Pauli matrices and 2D electron gas.

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Abstract

The Pauli matrices can be derived from the Boltzmann treatment of a two dimensional electron gas generated from two different crystal structures. The locally created electron gas in A and B wing of a Bell-type experiment enables entangled classical currents. We argue for a micro Boltzmann distribution interpretation of the EPR paradox.

Keywords: quantum foundation, solid state electronics, entanglement

1. INTRODUCTION

Providing a physical theory for local hidden variables [1] is difficult. If, however, a macroscopic (semi) classical treatment arrives at entanglement, this treatment can model local hidden variables in the micro domain. In this paper we will focus the attention on a two-dimensional electron gas. The distribution of a two dimensional electron gas can be obtained from the semiclassical Boltzmann electron transport theory.

1.1. Boltzmann transport

The Fermi-Dirac distribution of electrons in a two dimensional electron gas without electric and magnetic fields is [2]

$$f_0(\mathbf{k}) = \frac{1}{1+\exp\left[\frac{E(\mathbf{k})-\mu}{k_B T}\right]} \tag{1}$$

Here, $\mathbf{k} = (k_x, k_y)$ is the wave vector, k_B is the Boltzmann constant, T the absolute temperature and μ the chemical potential. If an electromagnetic source is available the acceleration $\frac{d}{dt}\mathbf{v}(\mathbf{k})$ of the electrons is influenced by the Lorentz force.

$$\frac{d}{dt}\mathbf{v}(\mathbf{k}) = -\frac{e}{m^*}\left(\mathbf{E}(\mathbf{k}) + \mathbf{v}(\mathbf{k}) \times \mathbf{B}(\mathbf{k})\right) \tag{2}$$

The acceleration is related to velocity in the wave vector $\frac{\hbar}{m^*}\frac{d\mathbf{k}}{dt} = \frac{\hbar}{m^*}\dot{\mathbf{k}}$. m^* is the effective mass of the electron.

1.2. Current

The distribution $f = f(\mathbf{k}, t)$ now differs from equation (1). It is still assumed that $f(\mathbf{k}, t) = f(\mathbf{k} + \dot{\mathbf{k}}dt, t + dt)$. If the difference between $f(\mathbf{k}, t)$ and $f_0(\mathbf{k})$ is denoted by $g(\mathbf{k}, t)$ the current can be approximated by

$$\mathbf{j}(t) = -2 \int \frac{d^2\mathbf{k}}{(2\pi)^2}\, e\mathbf{v}(\mathbf{k})g(\mathbf{k}, t) \tag{3}$$

The 2 in equation (3) arises from electron spin (considered phenomenologically). Note $\dot{\mathbf{k}} \neq \mathbf{0}$. The linear Boltzmann equation for $g(\mathbf{k}, t)$ is equal to

$$\frac{g(\mathbf{k}, t)}{\Delta\tau(\mathbf{k})} = (\nabla_{\mathbf{k}} f_0(\mathbf{k})) \cdot \frac{e}{\hbar}\mathbf{E}(\mathbf{k}) + (\nabla_{\mathbf{k}} g(\mathbf{k}, t)) \cdot \frac{e}{\hbar}\mathbf{v}(\mathbf{k}) \times \mathbf{B}(\mathbf{k}) \tag{4}$$

In a zero magnetic field or when $\nabla_{\mathbf{k}} g(\mathbf{k}, t) \sim \mathbf{v}(\mathbf{k}) + \mathbf{B}(\mathbf{k})$ the linear Boltzmann equation can be solved by

$$g(\mathbf{k}, t) = \Delta\tau(\mathbf{k})\frac{e}{\hbar}\left(\nabla_{\mathbf{k}} f_0(\mathbf{k})\right) \cdot \mathbf{E}(\mathbf{k}) \tag{5}$$

$\Delta\tau(\mathbf{k}) = \tau(\mathbf{k}) - \bar{t}_0$ the relative relaxation time. In the general theory [2] $\nabla_{\mathbf{k}} f_0 = \hbar\frac{\partial f_0}{\partial E(\mathbf{k})}\mathbf{v}(\mathbf{k})$ with

$$\lim_{T\to 0} \frac{\partial f_0}{\partial E(\mathbf{k})} = -\delta\left(E - E_F\right) \tag{6}$$

Because of non-zero velocities, (6) holds approximately [2]. The E_F is the Fermi energy. Given, $\mathbf{j} = \sigma\mathbf{E}$ the conductance 2×2 matrix can be written

$$\sigma = \frac{e^2}{2\hbar\pi^2} \int d^2\mathbf{k}\, \delta\left(E(\mathbf{k}) - E_F\right) \Delta\tau(\mathbf{k})\mathbf{v}(\mathbf{k}) \otimes \mathbf{v}(\mathbf{k}) \tag{7}$$

The tensor product $\otimes$ is: $(\mathbf{x} \otimes \mathbf{y})_{i,j} = x_i y_j$ for $i, j = 1, 2$ and $\mathbf{x} = (x_1, x_2)$ similarly $\mathbf{y} = (y_1, y_2)$.

2. PAULI MATRICES

Subsequently, the wave vector $\mathbf{k}$ is transformed with $k_x = k\cos(\varphi)$ and $k_y = k\sin(\varphi)$. The Jacobian is $k = \sqrt{k_x^2 + k_y^2}$. Hence

$$\sigma = \frac{e^2}{2\hbar\pi^2} \int k dk\, d\varphi\, \delta\left(E(k,\varphi) - E_F\right) \Delta\tau(k,\varphi)\mathbf{v}(k,\varphi) \otimes \mathbf{v}(k,\varphi) \qquad (8)$$

Note that, because $\dot{\mathbf{k}} \neq 0$, we also see $\dot{k} \neq 0$ and/or $\dot{\varphi} \neq 0$. The next step is to observe that $E(k,\varphi)$ in (8) depends on the velocity $\mathbf{v}(k,\varphi)$ (see [2]). The subsequent transformation is to write $E = E(k,\varphi) = \frac{1}{2}m^*\mathbf{v}^2(k,\varphi)$ together with $\phi = \varphi$. For completeness, $k = k(E,\phi)$ and $\varphi = \varphi(E,\phi)$. We do not use quadratic dispersion for E. The Jacobian for the transformation is

$$J = \begin{vmatrix} \partial k/\partial E & \partial k/\partial\phi \\ \partial\varphi/\partial E & \partial\varphi/\partial\phi \end{vmatrix} \qquad (9)$$

With this particular transformation $\partial\varphi/\partial E = 0$ and $\partial\varphi/\partial\phi = 1$. Let us take radial kinetic energy only $E = \frac{1}{2}m^*\mathbf{v}^2(k)$. Hence,

$$1 = m^*||\mathbf{v}(k)||\frac{\partial||\mathbf{v}(k)||}{\partial k}\frac{\partial k}{\partial E} \qquad (10)$$

With $||.||$ the Euclidean norm like e.g. in $||\mathbf{k}|| = k = \sqrt{k_x^2 + k_y^2}$. Hence, the Jacobian in (9) is $J = w(E) = \partial k/\partial E$ which, in k and φ, looks like $\{m^*||\mathbf{v}(k)||\frac{\partial||\mathbf{v}(k)||}{\partial k}\}^{-1}$. Generally, the $||\mathbf{v}(k,\varphi)|| > 0$, despite $T \to 0$. The conductance in (8) now can be transformed into

$$\sigma = \frac{e^2}{2\hbar\pi^2} \int_0^\infty dE \int_{-\pi}^{+\pi} d\phi\, \delta\left(E - E_F\right) \Delta\tau'(E,\phi)\mathbf{v}(E,\phi) \otimes \mathbf{v}(E,\phi) \qquad (11)$$

with $\Delta\tau'(E,\phi) = k(E,\phi)\Delta\tau(E,\phi)w(E)$. Subsequent integration over E ($E_F > 0$) results into

$$\sigma = \frac{e^2}{2\hbar\pi^2} \int_{-\pi}^{+\pi} d\phi\ \Delta\tau'(E_F,\phi)\mathbf{v}(E_F,\phi) \otimes \mathbf{v}(E_F,\phi) \qquad (12)$$

The $\mathbf{v}$ generally depend on E and ϕ. The first step to the Pauli matrices for (12) is to transform the velocity vector to an 'associated' form: $\mathbf{u}(E_F,\phi) = \mathbf{v}(E_F,\phi)\sqrt{\Delta\tau'(E_F,\phi)}$. Note that even when $\mathbf{v}$ is independent of ϕ the $\mathbf{u}$ can vary with ϕ because of ϕ dependece in the relaxation time. $\mathbf{u}$ can be real or imaginary. We have $\Delta\tau'(E,\phi) = k(E,\phi)\Delta\tau(E,\phi)w(E)$, with real positive or negative, relative relaxation time $\Delta\tau(E,\phi)$ and, real positive or negative, Jacobian $w(E)$. Let us specify $u_i(E_F,\phi)$ for $i = 1,2$ and derive the first Pauli matrix.

2.1. Pauli's σ_x for conductance

Suppose, the Heaviside H is defined by, $H(x) = 1$ for all $x \geq 0$ and $H(x) = 0$ for all $x < 0$. Moreover, suppose $\alpha \in (0, \frac{\pi}{4})$. Then for $\phi \in (-\alpha, \alpha)$,

$$\begin{aligned} \delta u_{x,1} = u_1(E_F, \phi) &= \kappa(E_F)\sqrt{\tfrac{1}{2\alpha}}\sqrt{\tan(\phi)}H(\phi+\alpha)H(\alpha-\phi) \\ \delta u_{x,2} = u_2(E_F, \phi) &= \kappa(E_F)\sqrt{\tfrac{1}{2\alpha}}\sqrt{\cot(\phi)}H(\phi+\alpha)H(\alpha-\phi) \end{aligned} \tag{13}$$

Note that in this definition, suppressing the E_F dependence notation for the moment, we see $u_i^2(\phi) < 0$ when $\phi < 0$ and $u_i^2(\phi) \geq 0$ when $\phi \geq 0$. The indication $\delta u_{x,i}$ refers to the choice of relatively small changes in associated velocity for the Pauli matrix σ_x. For a given $\phi > 0$ and $\phi \in (0, \alpha)$ we assume in approximation the conservation of total kinetic energy for 'differential associated velocities' $\delta\mathbf{u}_x(E, \phi) = \mathbf{u}(E, \phi)$. I.e. the changes in kinetic energies when electron and hole are created occur 'balanced' in the $\mathbf{u}$.

$$\frac{1}{2}m^*||\mathbf{u}(E,\phi)||^2 + \frac{1}{2}m^*||\mathbf{u}(E,-\phi)||^2 = 0 \tag{14}$$

It is believed that electron escape from a crystal structure leaving behind a hole can be pictured in equation (13). Note that equation (14) is valid in $\mathbf{u}$ not in $\mathbf{v}$ terms. Integrating for u_1^2 for instance, using (12) then (punching $\phi = 0$) gives

$$\sigma_{1,1} = \frac{1}{2\alpha}\frac{e^2}{2\hbar\pi^2}\int_{-\pi}^{+\pi} d\phi\, \kappa^2(E_F)\tan(\phi)H(\phi+\alpha)H(\alpha-\phi) \tag{15}$$

Or, equally

$$\sigma_{1,1} = \frac{\kappa^2(E_F)}{2\alpha}\frac{e^2}{2\hbar\pi^2}\int_{-\alpha}^{+\alpha} d\phi\, \tan(\phi) = -C\left[\log|\cos(\phi)|\right]_{-\alpha}^{+\alpha} \tag{16}$$

with $C = \frac{\kappa^2(E_F)}{2\alpha}\frac{e^2}{2\hbar\pi^2}$. From (16) follows $\sigma_{1,1} = -C\{\log|\cos(\alpha)| - \log|\cos(-\alpha)|\}$. Hence, $\sigma_{1,1} = 0$. A similar argument applies to $\sigma_{2,2}$ where the integration shows that $\sigma_{2,2} = -C\{\log|\sin(\alpha)| - \log|\sin(-\alpha)|\} = 0$. If we subsequently turn to $\sigma_{1,2} = \sigma_{2,1}$ then we see

$$\sigma_{1,2} = \frac{\kappa^2(E_F)}{2\alpha}\frac{e^2}{2\hbar\pi^2}\int_{-\alpha}^{+\alpha} d\phi\sqrt{\tan(\phi)}\sqrt{\cot(\phi)} = \frac{e^2\kappa^2(E_F)}{2\hbar\pi^2} \tag{17}$$

Hence, with $\kappa(E_F) = \frac{\pi\sqrt{2\hbar}}{e}$ the Pauli matrix σ_x can be obtained for the conductance matrix. However, because of infinitesimal changes in 'associated velocity' (13), the factor is maintained. Hence,

$$\Sigma_x = \frac{e^2\kappa^2(E_F)}{2\hbar\pi^2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{18}$$

κ sufficiently small. Note that 'going through a singularity' for the cot integral provides a zero result of the integral when the $\phi = 0$ is 'cut out' of the integration by left-hand $-\epsilon$ and right-hand ϵ for $\epsilon \to 0^+$. This is so because $|\sin(-\epsilon)|$ equals $|\sin(\epsilon)|$. Hence, the integral of both the tan as well as the cot containing associated 'velocities' in equation (13) can be given by the $\epsilon \to 0^+$ sum of $\int_{-\pi}^{-\epsilon}$ and $\int_{\epsilon}^{\pi}$ integration operations and is written $\oint_{-\pi}^{+\pi}$. The Heavisides in (13) convert the integrals into $\oint_{-\alpha}^{+\alpha}$.

2.2. σ_y conductance

The σ_y Pauli matrix can be similarly derived for the conductance. Let us define the 'associated velocity' entries

$$\begin{aligned} \delta u_{y,1} = u_1(E_F,\phi) &= \lambda(E_F)\cos(\phi/2)H(\phi-\beta)H(\tfrac{\pi}{2}-\phi) \\ \delta u_{y,2} = u_2(E_F,\phi) &= i\,\lambda(E_F)\sin(\phi/2)H(\phi-\tfrac{\pi}{2})H(\pi-\beta-\phi) \end{aligned} \tag{19}$$

with $\beta \in (0,\alpha)$. In the first place we may note that, dissimilar to the σ_x case in secion - 2.1, the canceling of 'associated' kinetic energy terms occur in a non-symmetrical way: i.e. $\frac{1}{2}m^*||\mathbf{u}(\phi)||^2 + \frac{1}{2}m^*||\mathbf{u}(\phi')||^2 = 0$ where (ϕ',ϕ) solves $\cos^2(\phi) - \sin^2(\phi') = 0$ with $\phi \in (\beta,\frac{\pi}{2})$ and $\phi' \in (\frac{\pi}{2},\pi-\beta)$. If the aim is to explain the geometric asymmetry in electron - hole creation geometry then perhaps the crystal structure, i.e. phonon hindrance can be held accountable for the asymmetry. Subsequently, the integral for $u_1(E_F,\phi)u_2(E_F,\phi)$ vanishes because the domains are disjoint. For the $\sigma_{1,1}$ term we see

$$\sigma_{1,1} = \frac{e^2\lambda^2(E_F)}{2\hbar\pi^2}\int_{\beta}^{\pi/2}\cos^2(\phi/2)\,d\phi \tag{20}$$

Because $\cos^2(\phi/2) = \frac{1}{2}(1+\cos(\phi))$ it follows that
$\sigma_{1,1} = \frac{e^2\lambda^2(E_F)}{4\hbar\pi^2}\left\{\frac{\pi}{2} - \beta + 1 - \sin(\beta)\right\}$. Similarly for $\sigma_{2,2}$ we may derive

$$\sigma_{2,2} = -\frac{e^2\lambda^2(E_F)}{2\hbar\pi^2}\int_{\pi/2}^{\pi-\beta}\sin^2(\phi/2)\,d\phi \tag{21}$$

Because $\sin^2(\phi/2) = \frac{1}{2}(1-\cos(\phi))$ it follows that $\sigma_{2,2} = -\frac{e^2\lambda^2(E_F)}{4\hbar\pi^2}\left\{\frac{\pi}{2} - \beta + 1 - \sin(\beta)\right\}$. Hence, when we take $\lambda(E_F)$ and $\kappa(E_F)$ such that

$$\kappa^2(E_F) = \frac{\lambda^2(E_F)}{2}\left\{\frac{\pi}{2} - \beta + 1 - \sin(\beta)\right\} \tag{22}$$

then we find

$$\Sigma_y = \frac{e^2\kappa^2(E_F)}{2\hbar\pi^2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{23}$$

3. ENTANGLED CURRENTS

Suppose in experiment, similar to Bell's [3], [4] we have a distant A and B wing where, at both sides, a localized but identical electric field with vecor

$$\mathbf{E}^T = \frac{2\hbar\pi^2}{e^2\kappa^2(E_F)}\hat{\mathbf{E}}^T = \frac{2\hbar\pi^2}{e^2\kappa^2(E_F)}(E_1, E_2) \tag{24}$$

reigns. Furthermore we insist that $\hat{\mathbf{E}}^2 = E_1^2 + E_2^2 = 1$. If we take the two 'crystal structures' we talked about in the previous two sections and generate electron-hole pairs, then at first instance, electrons in the 2D gas from the horizontal Σ_x related crystal and from the vertical Σ_y related crystal co-occur in the anglular interval (β, α). In this area per wing one can 'mix the two conductances' Σ_x and Σ_y with a unit parameter vector $\hat{\mathbf{a}} = (a_1, a_2)$ and/or $\hat{\mathbf{b}} = (b_1, b_2)$. E.g. $\Sigma_A(a_1, a_2) = a_1\Sigma_x + a_2\Sigma_y$. The mixed current at A is then equal to $\mathbf{j}_A = \Sigma_A(a_1, a_2)\mathbf{E}$. Similarly, one can define a $\Sigma_B(b_1, b_2)$ such that $\mathbf{j}_B = \Sigma_B(b_1, b_2)\mathbf{E}$. Note that we can take $a_1^2 \times 100\%$ to indicate the percentage at the A wing of Σ_x-crystal electrons and similarly for the $a_2^2 \times 100\%$. The same thing can be supposed for the entries of the $\mathbf{b}$ vector. Furthermore, the mixing percentages are transformed into angles. We suppose, $a_1 = \cos(\psi_A)$ and $a_2 = \sin(\psi_A)$ together with $b_1 = \cos(\psi_B)$, $b_2 = \sin(\psi_B)$ and project the ψ in an interval $[\gamma, \Delta\tau] \subset (0, \frac{\pi}{2})$. Subsequently, we may note that because of (24) and the expressions for Σ_x and Σ_y in resp. (18) and (23) the following inner-product for $\mathbf{j}_A$ and $\mathbf{j}_B$ employs Pauli matrices as in a quantum corelation (see e.g. [3]).

$$\mathbf{j}_A^T\,\mathbf{j}_B = \hat{\mathbf{E}}^T\left[\cos(\psi_A)\sigma_x + \sin(\psi_A)\sigma_y\right]\left[\cos(\psi_B)\sigma_x + \sin(\psi_B)\sigma_y\right]\hat{\mathbf{E}} \tag{25}$$

Note that, $\sigma_x^T = \sigma_x$ and $\sigma_y^T = \sigma_y$ together with $\sigma_x^2 = \sigma_y^2 = 1_{2\times 2}$. If we in addition inspect e.g. the term $\cos(\psi_A)\sin(\psi_B)\hat{\mathbf{E}}^T\sigma_x\sigma_y\hat{\mathbf{E}}$ it is noted that this equals $\cos(\psi_A)\sin(\psi_B)(E_1, E_2)\begin{pmatrix} -E_2 \\ E_1 \end{pmatrix} = 0$ Hence, the inner product of the A and B wing current is $\mathbf{j}_A^T\mathbf{j}_B = \cos(\psi_A - \psi_B)$. The inner product of the two current vectors are entangled.

4. CONCLUSION

It was demonstrated that under classical local conditions, entangled measurements can be simulated that commonly are thought only to occur in non-local quantum theory. Let us summarize the experiment: The source in a Bell-type experiment can be the cloning and subsequent sending in two different directions of the electric field vector $\mathbf{E}$. Two pairs of crystals are employed to generate the mixing of conductances Σ_x and Σ_y, in the A and conductances Σ_x and Σ_y, in the B wing. The electrons escape from the crystal surfaces (Σ_x parallel x^+ axis and Σ_y parallel y^+ axis and $x\perp y$) and locally mix in $\phi \in (\beta, \alpha)$. If the current vectors that are created from the 'cloned' electric fields vectors are transported to a measuring system O a current-current entangled inner product $\mathbf{j}_A^T\mathbf{j}_B = \mathbf{a}\cdot\mathbf{b}$ can be observed from $\mathbf{j}_A = [a_1\sigma_x + a_2\sigma_y]\hat{\mathbf{E}}_A$ and $\mathbf{j}_B = [b_1\sigma_x + b_2\sigma_y]\hat{\mathbf{E}}_B$. Hence, $\mathbf{j}_A^T\mathbf{j}_B$ can in principle violate the CHSH but note that the current vectors are created by local means. The parameters $\mathbf{a}$ and $\mathbf{b}$ refer to mixing percentages of electrons with $\phi \in (\beta, \alpha)$: i.e. in the A wing $a_1^2 \times 100\%$ of the electrons from the Σ_x crystal and $a_2^2 \times 100\%$ from the Σ_y crystal and similarly in the B wing. Perhaps one would like to argue against entanglement in currents but the outcome in the innerproduct $\mathbf{j}_A^T\mathbf{j}_B$ is the same as quantum mechanically. Entanglement refers to something unobservable and obtains its meaning from its use [5]. Essentially it is concluded from the experimental correlation. Unless we refered unwittingly to macroscopic quantum electron gas, Bell's correlation needs not be nonlocal in its origin. It can be claimed that the structure laid down here is the physics of locality and causality that Einstein had in mind. The question then of course, similar to Madelung's hydrodynamic interpreation of qm [6], is: 'if referred to the microphysics domain, what does the distribution in the Boltzmann equation (4) distribute'. On the mathematical side the result is in accordance with [7], [8], [9], [10].

An additional point arises if we note that mathematically there is a confinement to a mixing interval $\phi \in (\beta, \alpha)$, with $\beta > 0$. It can be that, macro-

scopically, the mathematical structure does not violate the CHSH because of the mixing angle restriction. This does not disqualify the mathematical structure used for local hidden variable purposes but shows the difference of macro vs micro Boltzmann distributions.

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