u$ for an integer l and $u \in U$. Let $a = \Delta^{k_1}u, b = \Delta^{k_2}v$ (for integers k_1, k_2 and $u, v \in U$) be the presentations. Since Z is the center of P_3 , we can see that a and b commute iff u and v commute. Since U is a free group with two generators, u and v commute iff $u = w^{l_1}$ and $v = w^{l_2}$ for some $w \in U$ and integers l_1 and l_2 . Since a and b are commutative pure 3-braids, $a = \Delta^{k_1}w^{l_1}$ and $b = \Delta^{k_2}w^{l_2}$.

We have a braid word transformation sequence as follows:

$$\begin{aligned} ab &= \Delta^{k_1}(w^{l_1}\Delta^{k_2})w^{l_2} \\ &\rightarrow \dots \rightarrow \Delta^{k_1}(\Delta^{k_2}w^{l_1})w^{l_2} = \Delta^{k_2}(\Delta^{k_1}w^{l_2})w^{l_1} \\ &\rightarrow \dots \rightarrow \Delta^{k_2}(w^{l_2}\Delta^{k_1})w^{l_1} = ba, \end{aligned}$$

where a transformation from $w\Delta$ to Δw (resp. from Δw to $w\Delta$) is applied l_1k_2 (resp. k_1l_2) times. Since Δ and w are pure braids, the triple linking numbers in the surface concerning a transformation from $w\Delta$ to Δw (resp. from Δw to $w\Delta$) have the same value with those of $\mathcal{S}_m(w, \Delta)$ (resp. $\mathcal{S}_m(\Delta, w)$). Hence

$$\begin{aligned} &\text{Tlk}_{i,j,k}(\mathcal{S}_3(a, b)) \\ &= l_1k_2\text{Tlk}_{i,j,k}(\mathcal{S}_3(w, \Delta)) + k_1l_2\text{Tlk}_{i,j,k}(\mathcal{S}_3(\Delta, w)). \end{aligned} \tag{3.1}$$

It is known [1] that for any oriented surface link S , $\text{Tlk}_{k,j,i}(S) = -\text{Tlk}_{i,j,k}(S)$ if i, j, k are mutually distinct, and $\text{Tlk}_{i,j,k}(S) = 0$ otherwise. Hence it suffices to calculate $\text{Tlk}_{i,j,k}(\mathcal{S}_3(a, b))$ for $(i, j, k) = (1, 2, 3), (2, 3, 1)$ and $(3, 1, 2)$. By Lemma 3.1, the triple linking numbers are presented by the exterior product as follows:

$$\begin{pmatrix} \text{Tlk}_{1,2,3}(\mathcal{S}_3(c, \Delta)) \\ \text{Tlk}_{2,3,1}(\mathcal{S}_3(c, \Delta)) \\ \text{Tlk}_{3,1,2}(\mathcal{S}_3(c, \Delta)) \end{pmatrix} = - \begin{pmatrix} \text{Lk}_{3,1}(\hat{c}) \\ \text{Lk}_{1,2}(\hat{c}) \\ \text{Lk}_{2,3}(\hat{c}) \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

where $c = A$ or B ; note that $(\text{Lk}_{3,1}(\hat{\Delta}), \text{Lk}_{1,2}(\hat{\Delta}), \text{Lk}_{2,3}(\hat{\Delta})) = (1, 1, 1)$. Since any element of U is presented by A and B , it follows that this equation holds for any $c \in U$. Put $\mathbf{w} = {}^t(\text{Lk}_{3,1}(\hat{w}), \text{Lk}_{1,2}(\hat{w}), \text{Lk}_{2,3}(\hat{w}))$. Since $w \in U$, by (3.1) and Lemma 3.1,

$$\begin{aligned} \begin{pmatrix} \text{Tlk}_{1,2,3}(\mathcal{S}_3(a, b)) \\ \text{Tlk}_{2,3,1}(\mathcal{S}_3(a, b)) \\ \text{Tlk}_{3,1,2}(\mathcal{S}_3(a, b)) \end{pmatrix} &= -l_1 k_2 \mathbf{w} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - k_1 l_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \mathbf{w} \\ &= -(l_1 \mathbf{w} \times k_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + k_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times l_2 \mathbf{w}) \\ &= -(k_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + l_1 \mathbf{w}) \times (k_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + l_2 \mathbf{w}) \\ &= - \begin{pmatrix} \text{Lk}_{3,1}(\hat{a}) \\ \text{Lk}_{1,2}(\hat{a}) \\ \text{Lk}_{2,3}(\hat{a}) \end{pmatrix} \times \begin{pmatrix} \text{Lk}_{3,1}(\hat{b}) \\ \text{Lk}_{1,2}(\hat{b}) \\ \text{Lk}_{2,3}(\hat{b}) \end{pmatrix}. \end{aligned}$$

Thus $\text{Tlk}_{i,j,k}(\mathcal{S}_m(a, b)) = -\text{Lk}_{i,j}(\hat{a})\text{Lk}_{j,k}(\hat{b}) + \text{Lk}_{i,j}(\hat{b})\text{Lk}_{j,k}(\hat{a})$. □

We can check that for the T^2 -link S of Lemma 3.1, $\text{Tlk}_{k,j,i}(S) = -\text{Tlk}_{i,j,k}(S)$ if i, j, k are mutually distinct, and otherwise $\text{Tlk}_{i,j,k}(S) = 0$. Thus we can see that this property holds for $\mathcal{S}_m(a, b)$ without referring to [1].

Lemma 3.1. *Put $A = \sigma_1^2$ and $B = \sigma_2^2$ as in the proof of Theorem 1.1. Then*

$$\text{Tlk}_{i,j,k}(\mathcal{S}_3(c, \Delta)) = -\text{Lk}_{i,j}(\hat{c}) + \text{Lk}_{j,k}(\hat{c}) \quad (3.2)$$

$$\text{Tlk}_{i,j,k}(\mathcal{S}_3(\Delta, c)) = -\text{Tlk}_{i,j,k}(\mathcal{S}_3(c, \Delta)) \quad (3.3)$$

where $c = A$ or B .

Proof. We show (3.2) for the case when $c = A = \sigma_1^2$. Since Δ is a full twist, we assume that Δ has a presentation $\Delta = \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2$. We have

the following braid word transformation sequence:

$$\begin{aligned}
\sigma_1 \Delta &= \sigma_1(\sigma_1 \sigma_2 \sigma_1) \sigma_2 \sigma_1 \sigma_2 & (3.4) \\
&\rightarrow \sigma_1(\sigma_2 \sigma_1 \sigma_2) \sigma_2 \sigma_1 \sigma_2 = \sigma_1 \sigma_2 \sigma_1 \sigma_2 (\sigma_2 \sigma_1 \sigma_2) \\
&\rightarrow \sigma_1 \sigma_2 \sigma_1 \sigma_2 (\sigma_1 \sigma_2 \sigma_1) = \Delta \sigma_1.
\end{aligned}$$

Since a Reidemeister move of type III is applied twice, there are two triple points in the surface diagram presented by (3.4): the first one is positive, and the second one is negative by Lemma 2.2. Since $A = \sigma_1^2$, repeating (3.4) twice, we have the braid word transformation sequence

$$A\Delta = \sigma_1^2 \Delta \rightarrow \sigma_1 D \rightarrow \sigma_1 \Delta \sigma_1 \rightarrow D \sigma_1 \rightarrow \Delta \sigma_1^2 = \Delta A, \quad (3.5)$$

where $D = \sigma_1 \sigma_2 \sigma_1 \sigma_2^2 \sigma_1 \sigma_2$. Thus there are four triple points. We denote them by t_1, \dots, t_4 . The sign of t_1 and t_3 (resp. t_2 and t_4) is positive (resp. negative) by Lemma 2.2.

Next we determine the type of t_l ($l = 1, \dots, 4$). For the m -braid group B_m and the symmetric group S_m of degree m , let $\phi : B_m \rightarrow S_m$ be a homomorphism which maps an m -braid c to a permutation associated with c , i.e. $\phi(\sigma_l) = (l \ l+1)$ for $l = 1, 2, \dots, m-1$. Let w_l ($l = 1, \dots, 4$) be the braid w of the transformation (2.1) presenting t_l (see Lemma 2.2). Then $w_1 = \sigma_1^2$, $w_2 = \sigma_1^2 \sigma_2 \sigma_1 \sigma_2$, $w_3 = \sigma_1^{-1} w_1$, and $w_4 = \sigma_1^{-1} w_2$. Put $\tau_i = \phi(w_i^{-1}) \in S_3$ ($i = 1, \dots, 4$). Since $\tau_1 = e$ (resp. $\tau_2 = (1 \ 3)$), the type of t_1 (resp. t_2) is $(\tau_1(3), \tau_1(2), \tau_1(1)) = (3, 2, 1)$ (resp. $(\tau_2(3), \tau_2(2), \tau_2(1)) = (1, 2, 3)$). Since $w_3^{-1} = w_1^{-1} \sigma_1$ (resp. $w_4^{-1} = w_2^{-1} \sigma_1$), we can obtain τ_3 (resp. τ_4) by applying the action of $\phi(\sigma_1) = (1 \ 2)$ on τ_1 (resp. τ_2); thus the type of t_3 (resp. t_4) is $(3, 1, 2)$ (resp. $(2, 1, 3)$).

Thus the sign and type of all the triple points are

$$+(3, 2, 1), -(1, 2, 3), +(3, 1, 2), -(2, 1, 3),$$

and hence

$$\text{Tlk}_{i,j,k}(\mathcal{S}_3(A, \Delta)) = \begin{cases} +1 & \text{if } (i, j, k) = (3, 2, 1) \text{ or } (3, 1, 2), \\ -1 & \text{if } (i, j, k) = (1, 2, 3) \text{ or } (2, 1, 3), \\ 0 & \text{otherwise.} \end{cases}$$

Since $A = \sigma_1^2 \in P_3$, $\text{Lk}_{i,j}(\hat{A}) = +1$ if $\{i, j\} = \{1, 2\}$ and otherwise zero. Hence $\text{Tlk}_{i,j,k}(\mathcal{S}_3(A, \Delta)) = -\text{Lk}_{i,j}(\hat{A}) + \text{Lk}_{j,k}(\hat{A})$ for any i, j, k .

Next we show (3.3) for the case when $c = A$. We have the braid word transformation sequence from ΔA to $A\Delta$ which is the reversed sequence of (3.5); thus the sign of each triple point is the reverse of the sign of t_l , and the type is the same with that of t_l ($l = 1, \dots, 4$). Hence $\text{Tlk}_{i,j,k}(\mathcal{S}_3(A, \Delta)) = -\text{Tlk}_{i,j,k}(\mathcal{S}_3(A, \Delta))$ for any i, j, k .

Since $B = \sigma_2^2$ (resp. Δ) is obtained from $A = \sigma_1^2$ (resp. Δ) by regarding the l th string as the $(3-l)$ th string for $l = 1, 2, 3$, we can see that (3.2) and (3.3) also hold for the case when $c = B$. \square

4 Triple point numbers

In this section, we show Theorem 1.2 by using Theorem 1.1. Further, we show Theorem 1.3 by using Theorem 1.2.

It is known [4] (see also [5]) that any oriented surface link is bordant to the split union of oriented “necklaces”, and [5] any surface link is unorientably bordant to the split union of necklaces and connected sums of standard projective planes; see also [17]. The triple point number of a necklace is easily determined to be $4n$, as follows. A “necklace” is a surface link introduced in [5] with non-trivial triple linking (see also [1, 3, 4]). An oriented necklace is a “Hopf 2-link without or with beads” ([4]). A necklace consists of two components homeomorphic to two tori or two Klein bottles (called a “strand”) and n components homeomorphic to 2-spheres (called “beads”); see [5]. By estimating the lower bound of the triple point number by using the triple linking numbers (see [4, 5], see [19] for the triple linking numbers of a non-orientable surface link), we can see that the triple point number concerning each bead is 4, and as a whole the triple point number is $4n$.

4.1 Proof of Theorem 1.2

Put $S = \mathcal{S}_m(b, \Delta^n)$. By definition, $t(S) \geq \sum_{i \neq j, j \neq k} |\text{Tlk}_{i,j,k}(S)|$. We show that $\sum_{i \neq j, j \neq k} |\text{Tlk}_{i,j,k}(S)| = 4n(\mu(m-2) - \nu)$, where $\mu = \sum_{i < j} |\text{Lk}_{i,j}(\hat{b})|$, and $\nu = \sum_{i < j < k} (\nu_{i,j,k} + \nu_{j,k,i} + \nu_{k,i,j})$, where $\nu_{i,j,k} = \min_{i,j,k} \{|\text{Lk}_{i,j}(\hat{b})|, |\text{Lk}_{j,k}(\hat{b})|\}$ if $\text{Lk}_{i,j}(\hat{b})\text{Lk}_{j,k}(\hat{b}) > 0$ and otherwise zero. Since $\text{Lk}_{i,j}(\Delta^n) = n$ for any $i \neq j$,

$$\text{Tlk}_{i,j,k}(S) = -n(\text{Lk}_{i,j}(\hat{b}) - \text{Lk}_{j,k}(\hat{b}))$$

for any i, j, k with $i \neq j$ and $j \neq k$ by Theorem 1.1.

It suffices to show that

$$\sum_{i \neq j, j \neq k} |\text{Lk}_{i,j}(\hat{b}) - \text{Lk}_{j,k}(\hat{b})| = 4(\mu(m-2) - \nu).$$

By definition of $\nu_{i,j,k}$,

$$|\text{Lk}_{i,j}(\hat{b}) - \text{Lk}_{j,k}(\hat{b})| = |\text{Lk}_{i,j}(\hat{b})| + |\text{Lk}_{j,k}(\hat{b})| - 2\nu_{i,j,k}.$$

We calculate the sum of $|\text{Lk}_{i,j}(\hat{b})| + |\text{Lk}_{j,k}(\hat{b})|$ as follows.

$$\begin{aligned} \sum_{i \neq j, j \neq k} |\text{Lk}_{i,j}(\hat{b})| + |\text{Lk}_{j,k}(\hat{b})| &= 2 \sum_{i \neq j, j \neq k} |\text{Lk}_{i,j}(\hat{b})| \\ &= 2(m-1) \sum_{i \neq j} |\text{Lk}_{i,j}(\hat{b})| \\ &= 4\mu(m-1). \end{aligned}$$

We calculate the sum of $\nu_{i,j,k}$ as follows.

$$\begin{aligned}
\sum_{i \neq j, j \neq k} \nu_{i,j,k} &= \sum_{i \neq j, j \neq k, k \neq i} \nu_{i,j,k} + \sum_{i \neq j} |\text{Lk}_{i,j}(\hat{b})| \\
&= 2 \sum_{i < j < k} (\nu_{i,j,k} + \nu_{j,k,i} + \nu_{k,i,j}) + 2\mu \\
&= 2(\nu + \mu).
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{i \neq j, j \neq k} |\text{Lk}_{i,j}(\hat{b}) - \text{Lk}_{j,k}(\hat{b})| &= 4\mu(m-1) - 2 \cdot 2(\nu + \mu) \\
&= 4(\mu(m-2) - \nu),
\end{aligned}$$

and hence $t(S) \geq 4n(\mu(m-2) - \nu)$. \square

4.2 Proof of Theorem 1.3

Put $S = \mathcal{S}_m(b, \Delta^n)$. By Theorem 1.2, $t(S) \geq 4n(\mu(m-2) - \nu)$. Since b consists of $\sigma_i^{2(-1)^i}$ ($i = 1, 2, \dots, m-1$), $\text{Lk}_{i,j}(\hat{b})$ is positive (resp. negative) if $j = i+1$ and i is even (resp. $j = i+1$ and i is odd), and otherwise zero. Thus $\nu_{i,j,k} = 0$ for any i, j, k , and it follows that $\nu = 0$. Hence $t(S) \geq 4n\mu(m-2)$, where $\mu = \sum_{i < j} |\text{Lk}_{i,j}(\hat{b})|$.

Now we show that the surface diagram of S in the form of a covering over T (see Section 2.2) has $4n\mu(m-2)$ triple points. By Lemma 2.2, it suffices to show that there is a braid word transformation sequence from $b\Delta^n$ to $\Delta^n b$ related $4n\mu(m-2)$ times by a Reidemeister move of type III.

By the presentation of b , b consists of 2μ standard generators; thus it suffices to show that there is a braid word transformation sequence from $\sigma_i\Delta$ to $\Delta\sigma_i$ related $2(m-2)$ times by a Reidemeister move of type III, as follows. We assume that a full twist Δ has the following presentation:

$$\Delta = (\sigma_1\sigma_2 \cdots \sigma_{m-1})^m.$$

The transformation sequence from $\sigma_i\Delta$ to $\Delta\sigma_i$ consists of the following parts:

- (i) A transformation from $\sigma_j(\sigma_1\sigma_2 \cdots \sigma_{m-1})$ to $(\sigma_1\sigma_2 \cdots \sigma_{m-1})\sigma_{j-1}$, where $2 \leq j \leq m-1$.
- (ii) A transformation from $\sigma_1(\sigma_1\sigma_2 \cdots \sigma_{m-1})^2$ to $(\sigma_1\sigma_2 \cdots \sigma_{m-1})^2\sigma_{m-1}$.

We can take a transformation sequence (i) (resp. (ii)) related once (resp. $(m-2)$ times) by a Reidemeister move of type III by Lemma 4.1. The transformation sequence from $\sigma_i\Delta$ to $\Delta\sigma_i$ is related $(i-1)$ times by the transformation (i), and then once by the transformation (ii), and then $(m-i-1)$ times by the transformation (i). Thus the total applied number is $(i-$

$1) + (m-2) + (m-i-1) = 2(m-2)$. Thus $t(S) = 4n(m-2)(\sum_{i < j} |\text{Lk}_{i,j}(\hat{b})|)$, and this is realized by the surface diagram in the form of a covering over the torus. \square

Lemma 4.1. *We can take a transformation sequence (i) (resp. (ii)) as in the proof of Theorem 1.3 which is related once (resp. $(m-2)$ times) by a Reidemeister move of type III.*

Proof. Let us regard two braids as equivalent if they are related by transformations other than Reidemeister moves of type III. Then we can take a transformation sequence (i) as follows.

$$\begin{aligned} \sigma_j(\sigma_1\sigma_2\cdots\sigma_{m-1}) &\sim \sigma_1\sigma_2\cdots\sigma_{j-2}(\sigma_j\sigma_{j-1}\sigma_j)\sigma_{j+1}\cdots\sigma_{m-1} \\ &\rightarrow \sigma_1\sigma_2\cdots\sigma_{j-2}(\sigma_{j-1}\sigma_j\sigma_{j-1})\sigma_{j+1}\cdots\sigma_{m-1} \\ &\sim (\sigma_1\sigma_2\cdots\sigma_{m-1})\sigma_{j-1}. \end{aligned}$$

Thus the transformation sequence (i) is related once by a Reidemeister move of type III.

We can take a transformation sequence (ii) as follows.

$$\begin{aligned} &\sigma_1(\sigma_1\sigma_2\cdots\sigma_{m-1})^2 \\ &\sim \sigma_1(\sigma_1)(\sigma_2\sigma_1)(\sigma_3\sigma_2)\cdots(\sigma_j\sigma_{j-1})\cdots(\sigma_{m-1}\sigma_{m-2})\sigma_{m-1} \\ &= \sigma_1(\sigma_1\sigma_2\sigma_1)\sigma_3\sigma_2\cdots\sigma_{m-1}\sigma_{m-2}\sigma_{m-1} \\ &\rightarrow \sigma_1(\sigma_2\sigma_1\sigma_2)\sigma_3\sigma_2\cdots\sigma_{m-1}\sigma_{m-2}\sigma_{m-1} \\ &= \sigma_1\sigma_2\sigma_1(\sigma_2\sigma_3\sigma_2)\cdots\sigma_{m-1}\sigma_{m-2}\sigma_{m-1} \\ &\rightarrow \sigma_1\sigma_2\sigma_1(\sigma_3\sigma_2\sigma_3)\cdots\sigma_{m-1}\sigma_{m-2}\sigma_{m-1} \\ &\rightarrow \cdots \\ &\rightarrow \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\cdots(\sigma_{m-2}\sigma_{m-1}\sigma_{m-2})\sigma_{m-1} \\ &\rightarrow \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\cdots(\sigma_{m-1}\sigma_{m-2}\sigma_{m-1})\sigma_{m-1} \\ &\sim (\sigma_1\sigma_2\cdots\sigma_{m-1})^2\sigma_{m-1}. \end{aligned}$$

Thus the transformation sequence (ii) is related $(m-2)$ times by Reidemeister moves of type III. \square

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