On the Area of Pedal and Antipedal Triangles

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Abstract

We give a new proof of the formula expressing the area of the triangle whose vertices are the projects of an arbitrary point in the plane onto the sides of a given triangle, in terms of the geometry of the the given triangle and the location of the projection point. Other related geometrical constructions and formulas are also presented.

pedal triangles, area, conics
Primary: 51M25, 51M16. Secondary 51M04, 51M15, 51N20.

1 Introduction

Recall that, given a triangle ABC, a triangle $A'B'C'$ is called a pedal triangle (with respect to ΔABC) if A', B', C' are the projections of a point P onto the sides of ΔABC. The point P will be called the pedal point of $\Delta A'B'C'$.

The main goal of this note is to give a new proof to the formula for the area of a pedal triangle of a point, relative to a fixed triangle. This formula takes into consideration, besides the geometrical characteristics of this fixed triangle, only the location of the pedal point. With the convention that $|\Delta XYZ|$ denotes the area of the triangle XYZ, the following holds:

Theorem 1 *Let ΔABC be a given triangle and denote by O and R the center and the radius of the circumcircle, respectively. Let P be a an arbitrary point and let $A' \in BC$, $B' \in AC$, $C' \in AB$ be the projections of P onto the sides of ΔABC (i.e., $\Delta A'B'C'$, is the pedal triangle of P with respect to ΔABC; cf. Figure 1). Then the following formula holds:*

$$\frac{|\Delta A'B'C'|}{|\Delta ABC|} = \frac{|R^2 - OP^2|}{4R^2}. \tag{1}$$

This is a classical result that has been around for many years. However, the proofs existing in the literature (we are aware of [1] and [2]) are rather complex and involved. Here we present an approach of algebraic nature, which is considerably more economical and direct. In addition, as consequences of Theorem 1 we note a couple of results, of independent interest.

Corollary 1 *The locus of all points for which the ratio of the area of the pedal triangle to the area of an arbitrary triangle ABC (with respect to which the pedal triangle is constructed) is constant is a circle concentric with the circle circumscribed to triangle ABC.*

Corollary 2 *The locus of all points with the property that their projections onto the sides of a given triangle ABC are three collinear points is the circumcircle of ΔABC.*

These are both obvious from (1). Corollary 2 is usually attributed to Simpson, and our contribution in this regard is to provide a conceptually new proof of this well-known fact.

As a natural counterpart to Theorem 1 we also derive a formula of a similar nature for the area of an antipedal triangle. Recall that ΔMNP is called the antipedal triangle of a point K with respect to $\Delta\, ABC$ if the lines KA, KB, KC are perpendicular to PN, MP and MN, respectively. We have:

Theorem 2 *If $\Delta\, MNP$ is the antipedal triangle of the point K with respect to $\Delta\, ABC$ then the following relation holds:*

$$\frac{|\Delta MNP|}{|\Delta ABC|} = \frac{4R^2}{|R^2 - OK_1{}^2|} \tag{2}$$

with O being the circumcenter and R being the circumradius of ΔABC, and K_1 being the isogonal of K (see Figure 4).

Recall that two points K, K_1 are said to be isogonal to one another with respect to ΔABC if K_1A is the reflection of KA across the median from A in ΔABC, plus similar conditions for the vertices B and C.

2 The Proof of Theorem 1

To simplify notation, we will use Δ for ΔABC and Δ_P for $\Delta A'B'C'$. Consider the lines AB, BC, AC, given by the equations $\alpha_C x + \beta_C y + \gamma_C = 0$, $\alpha_A x + \beta_A y + \gamma_A = 0$, $\alpha_B x + \beta_B y + \gamma_B = 0$, respectively. The signs of the corresponding coefficients for each line are selected such that if a point $P(x, y)$ is inside ΔABC, then $\alpha_C x + \beta_C y + \gamma_C > 0$, $\alpha_A x + \beta_A y + \gamma_A > 0$, $\alpha_B x + \beta_B y + \gamma_B > 0$. Also, for a point $P(x_1, y_1)$, we denote by d_C, d_A, and d_B the distance from P to AB, BC, and AC, respectively (see Figure 1). As a result, we have explicit formulas for d_C, d_A, and d_B. For example, $d_C = \frac{|\alpha_C x_1 + \beta_C y_1 + \gamma_C|}{\sqrt{\alpha_C^2 + \beta_C^2}}$, and similar expressions hold for d_A and d_B. In addition, by $\overline{d}_C$, etc., we denote the *directed* line segment of length d_C, i.e., $\overline{d}_C := \pm\, d_C$, with the choice of sign dictated by the location of P with respect to the line AB. In particular,

$$\overline{d}_C = \frac{\alpha_C x_1 + \beta_C y_1 + \gamma_C}{\sqrt{\alpha_C^2 + \beta_C^2}}. \tag{3}$$

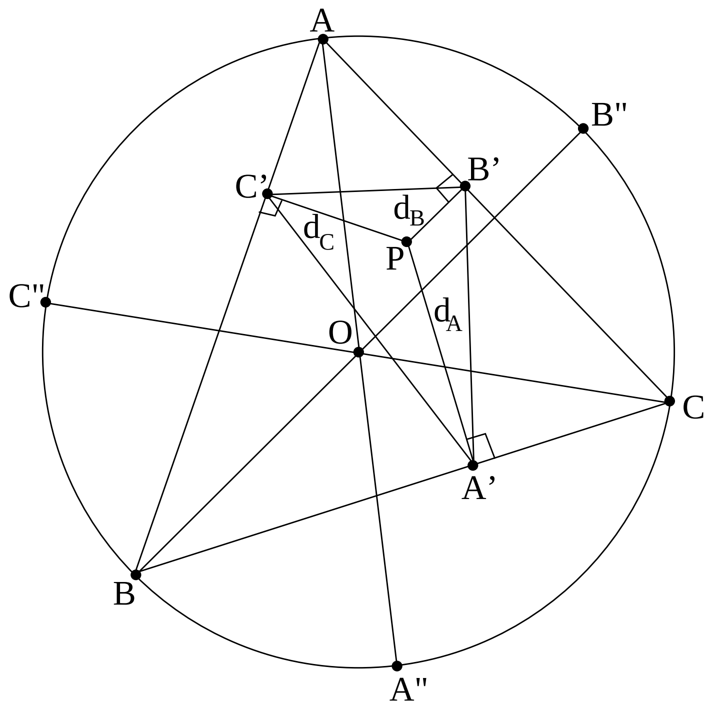

Figure 1

We can express $\overline{d}_A$ and $\overline{d}_B$ in the same manner. A direct computation shows that

$$|\Delta_P| = \pm\frac{d_B d_C \sin(\measuredangle A)}{2} \pm \frac{d_A d_C \sin(\measuredangle B)}{2} \pm \frac{d_A d_B \sin(\measuredangle C)}{2}, \tag{4}$$

where the selection of + or − is dictated by the location of the point P. With the convention that $(+,+,+)$ means that the signs of the three fractions on the right-hand side of (4) are positive, and similarly for all the other possible combinations, the picture below shows the regions in the plane which yield a particular combination.

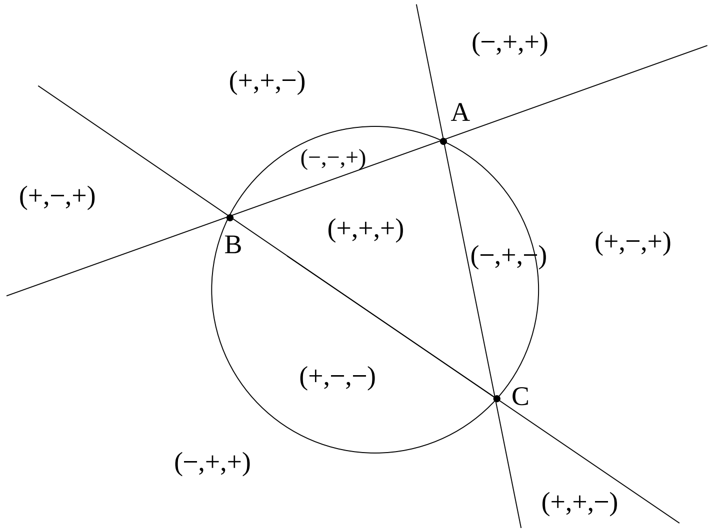

Figure 2

An analysis of this partitioning further implies that

$$\pm|\Delta_P| = \frac{\overline{d}_B\,\overline{d}_C \sin(\measuredangle A)}{2} + \frac{\overline{d}_A\,\overline{d}_B \sin(\measuredangle C)}{2} + \frac{\overline{d}_A\,\overline{d}_C \sin(\measuredangle B)}{2}, \tag{5}$$

where + corresponds to the case when P is contained in the circle of center O (denoted by (O)), and − corresponds to the case when P is outside (O). Making now use of (3) and the corresponding formulas for $\overline{d}_A$, $\overline{d}_B$, we can re-write (5) as

$$\begin{aligned}\pm|\Delta_P| \;=\; & \frac{\alpha_B x_1 + \beta_B y_1 + \gamma_B}{\sqrt{\alpha_B^2 + \beta_B^2}} \cdot \frac{\alpha_C x_1 + \beta_C y_1 + \gamma_C}{\sqrt{\alpha_C^2 + \beta_C^2}} \cdot \frac{\sin(\measuredangle A)}{2} \\ & + \frac{\alpha_A x_1 + \beta_A y_1 + \gamma_A}{\sqrt{\alpha_A^2 + \beta_A^2}} \cdot \frac{\alpha_C x_1 + \beta_C y_1 + \gamma_C}{\sqrt{\alpha_C^2 + \beta_C^2}} \cdot \frac{\sin(\measuredangle B)}{2} \\ & + \frac{\alpha_A x_1 + \beta_A y_1 + \gamma_A}{\sqrt{\alpha_A^2 + \beta_A^2}} \cdot \frac{\alpha_B x_1 + \beta_B y_1 + \gamma_B}{\sqrt{\alpha_B^2 + \beta_B^2}} \cdot \frac{\sin(\measuredangle C)}{2}. \end{aligned} \tag{6}$$

In addition, using the fact that $|\Delta|$ is a real constant that depends only on A, B and C, (6) yields

$$\begin{aligned}\pm\frac{|\Delta_P|}{|\Delta|} \;=\; & (ax_1 + by_1 + c)(dx_1 + ey_1 + f) \\ & + (gx_1 + hy_1 + i)(jx_1 + ky_1 + l) \\ & + (mx_1 + ny_1 + o)(px_1 + qy_1 + r), \end{aligned} \tag{7}$$

where $a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q$ and r are real constants that depend only on A, B and C.

At this point, we observe that (1) becomes

$$\pm\frac{|\Delta_P|}{|\Delta|} = \frac{R^2 - OP^2}{4R^2}, \tag{8}$$

provided we select $+$ when P is in (O) and $-$ when P is outside (O). Hence, if we now take into account (8) and (7), we obtain that (1) is equivalent with

$$\lambda_1 x_1^2 + \lambda_2 y_1^2 + \lambda_3 x_1 y_1 + \lambda_4 x_1 + \lambda_5 y_1 + \lambda_6 = 0, \tag{9}$$

where λ_1, λ_2, λ_3, λ_4, λ_5, and λ_6 are real constants that depend only on A, B, and C. Any points that satisfy (1) satisfy (9), and vice versa. We know that any quadratic equation in terms of x and y has as its graph a conic section. Since the graph of the quadratic equation is the locus of all points satisfying the equation, this means that the locus of points satisfying (9) has the shape of a conic. Thus, the shape of the locus of points satisfying (9) is a conic, meaning that the locus of points satisfying (1) is either a point, two intersecting lines, a parabola, a hyperbola, a circle, an ellipse, or the whole plane (if all the lambdas are zero). One can see that six points that satisfy (1) are as follows: the vertices A, B, and C, and the points diametrically opposed to the vertices, A'', B'', and C'', which all lie on the circumcircle of ΔABC. It is fairly easy to see that the point O also satisfies (1), as both sides of (1) will be $\frac{1}{4}$. Using these seven points, A, B, C, A', B', C' and O, one can eliminate all of the possible conics except for the whole plane. This means that for every P in the plane (1) holds.

3 The Area of an Antipedal Triangle

Theorem 1 provides us with an efficient formula to compute the area of a pedal triangle given the geometry of the reference triangle and the location of the pedal point. This is also useful for other purposes, such as computing the area of an *antipedal* triangle in terms of the geometry of the reference triangle and the location of the antipedal point. Before proceeding with the proof of Theorem 2, we prove a useful result on homotopic triangles.

Given a triangle $A_1A_2A_3$ along with a triangle $B_1B_2B_3$ inscribed in it, we describe a procedure for obtaining a triangle, $C_1C_2C_3$, that is inscribed in $\Delta B_1B_2B_3$ and is homotopic to $\Delta A_1A_2A_3$. Recall that two triangles are called homotopic if their sides are parallel.

Proposition 1 *Let $\Delta A_1A_2A_3$ be arbitrary and assume that $B_1 \in A_2A_3$, $B_3 \in A_2A_1$, $B_2 \in A_1A_3$ (see Figure 3 below). Take $C_1 \in B_2B_3$, $C_2 \in B_1B_3$, $C_3 \in B_1B_2$ such that*

$$\frac{A_2B_3}{B_3A_1} = \frac{B_2C_3}{C_3B_1}, \quad \frac{A_3B_1}{B_1A_2} = \frac{B_3C_1}{C_1B_2}, \quad \frac{A_1B_2}{B_2A_3} = \frac{B_1C_2}{C_2B_3}, \tag{10}$$

Then $\Delta A_1A_2A_3$ and $\Delta C_1C_2C_3$ are homotopic and, in addition, $|\Delta B_1B_2B_3|$ is the geometric mean of $|\Delta A_1A_2A_3|$ and $|\Delta C_1C_2C_3|$, i.e.

$$|\Delta B_1B_2B_3|^2 = |\Delta A_1A_2A_3| \cdot |\Delta C_1C_2C_3|. \tag{11}$$

Conversely, if $\Delta A_1A_2A_3$ and $B_1 \in A_2A_3$, $B_3 \in A_2A_1$, $B_2 \in A_1A_3$ are given and $C_1 \in B_2B_3$, $C_2 \in B_1B_3$, $C_3 \in B_1B_2$ are such that $\Delta A_1A_2A_3$ and $\Delta C_1C_2C_3$ are homotopic, then (10) and (11) hold.

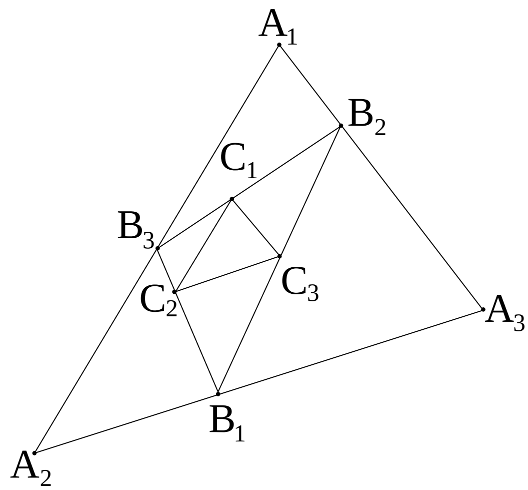

Figure 3

PROOF: Recall that an affine transformation of the plane into itself consists of a linear transformation followed by a translation. An affine transformation has the following properties: maps lines into lines, parallel lines into parallel lines, and preserves the ratio of line segments determined by points on a line.

Thus it suffices to prove Proposition 1 for the particular triangle $A_1A_2A_3$: $A_1 = (0,1), A_2 = (0,0), A_3 = (1,0)$, since any other triangle can be transformed via an affine transformation into this particular triangle while preserving the desired properties. In addition, let $B_1, B_2, B_3, C_1, C_2, C_3$ be as in Proposition 1. We set

$$k_1 := \frac{A_3B_1}{B_1A_2} = \frac{B_3C_1}{C_1B_2}, \quad k_2 := \frac{A_1B_2}{B_2A_3} = \frac{B_1C_2}{C_2B_3}, \quad k_3 := \frac{A_2B_3}{B_3A_1} = \frac{B_2C_3}{C_3B_1}. \tag{12}$$

Recall that if M, N, P are three collinear points, with coordinates $M(m_1, m_2)$, $P(p_1, p_2)$, and N between M and P, satisfying $\frac{MN}{NP} = k$, for some real, positive constant k, then N has coordinates

$$N = \Big(\frac{m_1 + kp_1}{1+k}, \frac{m_2 + kp_2}{1+k}\Big). \tag{13}$$

This fact, in combination with (12) yields

$$B_1 = \Big(\frac{1}{1+k_1}, 0\Big), \quad B_2 = \Big(\frac{k_2}{1+k_2}, \frac{1}{1+k_2}\Big), \quad B_3 = \Big(0, \frac{k_3}{1+k_3}\Big). \tag{14}$$

Furthermore,

$$C_1 = \Big(\frac{\frac{k_1k_2}{1+k_2}}{1+k_1}, \frac{\frac{k_3}{1+k_3} + \frac{k_1}{1+k_2}}{1+k_1}\Big), \quad C_2 = \Big(\frac{\frac{1}{1+k_1}}{1+k_2}, \frac{\frac{k_2k_3}{1+k_3}}{1+k_2}\Big),$$

$$C_3 = \Big(\frac{\frac{k_2}{1+k_2} + \frac{k_3}{1+k_1}}{1+k_3}, \frac{\frac{1}{1+k_2}}{1+k_3} \Big). \tag{15}$$

It is obvious that

$$|\Delta A_1 A_2 A_3| = \frac{1}{2}. \tag{16}$$

Next, using vector calculus, we will compute the areas of $\Delta B_1 B_2 B_3$ and $\Delta C_1 C_2 C_3$. More specifically,

$$\begin{aligned} |\Delta B_1 B_2 B_3| &= \tfrac{1}{2} \Big\| \overrightarrow{B_1 B_2} \times \overrightarrow{B_1 B_3} \Big\| \\ &= \frac{k_1 k_2 k_3 + 1}{2(1+k_1)(1+k_2)(1+k_3)}. \end{aligned} \tag{17}$$

A similar reasoning applies to $\Delta C_1 C_2 C_3$, namely

$$|\Delta C_1 C_2 C_3| = \tfrac{1}{2} \Big\| \overrightarrow{C_1 C_2} \times \overrightarrow{C_1 C_3} \Big\| = \frac{(k_1 k_2 k_3 + 1)^2}{2(1+k_1)^2 (1+k_2)^2 (1+k_3)^2}. \tag{18}$$

Identity (11) now follows by combining (16), (17), and (18), thus completing the proof of the first part of Proposition 1.

Finally, the converse statement (as recorded in the last part of the proposition) follows from the uniqueness of a triangle homotopic with $\triangle A_1 A_2 A_3$ and inscribed in $\triangle B_1 B_2 B_3$, plus what we have proved so far. The proof of the proposition is therefore complete. QED

After this preamble, we are ready to present the

Proof of Theorem 2. If $\Delta\, DEF$ is the pedal triangle of the point K_1 with respect to $\Delta\, ABC$, then

$$\measuredangle F K_1 A + \measuredangle K_1 A F = \frac{\pi}{2}. \tag{19}$$

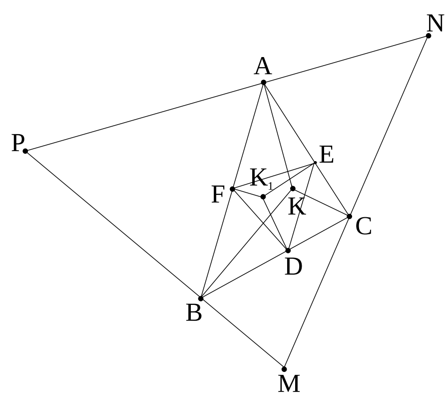

Figure 4

However, because K is the isogonal of K_1, this means that

$$\measuredangle FK_1A + \measuredangle KAE = \frac{\pi}{2}. \tag{20}$$

Keeping mind that the quadrilateral AFK_1E can be inscribed in a circle, (20) means that $AK \perp EF$, therefore $PN \| EF$. Similar reasoning can be done to show that $PM \| DF$ and $MN \| DE$. This implies that ΔMNP and ΔDEF are homotopic. From Proposition 1 we obtain

$$|\Delta DEF| \cdot |\Delta MNP| = |\Delta ABC|^2, \tag{21}$$

Theorem 1 implies

$$\frac{|\Delta DEF|}{|\Delta ABC|} = \frac{|R^2 - OK_1{}^2|}{4R^2}. \tag{22}$$

Therefore, $\frac{|\Delta MNP|}{|\Delta ABC|} = \frac{|\Delta ABC|}{|\Delta DEF|} = \frac{4R^2}{|R^2 - OK_1{}^2|}$, as claimed.

References

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