Log-concavity of Lucas sequences of first kind

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Abstract

In these notes we address the study of the log-concave operator acting on Lucas sequences of first kind. We will find for which initial values a generic Lucas sequence is log-concave, and using this we show when the same sequence is infinite log-concave. The main result will help to fix the log-concavity of some well known recurrent sequences like Fibonacci and Mersenne numbers. Some possible generalization for a complete classification of the log-concave operator applied to general linear recurrent sequences is proposed.

1 Introduction

Log-concave sequences arise in many areas of algebra, combinatorics, and geometry as detailed by the survey article of Brenti [1]. During the years there have been some studies on the log-operator $\mathcal{L}$ acting on recurrent sequences such as the work of Asai [2] on Bell numbers, Bóna [3] on sequences counting permutations, Liu [4] on combinatorial sequences and McNamara[5] with his work on Pascal's triangle. Lucas sequences where first introduced in 1874 by the French mathematician Edouard Lucas, an extensive reference is the book of Koshy [6]. By definition let P, Q two integer numbers such that $P^2 - 4Q \geq 0$, then the Lucas sequence of first kind $U_n(P, Q)$ is the recurrent sequence defined by $U_0 = 0, U_1 = 1, U_2 = p, U_n = PU_{n-1} - QU_{n-2}$. As special case for some P, Q the Lucas sequence associated becomes a well known sequence, for example $L(1, -1, n) = F_n$ where F_n is the Fibonacci sequence. In these notes we study the log-operator on these sequence to address the general problem to find which P, Q integer the corresponding Lucas sequence $U_n(P, Q)$ is log-concave or ∞-log concave . In section one we will introduce some basic definition and some basic results on log-operator acting on recurrent sequences. Section two will show a general result on how to solve the log-concavity problem on a generic Lucas sequence of first kind. Last section will propose a generalization of the methods used on Lucas sequence to generic linear recurrent sequences.

2 Basic definition

We now remark some definitions of the log-operator . We refer to the notation to McNamara [5]. Let us start with

Definition 1. Let $(a_n)_{n\in\mathbb{N}}$ a real sequence we define the log-operator as a function $\mathcal{L}:\mathbb{R}\to\mathbb{R}$ such that $b_n=\mathcal{L}(a_n)=a_n^2-a_{n-1}a_{n+1}$. If $b_n\geq 0$ for all $n\in\mathbb{N}$ then the sequence $(a_n)_{n\in\mathbb{N}}$ is said to be log-concave.

Considering that log-concavity can deals with negative indexes, by convention we will extend a sequence $(a_n)_{n\in\mathbb{N}}$ to a sequence $(a_n)_{n\in\mathbb{Z}}$ where by definition $a_n=0$ if $n<0$. In the same way if the sequence is finite so $n\leq m, m\in\mathbb{N}$ then all other indexes $n>m$ will be zero.

Definition 2. A real sequence $(a_n)_{n\in\mathbb{N}}$ is said to be $i-fold$ log-concave for $i\in\mathbb{N}, i\geq 1$ if $\mathcal{L}^i(a_n)$ is a nonnegative sequence. Where $\mathcal{L}^i(a_n)$ is the log-operator applied to a sequence $(a_n)_{n\in\mathbb{N}}$ i-times so $\mathcal{L}^i=\mathcal{L}\circ\mathcal{L}\circ\cdots\circ\mathcal{L}$.

Using McNamara [5] notation:

Definition 3. a real sequence $(a_n)_{n\in\mathbb{N}}$ is said to be ∞-log concave if $\mathcal{L}^i(a_n)$ is a nonnegative sequence for all $i\in\mathbb{N}, i\geq 1$.

So log-concavity in the ordinary sense is 1-fold log-concavity. To study log-concavity on Lucas sequences, we need some preliminary results, like the following:

Lemma 4. *let $(a_n)_{n\in\mathbb{N}}$ a sequence where $a_n=k$ for all $n\in\mathbb{N}$ and k is a real number, then $(a_n)_{n\in\mathbb{N}}$ is ∞-log concave .*

Proof. It is easy to check that

$$b_n=\mathcal{L}(a_n)=\mathcal{L}(a_n)=k^2-(k\cdot k)=0$$

for all $n\in\mathbb{N}$. It is also clear that the all zeros sequence b_n is invariant by the log-operator that is $\mathcal{L}(b_n)=b_n$. Being $b_n\geq 0$ that means that also $\mathcal{L}(b_n)\geq 0$ so the all zeros sequence is ∞-log concave . □

In the same way it is also easy to check that

Lemma 5. *let $(a_n)_{n\in\mathbb{N}}$ a sequence where for all $n\in\mathbb{N}$ $a_n=kb^n$ where $k,b\in\mathbb{R}, k\neq 0, b\neq 0$ then $(a_n)_{n\in\mathbb{N}}$ is ∞-log concave .*

Proof. By direct check

$$\mathcal{L}(a_n)=(a_n)^2-a_{n-1}a_{n+1}=k^2b^{2n}-k^2b^{n-1+n+1}=k^2b^{2n}-k^2b^{2n}=0$$

for all $n\in\mathbb{N}$. So a_n is 1-fold log-concave and the result sequence is the all zeros sequence than considering lemma 4 then the sequence a_n is also ∞-log concave . □

In next section we will detail our analysis of the log-operator to the Lucas sequence.

3 Log-operator and Lucas sequences

In these section we address the study of log-concavity, of a Lucas sequence of first kind Let start with the Lucas sequence definition:

Definition 6. Let $(P, Q) \in \mathbb{Z} \times \mathbb{Z}$ two non-zero integer such that $P^2 - 4Q \geq 0$ and let $n \in \mathbb{N}$ an index. A Lucas sequence $U_n(P, Q)$ of first kind is a recurrent sequence defined as follows:

$$\begin{aligned} U_0 &= 0 \\ U_1 &= 1 \\ U_n &= PU_{n-1} - QU_{n-2}. \end{aligned}$$

Choosing the correct P, Q it is possible to obtain some well known sequences for example:

- If $P = 1, Q = -1$ then the Lucas sequence $U_n(1, -1) = F_n$ where F_n is the Fibonacci sequence.
- If $P = 2, Q = -1$ then the Lucas sequence $U_n(2, -1)$ is the sequence of Pell numbers.
- If $P = 1, Q = -2$ then the Lucas sequence $U_n(1, 2)$ is the sequence of Jacobsthal numbers.
- If $P = 3, Q = 2$ then the Lucas sequence $U_n(3, 2)$ is the sequence of Mersenne numbers.

The main result of this section will prove for which initial P, Q the resulting Lucas sequence is ∞-log concave . Let us start by showing that in general if we choose a generic couple P, Q the Lucas sequence $U_n(P, Q)$ is not 1-fold log-concave .

We use the following proposition

Proposition 7. *The Fibonacci sequence F_n is not 1-fold log-concave .*

Proof. Considering the log-operator applied to F_n we have

$$b_n = \mathcal{L}(F_n) = F_n^2 - F_{n-1}F_{n+1};$$

now by the Cassini's identity

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \tag{1}$$

we obtain

$$F_n^2 - F_{n-1}F_{n+1} = (-1) \cdot (-1)^n = (-1)^{n+1}.$$

So

$$\mathcal{L}(F_n) = (-1)^{n+1}.$$

thus $F(n)$ is not 1-fold log-concave . If we applied the $\mathcal{L}$ operator to the sequence b_n and we calculate $\mathcal{L}^2(F(n)) = \mathcal{L}(\mathcal{L}(b_n)$ we obtain

$$\mathcal{L}^2(F(n)) = ((-1)^{n+1})^2 - (-1)^{n+2} \cdot (-1)^n = ((-1)^{n+1})^2 - (-1)^{2n+2} = 1 - 1 = 0$$

so after applying the log-operator more than once we obtain a sequence that is log-concave. □

We will now fix for what initial parameter P, Q the generate Lucas sequence $U_n(P, Q)$ is a 1-fold log-concave Lucas sequence, and in these cases where for what P, Q the Lucas sequence becomes ∞-log concave . Instead of trying to apply directly the log-operator to the generic expression of the Lucas sequence $U_n(P, Q)$, we will use a more treatable expression for $U_n(P, Q)$. To do this, we first need to recall [7] that:

Remark 8. let $U_n(P, Q)$ a Lucas sequence of first kind, than the characteristic equation of the recurrence relation is

$$x^2 - Px + Q = 0 \tag{2}$$

that has discriminant $D = P^2 - 4Q$. If the discriminant is positive so $D \geq 0$ then the roots of the characteristic equation are

$$a = \frac{P + \sqrt{D}}{2}, \; b = \frac{P - \sqrt{D}}{2} \tag{3}$$

and so if $D \geq 0$ it is possible to rewrite $U_n(P, Q)$ in the following way

$$U_n(P, Q) = \frac{a^n - b^n}{a - b} = \frac{a^n - b^n}{\sqrt{D}}. \tag{4}$$

Armed with this expression for Lucas sequence, we will divide our study in two main cases let us start with the simpler one.

Proposition 9. *Let $U_n(P, Q)$ a Lucas sequence where P, Q are two integer and the discriminant D of the characteristic equation associated with $U_n(P, Q)$ is zero then the Lucas sequence associated is 1-fold log-concave .*

Proof. It is easy to see that if $D = 0$ then $P^2 - 4Q = 0$ and so there exists and integer S such that $P = 2S$ and $Q = S^2$. Using this fact the Lucas sequence associated can be rewritten in the form

$$U_n = nS^{n-1}. \tag{5}$$

So now, applying the $\mathcal{L}$ operator, we see that

$$\begin{aligned} \mathcal{L}(U_n) &= \\ \mathcal{L}(nS^{n-1}) &= (nS^{n-1})^2 - [(n-1)S^{n-2} \cdot (n+1)S^n] \\ &= (n^2)S^{2n-2} - (n^2 - 1)S^{n-2+n} \\ &= (n^2)S^{2n-2} - (n^2 - 1)S^{2n-2} \\ &= (n^2 - n^2 + 1)S^{2n-2} \\ &= (S^{n-1})^2 \end{aligned}$$

and so $\mathcal{L}(U_n) \geq 0$ for all $S \in \mathbb{Z}$. This prove that U_n is 1-fold log-concave . □

From proposition 9 we have also the following corollary

Corollary 10. *Let $U_n(P, Q)$ a Lucas sequence where P, Q are two integer and there exist an $S \in \mathbb{Z}$ such that $P = 2S$ and $Q = S^2$ then the Lucas sequence associated is ∞-log concave .*

Proof. We have seen that under the hypothesis $\mathcal{L}(U_n) = (S^{n-1})^2 = (S^2)^{n-1}$. By changing the index we have that the original sequence become a sequence of the form $b_k = S^k$ where $k \in \mathbb{Z}, k = 2n-2, k \geq -2$. Considering that for negative indexes $b_k = 0$ we have that by lemma 5 the sequence b_k is ∞-log concave and so U_n. $\square$

Let now consider the general case

If $D = P^2 - 4Q > 0$ by remark 8 it is possible to rewrite $U_n(P,Q)$ in the following way

$$U_n(P,Q) = \frac{a^n - b^n}{a-b} = \frac{a^n - b^n}{\sqrt{D}} \tag{6}$$

where

$$a = \frac{P+\sqrt{D}}{2}, \; b = \frac{P-\sqrt{D}}{2} \tag{7}$$

we notice that, using direct calculation we have

$$\begin{aligned}
&\mathcal{L}(U_n) = \\
\mathcal{L}\left(\frac{a^n - b^n}{\sqrt{D}}\right) &= \left(\frac{a^n - b^n}{\sqrt{D}}\right)^2 - \left[\frac{a^{n-1} - b^{n-1}}{\sqrt{D}} \cdot \frac{a^{n+1} - b^{n+1}}{\sqrt{D}}\right] \\
&= \left(\frac{a^{2n} - 2a^nb^n + b^{2n}}{D}\right) - \left(\frac{a^{n-1+n+1} - a^{n-1}b^{n+1} - a^{n+1}b^{n-1} + b^{n+1+n-1}}{D}\right) \\
&= \frac{a^{2n} - 2a^nb^n + b^{2n} - a^{2n} + a^{n-1}b^{n+1} + a^{n+1}b^{n-1} - b^{2n}}{D} \\
&= \frac{a^{n+1}b^{n-1} - 2a^nb^n + a^{n-1}b^{n+1}}{D} \\
&= \frac{a^{n-1}b^{n-1}(a^2 - 2ab + b^2)}{D} \\
&= \frac{(ab)^{n-1}(a^2 - 2ab + b^2)}{D} \\
&= \frac{(ab)^{n-1}(a-b)^2}{D}
\end{aligned}$$

now then by definition

$$ab = \frac{P+\sqrt{D}}{2} \cdot \frac{P-\sqrt{D}}{2} = \frac{1}{4}(P^2 - D) = \frac{1}{4}(P^2 - P^2 + 4Q) = Q \tag{8}$$

and

$$a - b = \frac{P+\sqrt{D}}{2} - \frac{P-\sqrt{D}}{2} = \frac{2P}{2} = P. \tag{9}$$

So finally we have

$$\mathcal{L}(U_n) = \frac{Q^{n-1}P^2}{D} \tag{10}$$

So $\mathcal{L}(U_n) \geq 0$ if $Q \geq 0$. Combining this with the assumption that $P^2 - 4Q \geq 0$ we have that $U_n(P, Q)$ is 1-fold log-concave if

$$\begin{cases} Q \geq 0 \\ P^2 - 4Q > 0 \end{cases}$$

that gives the following set of solutions $Q \geq 0 \wedge P > 2\sqrt{Q}$ or $Q \geq 0 \wedge P < -2\sqrt{Q}$.

We can summarize the result in the following

Theorem 11. *Let P, Q two integer such that $Q \geq 0 \wedge P > 2\sqrt{Q}$ or $Q \geq 0 \wedge P < -2\sqrt{Q}$, then the associated Lucas sequence $U_n(P, Q)$ is 1-fold log-concave .*

using theorem 11 and the lemma 5, it is easy to check that

Corollary 12. *Let P, Q two integer such that $Q \geq 0 \wedge P > 2\sqrt{Q}$ or $Q \geq 0 \wedge P < -2\sqrt{Q}$. Then the Lucas sequence $U_n(P, Q)$ is ∞-log concave .*

Proof. Under the hypothesis we have that

$$b_n = \mathcal{L}(U_n) = \frac{Q^{n-1}P^2}{D}$$

that is a sequence of the form $k_n a^n$ and by lemma 5 $U_n(P, Q)$ is ∞-log concave . □

At the end using the corollary 12 we can check that:

- $U_n(1, -1)$ is the Fibonacci sequence that is not 1-fold log-concave and so neither ∞-log concave .
- $U_n(2, -1)$ is the sequence of Pell numbers that is not 1-fold log-concave and so neither ∞-log concave .
- $U_n(1, -2)$ is the sequence of Jacobsthal numbers that is not 1-fold log-concave and so neither ∞-log concave .
- $U_n(3, 2)$ is the sequence of Mersenne numbers that is ∞-log concave .

4 Conclusion

In these notes we have studied the log-operator applied to a generic Lucas sequence of first kind U_n. We have shown that for initial parameter $Q \geq 0, P \geq 2Q$ or $Q \geq 0, P \leq 2Q$, the associate Lucas sequence of first kind is ∞-log concave . As result we find that Fibonacci, Pell and Jacobsthal sequences are not ∞-log concave but the Mersenne numbers sequence is ∞-log concave . There is a natural question that arise from these results. As shown the key fact, that a sequence is recurrent, allow the sequence to be expressed in a more treatable way before applying the log-operator . It would be interesting giving a generic linear recurrent sequence that satisfy a generic characteristics equation of order k, to find sufficient condition on the coefficient of the equation to be sure that the sequence is 1-fold log-concave and after this fix which conditions leads to a ∞-log concave sequence. Formalizing a little, giving a recurrent sequence define as $a_n = k_1a_{n-1} + k_2a_{n-2} + \dots k_ma_{n-m}$ that has a characteristic equation $a_n - k_1a_{n-1} - k_2a_{n-2} - \cdots - k_ma_{n-m} = 0$ is there is a sufficient condition on the $k_1, k_2, \dots, k_m$ integer coefficient such that a_n is 1-fold log-concave and ∞-log concave . This question would be subject of further study.

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