Simply Explicitly Invertible Approximations to 4 Decimals of Error Function and Normal Cumulative Distribution Function

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Abstract. We improve the Winitzki's Approximation of the error function $erf(x) \cong \sqrt{1 - e^{-x^2 \frac{\frac{4}{\pi} + 0.147x^2}{1+0.147x^2}}}$ which has error $|\varepsilon(x)| < 1.25 \cdot 10^{-4}$ $\forall x \geq 0$ till reaching 4 decimals of precision with $|\varepsilon(x)| < 2.27 \cdot 10^{-5}$; also reducing slightly the relative error. Old formula and ours are both explicitly invertible, essentially solving a biquadratic equation, after obvious substitutions. Then we derive approximations to 4 decimals of normal cumulative distribution function $\Phi(x)$, of erfc(x) and of the Q function (or cPhi).
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In this note we improve the Winitzki's Approximation of the error function till reaching 4 decimals of precision both for erf and Φ, reducing about 5.5 times the respective absolute errors (and reducing the relative errors too).

Lemma (Winitzki's Approximation of erf). (See [1])

$$erf(x) \cong \sqrt{1 - e^{-x^2 \frac{\frac{4}{\pi} + 0.147x^2}{1+0.147x^2}}} \quad |\varepsilon(x)| < 1.25 \cdot 10^{-4} \quad |\varepsilon_r(x)| < 1.28 \cdot 10^{-4} \ \forall x \geq 0$$

Theorem (Improving of Winitzki's Approximation of erf).

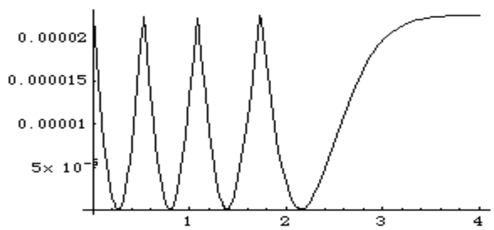
$$erf(x) \cong \sqrt{1 - e^{\frac{-1.2735457x^2 - 0.1487936x^4}{1+0.1480931x^2+0.0005160x^4}}} \tag{1}$$

$$|\varepsilon(x)| < 2.27 \cdot 10^{-5} \ \forall x \geq 0 \qquad |\varepsilon_r(x)| < 1.21 \cdot 10^{-4} \ \forall x \geq 0$$

Proof (only for absolute error). For $0 \leq x \leq 4$, see the figure above [1].
For $x > 4$ let's consider that it is $erf(4) = 0.99999998458...$ and $erf(x) \to 1$ and erf is increasing, then

$$\big(\forall x > 4\big) \qquad |1 - erf(x)| < 10^{-7}. \tag{2}$$

[1] The figure represents the graph of $2.27 \cdot 10^{-5} - \left| erf(x) - \sqrt{1 - e^{\frac{-1.2735457x^2 - 0.1487936x^4}{1+0.1480931x^2+0.0005160x^4}}} \right|$, made by the software Mathematica, showing that the quantity is positive.
To verify that the graph do not intersect the x axis, you may make zooms in the subdomains $[0.2, 0.3]$, $[0.75, 0.85]$, $[1.35, 1.45]$ and $[2.1, 2.2]$.



Let's $\eta(x)$ our approximation of erf(x) as in (1): $erf(x) \cong \eta(x) := \sqrt{1-e^{E(x)}}$ being $E(x) := \frac{-1.2735457x^2-0.1487936x^4}{1+0.1480931x^2+0.0005160x^4}$. It is:

$$\big(\forall x>4\big) \quad 0<3=\Big(\frac{1}{3}15-1\Big)^2-1-12<(\frac{1}{3}\cdot 4^2-0.795)^2-0.795^2-12<$$

$$=\Big(\frac{1}{3}x^2-\frac{3}{2}0.53\Big)^2-\Big(\frac{3}{2}0.53\Big)^2-12=\frac{1}{9}x^4-0.53x^2-12<$$

$$<0.14176x^4-0.53x^2-12=(0.148x^4-0.00624)x^4+(1.27x^2-1.8)x^2-12$$

$$\Rightarrow 1.27x^2+0.148x^4>12+1.8x^2+0.00624x^4$$

$$\Rightarrow \frac{1.27x^2+0.148x^4}{12}>1+0.15x^2+0.00052x^4$$

$$\Rightarrow E(x)=\frac{-1.2735457x^2-0.1487936x^4}{1+0.1480931x^2+0.0005160x^4}<\frac{-1.27x^2-0.148x^4}{1+0.15x^2+0.00052x^4}<-12$$

$$\Rightarrow e^{R(x)}<e^{-12}<e^{-12}+(e^{-12}-e^{-24})=2e^{-12}-e^{-24}=1-(1-e^{-12})^2$$

$$\Rightarrow (1-e^{-12})^2<1-e^{R(x)} \quad \Rightarrow \quad 1-e^{-12}<\sqrt{1-e^{R(x)}}$$

$$\Rightarrow 0<1-\eta(x)=1-\sqrt{1-e^{R(x)}}<e^{-12}\Rightarrow$$

$$\big(\forall x>4\big) \qquad |1-\eta(x)|<e^{-12}. \tag{3}$$

By (2) and (3) it is

$$\big(\forall x>4\big) \qquad |erf(x)-\eta(x)|\le|1-erf(x)|+|1-\eta(x)|<10^{-7}+e^{-12}<10^{-5}.$$

□

TABLE ERF2.27E-5

Simply explicitly invertible approximations to 4 decimals of the error function erf, of complementary error function erfc, of the normal cumulative distribution function Φ, and of Q function, for $x \geq 0$. With majorizations of absolute and relative errors, and definitions.

$$erf(x) := \int_0^x \frac{2}{\sqrt{\pi}} e^{-t^2} dt = 1 - erfc(x) \qquad \Phi(x) := \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} = 1 - Q(x)$$

A. $erf(x) \cong$ $\cong \sqrt{1 - e^{\frac{-1.2735457x^2 - 0.1487936x^4}{1 + 0.1480931x^2 + 0.0005160x^4}}}$	$\lvert\varepsilon(x)\rvert < 2.27 \cdot 10^{-5}\ \forall x \geq 0$ [2] $\lvert\varepsilon_r(x)\rvert < 1.21 \cdot 10^{-4}\ \forall x \geq 0$
B. $erfc(x) \cong$ $\cong 1 - \sqrt{1 - e^{\frac{-1.2735457x^2 - 0.1487936x^4}{1 + 0.1480931x^2 + 0.0005160x^4}}}$	$\lvert\varepsilon(x)\rvert < 2.27 \cdot 10^{-5}\ \forall x \geq 0$ [3] $\lvert\varepsilon_r(x)\rvert < 1\%\ \forall x \in [0, b],\ b > 2.1588$
C. $\Phi(x) \cong$ $\cong \frac{1}{2} + \frac{1}{2}\sqrt{1 - e^{\frac{-1.2735457x^2 - 0.0743968x^4}{2 + 0.1480931x^2 + 0.0002580x^4}}}$	$\lvert\varepsilon(x)\rvert < 1.14 \cdot 10^{-5}\ \forall x \geq 0$ [4] $\lvert\varepsilon_r(x)\rvert < 1.78 \cdot 10^{-5}\ \forall x \geq 0$
D. $Q(x) \cong$ $\cong \frac{1}{2} - \frac{1}{2}\sqrt{1 - e^{\frac{-1.2735457x^2 - 0.0743968x^4}{2 + 0.1480931x^2 + 0.0002580x^4}}}$	$\lvert\varepsilon(x)\rvert < 1.14 \cdot 10^{-5}\ \forall x \geq 0$ [5] $\lvert\varepsilon_r(x)\rvert < 1\%\ \forall x \in [0, b],\ b > 3.053$

References

[1] S. Winitzki: *A handy approximation for the error function and its inverse*, http://docs.google.com/viewer?a=v&pid=sites&srcid=ZGVmYXVsdGRvbWFpbnx3aW5pdHpraXxneDoxYTUzZTEzNWQwZjZlOWY2
http://sites.google.com/site/winitzki
http://sites.google.com/site/winitzki/sergei-winitzkis-files
(2008) (read 2011, December)

[2] For $x \gtrsim 4.125$ the approximation $erf(x) \cong 1$ has less absolute error.
[3] For $x \gtrsim 4.125$ the approximation $erfc(x) \cong 0$ has less absolute error.
[4] For $x \gtrsim 5.834$ the approximation $\Phi(x) \cong 1$ has less absolute error.
[5] For $x \gtrsim 5.834$ the approximation $Q(x) \cong 0$ has less absolute error.