The cone of Hilbert Nullforms

Vladimir L. Popov

Steklov Mathematical Institute
Russian Academy of Sciences
Gubkina 8, Moscow, 119991, Russia

`popovvl@mi.ras.ru`

Abstract

We describe a geometric–combinatorial algorithm that allows one, using solely the system of weights and roots, to determine the Hesselink strata of the null-cone of a linear representation of a reductive algebraic group and calculate their dimensions. In particular, it provides a constructive approach to calculating the dimension of the null-cone and determining all its irreducible components of maximal dimension. In the case of the adjoint representation (and, more generally, a θ-representation), the algorithm turns into the classification algorithm for the conjugacy classes of nilpotent elements in a semisimple Lie algebra (respectively, homogeneous nilpotent elements in a cyclically graded semisimple Lie algebra).

1. Introduction

Below all algebraic varieties are taken over an algebraically closed field k of characteristic zero.

Let G be a reductive algebraic group and let V be a finite-dimensional algebraic G-module. A vector $v \in V$ is called a *Hilbert nullform* (cf. [He$_2$]; other terms are *unstable* vector, [Mu], *nilpotent* vector, [PV]), if any G-invariant polynomial function f on V that vanishes at 0 also vanishes at v. The closed subscheme $\mathcal{N}_{G,V}$ of V defined by vanishing of all such f is called the *null-cone of V*. Hilbert was the first who found that the null-cones play a fundamental role in the theory of invariants and its applications, see [Hi]. In the general case, the null-cone has a complicated structure; it may be irreducible, reducible, or nonequidimensional, and may have complex singularities.

According to the Hilbert–Mumford criterion, [Hi], [Mu], a vector $v \in V$ is a nullform if and only if there is a one-parameter subgroup $\lambda : k^\times \to G$ that steers v to 0 (i.e., such that $\lim_{t \to 0} \lambda(t) \cdot v = 0$). In 1978, when solving different problems, several authors simultaneously revealed the key role of those λ that do it "most optimally", see [Bo], [Ke], [He$_1$], [He$_2$], [Ro] (however, for $V = \mathrm{Lie}(G)$, where the null-cone coincides with the cone of nilpotent elements, this fact was revealed in a different form already in [Dy]: indeed, it

1

was found out later, see [He$_2$], [He$_1$], [Sl], that the "characteristics" of nilpotent elements introduced in [Dy] can be equivalently defined in terms of the optimality property). The formalization of this property is performed by introducing an Ad(G)-invariant norm on the set of all one-parameter subgroups: optimal subgroups are those that have the minimal norm. Technically, it is more convenient to consider a wider class of virtual one-parameter subgroups, [He$_1$], [He$_2$], which is actually the set Lie(G)($\mathbb{Q}$) of rational semisimple elements in the Lie algebra Lie(G). The norm is defined by the choice of an Ad(G)-invariant inner product $\langle \, , \, \rangle$ on Lie(G) that is rational-valued and positive definite on the $\mathbb{Q}$-vector space Lie(T)($\mathbb{Q}$) for a certain (and thereby for any) maximal torus T in G. If $\Lambda(v)$ is a family of all optimal virtual one-parameter subgroups corresponding to a nullform v, then, following [He$_2$], we can consider the following equivalence relations on the set of all nullforms:

$$x \sim y \iff \Lambda(g \cdot x) = \Lambda(y) \ \text{ for a certain } \ g \in G.$$

The equivalence classes form a finite partition of the null-cone $\mathcal{N}_{G,V}$ and are called *strata*. Note that, in general, the strata depend on the choice of the norm. It is remarkable that, although the null-cone $\mathcal{N}_{G,V}$ itself has a complicated structure, the configuration of the strata turns out to be simple: Hesselink [He$_2$] proved that each stratum S is an irreducible subvariety of the null-cone $\mathcal{N}_{G,V}$, open in its closure $\overline{S}$, and there exists an isomorphism φ of the stratum S onto an invariant open subset of a homogeneous vector bundle E over G/P, where P is a parabolic subgroup in G (all these objects depend on S); in particular, S is a smooth rational variety. Moreover, there exists a morphism $\pi : E \to \overline{S}$, which is a resolution of singularities of the variety $\overline{S}$ and is such that $\pi|_{\pi^{-1}(S)} = \varphi^{-1}$. In particular, this gives a resolution of singularities for each irreducible component of the null-cone $\mathcal{N}_{G,V}$ and shows that, using Kempf's terminology, [Ke], the latter is obtained by collapsing of a homogeneous vector bundle over a generalized flag variety of the group G. Note that, in general, the closure of a stratum is not a union of strata.

Developing the idea outlined in [Po$_3$, (2.2.22)] we consider in this paper the problem of constructive description of strata. This problem has been already discussed in [He$_2$, Section 5], but has not been solved there: concerning the directly related problem of finding out if a given saturated set is the closure of a blade (see [He$_2$] for the meaning of these terms; we will not use them any longer), Hesselink writes that it "is more difficult" and restricts the analysis to a certain necessary condition. In the present paper we show that the family of all strata of the null-cone $\mathcal{N}_{G,V}$ is completely determined only by the geometric configuration of two finite subsets of the Euclidean space Lie(T)($\mathbb{Q}$), namely, the system of weights of the G-module V (endowed with their multiplicities) and the system of roots of the group G, and we obtain a simple geometric-combinatorial algorithm for describing this family; this algorithm employs only elementary geometric operations, namely, the orthogonal projection of a finite system of points onto a linear space and taking its convex hull. Technically, the procedure is reduced to constructing certain oriented rooted trees by means of these operations and assigning plus or minus signs to the vertices of these trees

according to a certain rule. The strata are defined precisely by the trees to whose roots is assigned a plus sign.

More precisely, this algorithm allows one to describe constructively the following objects:

(a) a certain finite subset $\mathfrak{S}$ in $\mathrm{Lie}(T)(\mathbb{Q})$;

(b) a certain linear subspace $V[l^+]$ in V for every element $l \in \mathfrak{S}$; and

(c) a certain open subset $V[l^+]^0$ in $V[l^+]$,

such that the sets $G \cdot V[l^+]^0$, where l runs over $\mathfrak{S}$, exhaust, without repetition, the family of all strata of the null-cone $\mathcal{N}_{G,V}$.

In this case, the dimension of the stratum $G \cdot V[l^+]^0$ turns out to be equal to the sum of the number of roots lying the half-space $\{v \in \mathrm{Lie}(T)(\mathbb{Q}) \mid \langle l, v \rangle < 0\}$ and the number of weights (taken with their multiplicities) lying the half-space $\{v \in \mathrm{Lie}(T)(\mathbb{Q}) \mid \langle l, v \rangle \geqslant 1\}$. This yields a simple constructive approach to calculating the dimension of the null-cone $\mathcal{N}_{G,V}$: it is equal to the maximum of the above sums taken over all l from $\mathfrak{S}$. Calculating $\dim \mathcal{N}_{G,V}$ is a nontrivial problem; for example, the problem of classifying equidimensional representations and that of classifying representations with a free module of covariants, [Po$_1$], [Ad], [Sc$_1$], [Li$_1$], are reduced to the above problem; their solutions are actually based on developing various ad hoc methods for estimating $\dim \mathcal{N}_{G,V}$. Until now, general information on $\dim \mathcal{N}_{G,V}$ has been restricted to various a priori bounds of this number based on the Hilbert–Mumford criterion, see [Po$_1$], [Sc$_1$], [Sc$_2$], [Ad], [Ka$_2$].

Moreover, this also yields a simple and constructive method for describing all irreducible components of maximal dimension of the null-cone $\mathcal{N}_{G,V}$: they are exhausted, without repetition, by varieties $G \cdot V[l^+]$, where l is an element for which the above maximum is attained.

The algorithm can be applied for composing a computer program for the constructive description of all strata, calculating their dimensions, calculating the dimension of the null-cone, and describing all its irreducible components of maximal dimension. The operating rate of such a program is determined by its part related to sorting out all the subsets of the set of weights. For $G = T$, the system of weights may be any finite set of characters, so such looking over is inevitable. But if G is noncommutative, the Weyl group provides nontrivial symmetries of the system of weights, which can be used for improving the enumerative part of the program.[1]

In the case of the adjoint representation, and, more generally, of a so-called θ-representation (see [Vi$_1$], [Ka$_2$], as well as [PV, 8.5] and [Li$_1$]) our algorithm turns into the algorithm of classifying conjugacy classes of nilpotent elements in a semisimple Lie algebra and, respectively, of homogeneous nilpotent elements in a cyclically graded semisimple Lie algebra

---

[1] *Added in September* 2010: Jointly with N. A'Campo a computer implementation of this algorithm has been composed in 2005. Pro tempore the code can be obtained on request sent to the address norbertacampo@gmail.com. But we hope to compose also the relevant comments for publishing and posting both on the web.

(see below Proposition 5). Although these classifications, obtained in [Dy] and [Vi₁], [Vi₂] by a different method, are known, it would be interesting to derive (and check) them again in nonserial cases by a computer program based on our algorithm. This, in particular, applies to the classification of nullforms in the spaces of 3-vectors of a 9-dimensional space, [VE], 4-vectors of an 8-dimensional space, [An], spinors of 14- and 16-dimensional spaces, [Po₂], [AE], and to certain other classifications. In fact, each of the cited results can be considered as the solution, in the case under consideration, of the classification problem for the strata of the null-cone. Let us mention another important case when the classification problem has also been solved, [LB]: it is the case of the diagonal action of the group GL_n by conjugation on the space of m-tuples of $n \times n$-matrices.

The final section contains examples of application of the algorithm. We discuss the cases of rank $\leqslant 2$, ternary forms, and nilpotent elements in semisimple Lie algebras (the case of a special simple Lie algebra of type G_2 is considered in detail) and in spaces of θ-representations. In this section we also analyze an example of a certain action of GL_2 which shows that, in general, strata depend on the choice of the norm and may not coincide with orbits if there are only finitely many orbits in the null-cone.

2. Strata

Since the nullforms for the actions of the group G and its identity component in V are the same, we will assume that G is a *connected* group. The space V is assumed to be *nonzero*.

2.1. First, we shall introduce certain notation and conventions.

Let S be an algebraic torus. The group $\mathrm{Hom}(S, k^\times)$ of its rational characters is a free abelian group of finite rank. We consider it in additive notation and denote by s^μ the value of the character $\mu \in \mathrm{Hom}(S, k^\times)$ at point $s \in S$.

Let $\mathfrak{s} := \mathrm{Lie}(S)$. The map $\mathrm{Hom}(S, k^\times) \to \mathfrak{s}^*$, $\mu \mapsto d_e\mu$, is an embedding. We identify $\mathrm{Hom}(S, k^\times)$ with the image of this embedding. Then, $\mathrm{Hom}(S, k^\times)$ is a lattice of full rank in $\mathfrak{s}^*$, so that its linear span over $\mathbb{Q}$ is a $\mathbb{Q}$-form of the vector space $\mathfrak{s}^*$. Let $\mathfrak{s}(\mathbb{Q})$ be the $\mathbb{Q}$-form of the Lie algebra $\mathfrak{s}$ defined by this span:

$$\mathfrak{s}(\mathbb{Q}) := \{s \in \mathfrak{s} \mid l(s) \in \mathbb{Q} \ \forall \ l \in \mathrm{Hom}(S, k^\times)\}. \tag{1}$$

For any algebraic S-module L and character $\mu \in \mathrm{Hom}(S, k^\times)$ we set

$$L_\mu := \{l \in L \mid s \cdot l = s^\mu l \ \forall \ s \in S\}$$

and denote by $\Delta_{L,S}$ the system of weights of S in L,

$$\Delta_{L,S} := \{\mu \in \mathrm{Hom}(S, k^\times) \mid L_\mu \neq 0\}.$$

Then, $L = \bigoplus_{\mu \in \Delta_{L,S}} L_\mu$. The multiplicity of the weight μ in the S-module L is the number $\dim L_\mu$.

4

2.2. Let T be a maximal torus in G and $\mathfrak{t} := \mathrm{Lie}(T)$. Fix an $\mathrm{Ad}(G)$-invariant inner product $\langle \ , \ \rangle$ on $\mathfrak{g} := \mathrm{Lie}(G)$ whose restriction on $\mathfrak{t}(\mathbb{Q})$ is rational-valued and positive definite. The Weyl group $W_{G,T} := \mathrm{Nor}_G(T)/T$, considered as a group of transformations of the space $\mathfrak{t}$, carries $\mathfrak{t}(\mathbb{Q})$ into itself, and the restriction $\langle \ , \ \rangle|_{\mathfrak{t}(\mathbb{Q})}$ is $W_{G,T}$-invariant. This restriction uniquely defines the inner product $\langle \ , \ \rangle$. Any rational-valued positive definite $W_{G,T}$-invariant inner product on $\mathfrak{t}(\mathbb{Q})$ is obtained as such a restriction. In what follows, we assume that the strata of the null-cone $\mathcal{N}_{G,V}$ are defined by the norm on $\mathrm{Lie}(G)(\mathbb{Q})$, which is defined by the inner product $\langle \ , \ \rangle$.

We identify $\mathfrak{t}$ with $\mathfrak{t}^*$ by means of the map $\mathfrak{t} \to \mathfrak{t}^*$, $t \mapsto \langle t, \cdot \rangle$. Then $\mathrm{Hom}(T, k^\times)$ becomes a lattice of full rank in $\mathfrak{t}(\mathbb{Q})$. Henceforth, we consider $\mathfrak{t}(\mathbb{Q})$ as a Euclidean linear space over $\mathbb{Q}$ with respect to the inner product $\langle \ , \ \rangle|_{\mathfrak{t}(\mathbb{Q})}$.

2.3. We have the following decomposition:

$$V = \bigoplus_{\mu \in \Delta_{V,T}} V_\mu. \tag{2}$$

Since maximal tori are conjugate and their union is dense in G, the action of G on V is trivial if and only if $\Delta_{V,T} = \{0\}$.

If V is the adjoint G-module $\mathfrak{g}$, then $\Delta_{\mathfrak{g},T} = \Phi_{\mathfrak{g},T} \sqcup \{0\}$, where $\Phi_{\mathfrak{g},T}$ is the system of roots of the Lie algebra $\mathfrak{g}$ with respect to the torus T. We have $\dim \mathfrak{g}_\alpha = 1 \ \forall \ \alpha \in \Phi_{\mathfrak{g},T}$, $\mathfrak{g}_0 = \mathfrak{t}$ and

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_{\mathfrak{g},T}} \mathfrak{g}_\alpha. \tag{3}$$

2.4. Let $l \in \mathfrak{t}(\mathbb{Q})$ be a nonzero element. For any number $\gamma \in \mathbb{Q}$ we set

$$\{l = \gamma\} := \{t \in \mathfrak{t}(\mathbb{Q}) \mid \langle l, t \rangle = \gamma\}.$$

$\{l \geqslant \gamma\}$, etc. are defined similarly.

The element l determines in $\mathfrak{g}$ the following algebraic subalgebras:

$$\mathfrak{t}[l] := \{t \in \mathfrak{t} \mid (l,t) = 0\}, \quad \mathfrak{g}[l] := \mathfrak{t}[l] \oplus \bigoplus_{\alpha \in \{l=0\}} \mathfrak{g}_\alpha, \quad \mathfrak{p}[l] := \mathfrak{t} \oplus \bigoplus_{\alpha \in \{l \geqslant 0\}} \mathfrak{g}_\alpha. \tag{4}$$

Let $T[l]$, $G[l]$ and $P[l]$ be connected algebraic subgroups of the group G with the Lie algebras $\mathfrak{t}[l]$, $\mathfrak{g}[l]$ and $\mathfrak{p}[l]$, respectively. $G[l]$ is a reductive group, $T[l]$ is its maximal torus, and $P[l]$ is a parabolic subgroup in G. It follows from (1) and (4) that

$$\mathfrak{t}[l](\mathbb{Q}) = \{l = 0\}, \quad \Phi_{\mathfrak{g}[l],T[l]} = \Phi_{\mathfrak{g},T} \cap \mathfrak{t}[l](\mathbb{Q}), \quad \mathfrak{g}[l]_\alpha = \mathfrak{g}_\alpha \ \forall \ \alpha \in \Phi_{\mathfrak{g}[l],T[l]}. \tag{5}$$

The element l also determines the following linear subspaces in V:

$$V[l] := \bigoplus_{\mu \in \{l=1\}} V_\mu, \quad V[l^+] := \bigoplus_{\mu \in \{l \geqslant 1\}} V_\mu, \tag{6}$$

5

Relations (4) and (6) imply the $G[l]$-invariance of $V[l]$ and the $P[l]$-invariance of $V[l^+]$.

2.5. Let

$$V[l^+] \to V[l], \ v \mapsto v',$$

be the projection parallel to $\bigoplus_{\mu \in \{l > 1\}} V_\mu$. Consider in $V[l]$ the null-cone $\mathcal{N}_{G[l],V[l]}$ and in $V[l^+]$ the open $P[l]$-invariant subset

$$V[l^+]^0 := \left\{ v \in V[l^+] \mid v' \notin \mathcal{N}_{G[l],V[l]} \right\} \tag{7}$$

In general, the set $V[l^+]^0$ may be empty. By (7), its noneptiness is equivalent to the inequality

$$\mathcal{N}_{G[l],V[l]} \neq V[l]. \tag{8}$$

Definition. A nonzero element $l \in \mathfrak{t}(\mathbb{Q})$ is called a *stratifying* element if condition (8) holds.

2.6. Suppose that l is a stratifying element from $\mathfrak{t}(\mathbb{Q})$. Then, the variety $V[l^+]^0$ is nonempty. Let $G \times^{P[l]} V[l^+]^0$ be the space of the homogeneous bundle over $G/P[l]$ with the fiber $V[l^+]^0$. Consider the morphism

$$G \times^{P[l]} V[l^+]^0 \longrightarrow V, \ \ [g, v] \mapsto g \cdot v, \tag{9}$$

where $[g, v]$ is the image of the point $g \times v$ under the quotient morphism $G \times V[l^+]^0 \to G \times^{P[l]} V[l^+]^0$. Let $\mathcal{H}[l]$ be the image of morphism (9),

$$\mathcal{H}[l] := G \cdot V[l^+]^0. \tag{10}$$

Theorem 1.
  (1) *If $l \in \mathfrak{t}(\mathbb{Q})$ is a stratifying element, then*
    (a) *$\mathcal{H}[l]$ is a stratum of the null-cone $\mathcal{N}_{G,V}$ that is different from $\{0\}$;*
    (b) *morphism (9) isomorphically maps $G \times^{P[l]} V[l^+]^0$ onto $\mathcal{H}[l]$;*
    (c) *$G \cdot V[l^+]$ is the closure of $\mathcal{H}[l]$.*

  (2) *Any stratum of the null-cone $\mathcal{N}_{G,V}$ that is different from $\{0\}$ has the form $\mathcal{H}[l]$ for a certain stratifying element l.*
  (3) *$\mathcal{H}[l_1] = \mathcal{H}[l_2]$ if and only if l_1 and l_2 lie on the same $W_{G,T}$-orbit.*

Proof. See [PV, Theorem 5.6] (see also [Sl, Prop. 1], [Ne, Proof of Theorem 9.2], [Ki, Remark 12.21]). $\qquad\square$

Since $\dim G \times^{P[l]} V[l^+]^0 = \dim G/P[l] + \dim V[l^+]^0$, formulas (2), (3), (4), (6), (7), claim (1)(b) of Theorem 1 and the one-dimensionality of the root subspaces $\mathfrak{g}_\alpha$ imply the following corollary.

Corollary 1 (formula for the dimension of a stratum)**.**

$$\dim \mathcal{H}[l] = \#\{l < 0\} \cap \Phi_{\mathfrak{g},T} + \sum_{\mu \in \{l \geqslant 1\}} \dim V_\mu. \tag{11}$$

In turn, since the strata are open in their closures and the number of strata is finite, we obtain the following corollary from (2) and (11).

Corollary 2 (numerical criterion for the openness of a stratum in V)**.** *The following properties of a stratifying element* $l \in \mathfrak{t}(\mathbb{Q})$ *are equivalent:*

    (a) *the stratum* $\mathcal{H}[l]$ *is open in* V,

    (b) $\dim \mathcal{H}[l] = \dim V$,

    (c) $\#\{l < 0\} \cap \Phi_{\mathfrak{g},T} = \sum_{\mu \in \{l < 1\}} \dim V_\mu$.

A stratifying element l *with these properties exists if and only if* $\mathcal{N}_{G,V} = V$.

Finally, the inclusion $\mathcal{H}(l) \subseteq V$ yields the following corollary.

Corollary 3. *If an element* l *of* $\mathfrak{t}(\mathbb{Q})$ *is a stratifying element, then*

$$\#\{l < 0\} \cap \Phi_{\mathfrak{g},T} \leqslant \sum_{\mu \in \{l < 1\}} \dim V_\mu. \tag{12}$$

2.7. In the next section we will show that the family of all stratifying elements in $\mathfrak{t}(\mathbb{Q})$ is completely determined only by the geometric configuration of the system of weights $\Delta_{V,T}$ (endowed with their multiplicities) and the system of roots $\Phi_{\mathfrak{g},T}$ in the Euclidean space $\mathfrak{t}(\mathbb{Q})$, and obtain a simple geometric–combinatorial algorithm for describing this family.

Namely, this algorithm allows one to describe constructively a finite set $\mathfrak{S}$ of nonzero elements $l \in \mathfrak{t}(\mathbb{Q})$ such that $V[l^+] \neq \varnothing$ for every $l \in \mathfrak{S}$ and that the sets $G \cdot V[l^+]^0$, where l runs over $\mathfrak{S}$, exhaust, without repetition, the family of all strata in $\mathcal{N}_{G,V}$ that are different from $\{0\}$. By Theorem 1 this yields the following description for the null-cone:

$$\mathcal{N}_{G,V} = \bigcup_{l \in \mathfrak{S}} G \cdot V[l^+]. \tag{13}$$

This description is constructive in the sense that our algorithm constructively describes all linear subspaces $V[l^+]$ from (13). Since the application of this algorithm to the $G[l]$-module $V[l]$ for $l \in \mathfrak{S}$ yields a constructive description, analogous to (13), of the null-cone $\mathcal{N}_{G[l],V[l]}$, we obtain from (7) a constructive description for every stratum $G \cdot V[l^+]^0$.

In view of Corollary 1 to Theorem 1, the constructive description of the set $\mathfrak{S}$ also yields the following constructive method for calculating the dimension of the null-cone $\mathcal{N}_{G,V}$ by means of calculating the number of roots and weights (with multiplicities) in appropriate half-spaces of the space $\mathfrak{t}(\mathbb{Q})$:

Proposition 1.

$$\dim \mathcal{N}_{G,V} = \max_{l \in \mathfrak{S}}\big\{ \#\{l < 0\} \cap \Phi_{\mathfrak{g},T} \ + \sum_{\mu \in \{l \geqslant 1\}} \dim V_\mu \big\}. \tag{14}$$

In addition, Corollary 1 to Theorem 1 yields a constructive method for describing all irreducible components of maximal dimension of the null-cone $\mathcal{N}_{G,V}$: these components are exhausted, without repetition, by the varieties $G \cdot V[l^+]$, where $l \in \mathfrak{S}$ is an element for which the sum $\#\{l < 0\} \cap \Phi_{\mathfrak{g},T} \ + \sum_{\mu \in \{l \geqslant 1\}} \dim V_\mu$ attains its maximum.

2.8. In some important cases, it is a priori known that the strata are orbits. For example, this is so for adjoint representations and, more generally, for θ-representations (see below Proposition 4). In these cases, the number of orbits in the null-cone $\mathcal{N}_{G,V}$ is finite, and our algorithm turns into the classification algorithm for these orbits. The following proposition can be used (see below Example 5) for describing the representatives of such orbits.

Proposition 2. *Consider a T-weight basis $\{v_i\}_{i \in I}$ in V and a set $\{X_\alpha \in \mathfrak{g}_\alpha \mid X_\alpha \neq 0, \alpha \in \Phi_{\mathfrak{g},T}\}$ such that, for any i and α, the coordinates of the vector $X_\alpha \cdot v_i$ in the basis $\{v_i\}_{i \in I}$ lie in $\mathbb{Q}$. Let l be a stratifying element from $\mathfrak{t}(\mathbb{Q})$ and $J := \{i \in I \mid v_i \in V[l]\}$. Then, for any family of constants $\{c_j \in k \mid j \in J\}$ that are algebraically independent over $\mathbb{Q}$, the vector $v := \sum_{j \in J} c_j v_j$ lies in the stratum $\mathcal{H}_{G,V}$.*

Proof. By construction, $v \in V[l]$. By (7) and Theorem 1 the proof boils down to checking that $v \notin \mathcal{N}_{G[l],V[l]}$.

The family $\{v_j\}_{j \in J}$ is a basis in $V[l]$. Let $\{x_j\}_{j \in J}$ be the dual basis in $V[l]^*$. The monomials of degree d in $\{x_j\}_{j \in J}$ form the basis of the space $k[V[l]]_d$ of homogeneous polynomials of degree d on $V[l]$. For $\alpha \in \{l = 0\}$, the element X_α acts linearly on $k[V[l]]_d$, and it follows from the hypothesis that its matrix has rational coefficients in this basis. Since the algebra of $G[l]$-invariants on $V[l]$ coincides with $\{f \in k[V[l]] \mid \mathrm{Lie}(G[l]) \cdot f = 0\}$, this implies that this algebra is generated by certain homogeneous polynomials in $\{x_j\}_{j \in J}$ with coefficients in $\mathbb{Q}$. By (8), there is a nonconstant polynomial f among them. It follows from the algebraic independence condition that $f(v) \neq 0$. Hence, $v \notin \mathcal{N}_{G[l],V[l]}$. $\qquad\square$

Note that the basis and the set mentioned in Proposition 2 always exist because there exists a $\mathbb{Z}$-form of $\mathfrak{g}$ defined by the Chevalley basis and an admissible $\mathbb{Z}$-form of V (see, for example, [Hu]). For the adjoint representation, these basis and set are given by the $\mathbb{Z}$-form of the algebra $\mathfrak{g}$.

2.9. If the number of orbits in a null-cone is finite, then the strata do not necessarily coincide with orbits (see Example 3 in Section 4). It would be interesting to find out whether it is always possible turn strata into orbits in such cases by changing the inner product $\langle \ , \ \rangle$. If this is so, then our algorithm yields a classification algorithm for the nullforms of any visible representation [Ka$_1$], [Ka$_2$].

3. Algorithm

First, we present a number of additional notations and definitions concerning geometry and certain special graphs.

3.1. Let E be a finite-dimensional linear Euclidean space over $\mathbb{Q}$ and let M be its nonempty subset. Introduce the following notation (Fig. 1):

- aff M is the affine hull of the subset M in E (i.e., the minimal linear variety in E that contains M);
- conv M is the convex hull of M in E,
- perp M is the perpendicular dropped from zero onto aff M (i.e., the unique vector from aff M that is orthogonal to the difference of any two vectors from aff M).
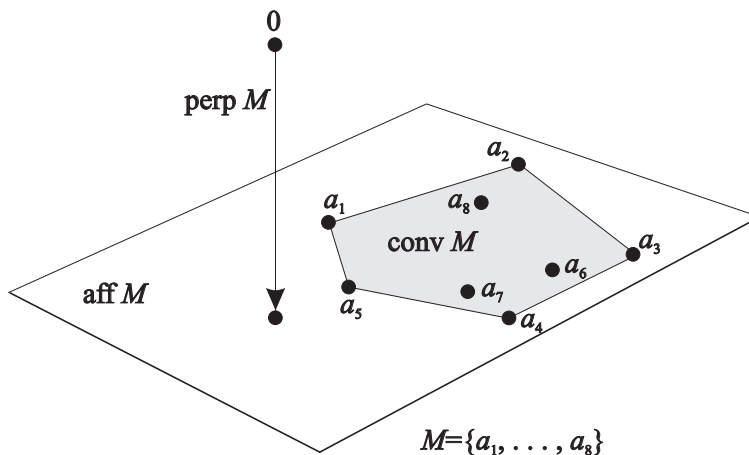

Fig. 1

3.2. Consider a rooted tree Γ (i.e., a connected graph without cycles and with a selected vertex called a root). Let $v_0, \ldots, v_d$ be the set of its vertices and let v_0 be its root. We endow each edge with an orientation (by specifying its beginning and end) directed "from" v_0, i.e., so that any vertex different from the root is the end of precisely one edge (this requirement uniquely defines the orientation of edges). Figure 2 on the next page shows precisely such an orientation.

For any two different vertices v_i and v_j, there exists at most one route with the beginning at v_i and the end at v_j, that passes along edges. If such a route exists, we call it a *geodesic* connecting v_i with v_j. The number of edges in a geodesic is called its *length*. For instance, the vertices v_1 and v_{20} in Fig. 2 are not connected, but the vertices v_1 and v_{21} are connected by the geodesic, and its length is 4. The root v_0 is connected by a geodesic with any vertex.

A vertex is called the *end* vertex if it is not the beginning of any edge. For instance, the following vertices are end vertices in Fig. 2: v_2, v_4, v_6, v_8, v_{10}, v_{11}, v_{13}, v_{14}, v_{15}, v_{17}, v_{18}, v_{20} v_{21}.

Each vertex is connected by a geodesic with at least one end vertex. The *height* of a vertex v_i is the greatest length of geodesics that connect v_i with end vertices. For instance,

in Fig. 2 the root v_0 has the height 5, the vertex v_3 has the height 3, and the vertex v_{11} has the height 0. The end vertices are exactly the vertices of height 0.
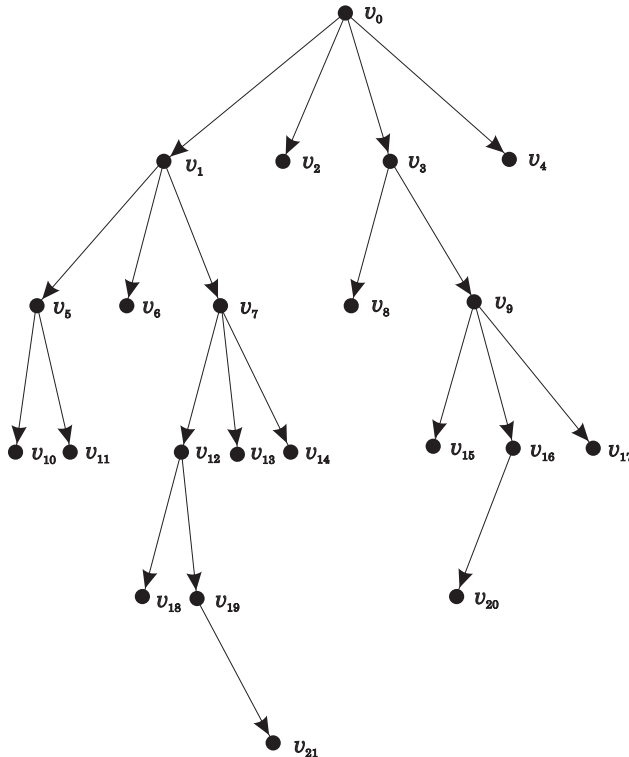

Fig. 2

If, moving along a geodesic γ, we write out all successively encountered vertices, then the sequence of vertices obtained is called a *geodesic sequence* corresponding to γ. For instance, if γ in Fig. 2 is a geodesic connecting the root v_0 with the vertex v_{21}, then γ corresponds to the the geodesic sequence of vertices v_0, v_1, v_7, v_{12}, v_{19}, v_{21}. Maximal geodesic sequences of vertices (i.e., those that cannot be embedded into geodesic sequences of greater length) correspond to γ such that start at the root and end at an end vertex. For example, v_0, v_3, v_8 in Fig. 2 is a maximal geodesic sequence of vertices.

3.3. Passing to the construction of the algorithm, consider a nonzero element $l \in \mathfrak{t}(\mathbb{Q})$. Suppose that the set of weights

$$\Delta_{V,T}[l] := \Delta_{V,T} \cap \{l = 1\} \tag{15}$$

that lies on the hyperplane $\{l = 1\}$ is nonempty. By (6), this is equivalent to the condition $V[l] \neq \{0\}$.

Consider an orthogonal projection

$$\mathrm{pr}_l : \mathfrak{t}(\mathbb{Q}) \longrightarrow \mathfrak{t}[l](\mathbb{Q}).$$

10

By (5) and (6), the system of weights $\Delta_{V[l],T[l]}$ of the torus $T[l]$ in $V[l]$ has the form

$$\Delta_{V[l],T[l]} = \mathrm{pr}_l(\Delta_{V,T}[l]) \tag{16}$$

and equality (16) holds with the mutiplicities of the weights taken into account if we assume that the latter are preserved under the projection pr_l.

Proposition 3. *Let $M := \Delta_{V,T}[l]$. If l is a stratifying element, then* (*see Fig.* 3)

$$\mathrm{perp}\, M \in \mathrm{conv}\, M, \tag{17}$$

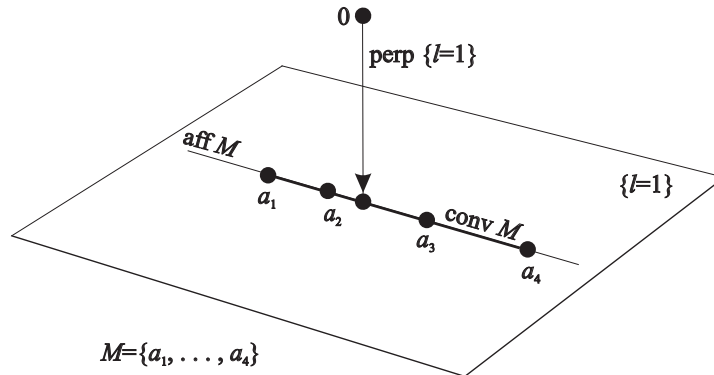$$\mathrm{perp}\, M = \mathrm{perp}\, \{l = 1\}. \tag{18}$$

Fig. 3

Proof. If l is a stratifying element, then

$$0 \in \mathrm{conv}\, \mathrm{pr}_l M. \tag{19}$$

Indeed, if inclusion (19) did not hold, then, by (16), each $T[l]$-orbit in $V[l]$ would contain zero in its closure (see, for example, [PV, 5.4]), and, hence, $\mathcal{N}_{G[l],V[l]} = V[l]$ in spite of the fact that l is a stratifying element.

Since the projection pr_l commutes with taking convex hull and the only vector of the hyperplane $\{l = 1\}$ that is mapped by pr_l to zero is $\mathrm{perp}\, \{l = 1\}$, inclusion (19) is equivalent to the inclusion

$$\mathrm{perp}\, \{l = 1\} \in \mathrm{conv}\, M. \tag{20}$$

Since $\mathrm{aff}\, M \subseteq \{l = 1\}$, the vector $\mathrm{perp}\, \{l = 1\}$ is orthogonal to the difference of any two vectors from $\mathrm{aff}\, M$. By the inclusion $\mathrm{conv}\, M \subset \mathrm{aff}\, M$ and the uniqueness of the perpendicular dropped from zero onto a linear variety, inclusion (20) implies (18). In turn, (18) and (20) imply (17). $\square$

3.4. We now introduce certain finite subsets in $\mathfrak{t}(\mathbb{Q})$ determined by the G-module V. Namely, consider the orbits of the Weyl group $W_{G,T}$ on the set of all nonempty subsets of the system of weights $\Delta_{V,T}$ and fix a representative in each subset. Among these representatives, we choose those representatives M, that share the following properties:

(i) $\operatorname{perp} M \neq 0$,

(ii) $\operatorname{perp} M \in \operatorname{conv} M$,

(iii) M is the intersection of the system of weights $\Delta_{V,T}$ with the hyperplane $\mathfrak{t}(\mathbb{Q})$ passing through the end of the vector $\operatorname{perp} M$ and orthogonal to this vector, i.e.,

$$M = \Delta_{V,T} \cap \{l_M = 1\}, \quad \text{where} \;\; l_M := \frac{\operatorname{perp} M}{\|\operatorname{perp} M\|^2},$$

(iv) for $l = l_M$, inequality (12) holds.

When M runs over the whole family of representatives possessing properties (i)–(iv), the vector l_M runs over a certain finite subset $\mathfrak{L}_{G,T,V}$ in $\mathfrak{t}(\mathbb{Q})$.

If the G-module V is trivial (i.e., $\Delta_{V,T} = \{0\}$), then $\mathfrak{L}_{G,T,V} = \varnothing$ by (i). In particular, this is so if G is the identity group $\{e\}$. If the G-module V is nontrivial, then there exist nonzero nullforms in V (for example, vectors from V_μ for nonzero $\mu \in \Delta_{V,T}$)); hence, there are strata in $\mathcal{N}_{G,V}$ that are different from $\{0\}$. According to the previous discussion, these strata are exhausted, without repetition, by varieties of the form $\mathcal{H}[l]$, where l is a stratifying vector from $\mathfrak{L}_{G,T,V}$. In particular, $\mathfrak{L}_{G,T,V} \neq \varnothing$ in this case.

Thus, the algorithm for describing all strata of the null-cone $\mathcal{N}_{G,V}$ of any nontrivial G-module V will be obtained if we find an algorithm for verifying whether or not $l \in \mathfrak{L}_{G,T,V}$ is a stratifying element. That is what we shall now do.

3.5. To this end, we consider the following subset in $\mathfrak{L}_{G,T,V}$:

$$\mathfrak{M}_{G,T,V} := \left\{ l \in \mathfrak{L}_{G,T,V} \; \Big| \; \#\{l < 0\} \cap \Phi_{\mathfrak{g},T} \; = \sum_{\mu \in \{l < 1\}} \dim V_\mu \right\}. \tag{21}$$

This subset "governs" the coincidence of $\mathcal{N}_{G,V}$ with V. Namely, if the G-module V is nontrivial, then (21) and Corollary 2 of Theorem 1 imply that the existence of a stratifying vector l in $\mathfrak{M}_{G,T,V}$ is equivalent to the equality $\mathcal{N}_{G,V} = V$, and that, for such l, the stratum $\mathcal{H}[l]$ is open in V. Since different elements from $\mathfrak{M}_{G,T,V}$ lie in different $W_{G,T}$-orbits, the openness of $\mathcal{H}[l]$ in V and claim (3) of Theorem 1 imply that *only one* such an element l may exist. Note the particular case:

$$\mathfrak{M}_{G,T,V} = \varnothing \implies \mathcal{N}_{G,V} \neq V. \tag{22}$$

Implication (22) also holds for trivial G-modules V, since, for these modules, $\mathcal{N}_{G,V} = \{0\} \neq V$ and $\mathfrak{M}_{G,T,V} = \varnothing$ by virtue of $\mathfrak{L}_{G,T,V} = \varnothing$.

3.6. Henceforth, we assume that the G-module V is nontrivial and pass to the algorithm for verifying whether or not $l \in \mathfrak{L}_{G,T,V}$ is a stratifying element. Since $\mathcal{N}_{G[l],V[l]}$ is a closed subset in $V[l]$, condition (8) is equivalent to the fact that the element l *satisfies* the inequality

$$\dim \mathcal{N}_{G[l],V[l]} < \dim V[l]. \tag{23}$$

12

In turn, because the number of strata is finite and these strata are open in their closures, inequality (23) is equivalent to the fact that the dimension of all strata of the variety $\mathcal{N}_{G[l],V[l]}$ is strictly less than $\dim V[l]$. As is shown above, the latter is equivalent to the fact that not a single element of the set $\mathfrak{M}_{G[l],T[l],V[l]}$ determines the stratum of the null-cone $\mathcal{N}_{G[l],V[l]}$, i.e., *no element* element $l' \in \mathfrak{M}_{G[l],T[l],V[l]}$ *satisfies* the following inequality:

$$\dim \mathcal{N}_{G[l][l'],V[l][l']} < \dim V[l][l']. \tag{24}$$

Applying these arguments to the $G[l][l']$-module $V[l][l']$, we obtain that (24) *is not fulfilled* precisely when there exists an element l'' in the set $\mathfrak{M}_{G[l][l'],T[l][l'],V[l][l']}$ that determines the stratum of the null-cone $\mathcal{N}_{G[l][l'],V[l][l']}$, i.e., such stratum that the following inequality *is fulfilled*:

$$\dim \mathcal{N}_{G[l][l'][l''],V[l][l'][l'']} < \dim V[l][l'][l'']. $$

An so on.

3.7. This brings us to the consideration of all possible sequences of the form

$$l_1, \ldots, l_d, \quad \text{where} \ \ l_1 = l, \ \text{and} \ \ l_{i+1} \in \mathfrak{M}_{G[l_1]\ldots[l_i],T[l_1]\ldots[l_i],V[l_1]\ldots[l_i]} \ \text{for} \ \ i = 1, \ldots, d-1. \tag{25}$$

By (4) and (6), for each of these sequences, the space $V[l_1]...[l_d]$ is nonzero and $G[l_1]...[l_d]$ is a connected reductive group of rank $\mathrm{rk}G - d$. Thus, $d \leqslant \mathrm{rk}\,G$. The sequence (25) is maximal (i.e., nonextendable to the right) if and only if $\mathfrak{M}_{G[l_1]\ldots[l_d],T[l_1]\ldots[l_d],V[l_1]\ldots[l_d]} = \varnothing$; in this case, it follows from (22) that

$$\mathcal{N}_{G[l_1]\ldots[l_d],V[l_1]\ldots[l_d]} \neq V[l_1]...[l_d]. \tag{26}$$

3.8. Let us associate with vector $l \in \mathfrak{L}_{G,T,V}$ a rooted tree Γ_l, whose edges are directed "from" the root (see Subsection 3.2). This tree is successively constructed by the following procedure (see Fig. 4 on the next page):

- l is the root of Γ_l.
- If the set $\mathfrak{M}_{G[l],T[l],V[l]}$ is empty, then Γ_l consists only of the root.
- If this set is nonempty and, say, consists of elements $a_1, \ldots, a_s$, then these elements are vertices of Γ_l, and exactly s edges emanate from the root that have the ends $a_1, \ldots, a_s$.
- Similarly, if the set $\mathfrak{M}_{G[l][a_i],T[l][a_i],V[l][a_i]}$ is empty, then there are no edges emanating from vertex a_i.
- If this set is nonempty and, say, consists of elements $b_1, \ldots, b_t$, then these elements are vertices of Γ_l , and exactly t edges emanate from a_i that have the ends $b_1, \ldots, b_t$.
- And so on.
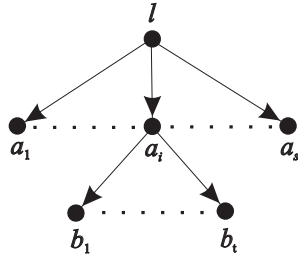- There are no other vertices and edges in the tree Γ_l.

13

Fig. 4

3.9. The decision rule can be now formulated as follows.

Theorem 2. *Let l be an element of the set $\mathfrak{L}_{G,T,V}$. We assign plus or minus signs to the vertices of the tree Γ_l (see Fig. 5) by the following rule (which uniquely defines this sign allocation):*

> *A minus sign is assigned a vertex if and only if there exists an edge emanating from this vertex whose end has a plus sign (in particular, all end vertices have a plus sign).*

Then, l is a stratifying element if and only if the root of the graph Γ_l has a plus sign.

Remark. In fact, the proof and what has been said in Subsection 3.5 imply that the tree Γ_l automatically possesses the following special property: *at most one* edge emanates from any vertex of Γ_l whose other end has a plus sign. In particular, at most one edge leading to an end vertex emanates from any vertex.
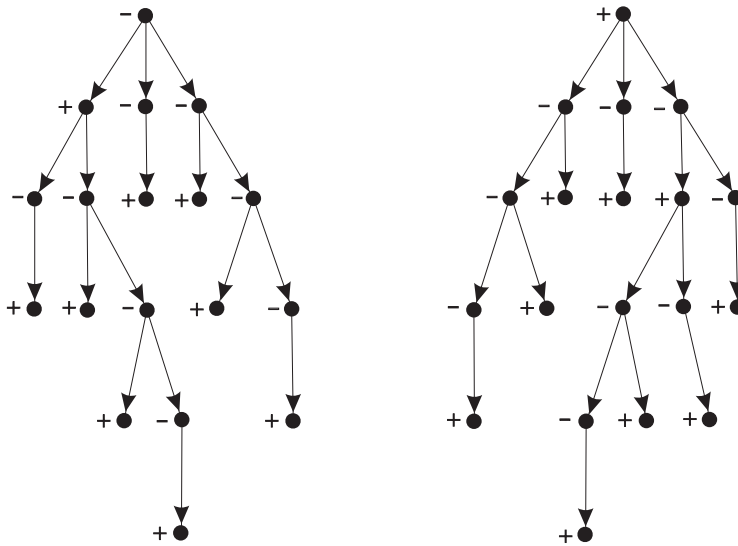

Fig. 5

Proof. Sequences (25) are geodesic sequences of vertices of the tree Γ_l, that start at the root. Consider a vertex a and the geodesic γ connecting the root l with a. Let $l_1, \ldots, l_d$,

14

where $l_d = a$, be sequence (25) corresponding to γ. Let us prove by induction on the height of the vertex a that a plus sign is assigned the vertex a if and only if condition (26) holds. By the definition of the stratifying element, for a vertex of maximal height, i.e., for $a = l$, this gives the assertion of the theorem.

If a has a zero height, then a is an end vertex. Therefore, the sequence $l_1, \ldots, l_d$ is maximal, and, hence, as was mentioned above, inequality (26) holds. On the other hand, by the sign allocation rule, the vertex a has a plus sign in this case. Thus, the assertion of the theorem holds for vertices of zero height.

Now, suppose that a is a vertex of nonzero height. Then, sequence (25) is nonmaximal, i.e., the set $\mathfrak{M}_{G[l_1]\ldots[l_d],T[l_1]\ldots[l_d],V[l_1]\ldots[l_d]}$ is nonempty. As was shown in Subsection 3.6, condition (26) holds if and only if any element $b \in \mathfrak{M}_{G[l_1]\ldots[l_d],T[l_1]\ldots[l_d],V[l_1]\ldots[l_d]}$ does not satisfy the condition

$$\mathcal{N}_{G[l_1]\ldots[l_d][b],V[l_1]\ldots[l_d][b]} \neq V[l_1]\ldots[l_d][b]. \tag{27}$$

On the other hand, it follows from the definition of the graph Γ_l that b is its vertex and that the geodesic connecting the root with b corresponds to the sequence $l_1, \ldots, l_d, b$. It follows from the definition of height that b has a smaller height than a. Hence, by the induction hypothesis, condition (27) is violated if and only if b has a minus sign. Thus, condition (26) is fulfilled if and only if all vertices $b \in \mathfrak{M}_{G[l_1]\ldots[l_d],T[l_1]\ldots[l_d],V[l_1]\ldots[l_d]}$ have a minus sign. However, according to the sign allocation rule, the latter is equivalent to the fact that the vertex a has a plus sign. This completes the proof. $\square$

Corollary 1. *Suppose that an element $l \in \mathfrak{L}_{G,T,V}$ is not orthogonal to any root. Then l is a stratifying element.*

Proof. By (15) and (16) and conditions (ii) and (iii) in the definition of the set $\mathfrak{L}_{G,T,V}$ (see Subsection 3.4), zero belongs to conv $\Delta_{V[l],G[l]}$. Therefore, for any nonzero element $l' \in V[l]$, the half-space $\{l' < 1\}$ contains at least one weight. Since, by the hypothesis, there are no roots in $V[l]$ condition (21) implies that $\mathfrak{M}_{G[l],G[l],V[l]} = \varnothing$. Thus, the tree Γ_l with the allocation of signs defined in Theorem 2 has the form shown in Fig. 6.

$$l \\ \bullet \\ +$$

Fig. 6

Hence, l is a stratifying element. $\square$

Since $\Phi_{\mathfrak{g},T} = \varnothing$ for $G = T$, this and (10) imply the following corollary.

Corollary 2. (the case of a torus) *If G is a torus, $G = T$, then*

(a) *all elements of the set $\mathfrak{L}_{G,T,V}$ are stratifying elements;*

(b) *the closures of the strata are linear subspaces of the form $\bigoplus_{\mu \in \text{conv } M} V_\mu$, where M runs over those subsets in the system of weights $\Delta_{V,T}$, that possess properties (i), (ii),*

and (iii) *pointed out in Subsection* (3.4). *The strata themselves are complements, in these subspaces, to unions of finite sets of linear spaces of a similar form.*

4. Examples

Denote by $F_{d,n}$ the space of forms of degree $d \geqslant 0$ in variables $x_1, \ldots, x_n$ over k. We consider this space as the G-module for $G = \mathrm{GL}_n$ or SL_n with respect to the natural action by linear transformations of variables.

We put $\mathbb{N} := \{0, 1, 2, \ldots\}$ and $\mathbb{Z}_+ := \{1, 2, 3, \ldots\}$.

Example 1 (the case of rank 1)**.** If $\dim T = 1$, then, by Corollary 1 to Theorem 2, all elements of the set $\mathfrak{L}_{G,T,V}$ are stratifying elements.

Suppose that, in view of Corollary 2 to Theorem 2, the group G is not a torus. Then, passing to the universal covering, we can assume that $G = \mathrm{SL}_2$ and T is the diagonal maximal torus in G. The group $\mathrm{Hom}(T, k^\times)$ is generated by the character $\mathrm{diag}(a, a^{-1}) \mapsto a$. Denote it by ε. Then, $\Phi_{\mathfrak{g},T} = \{2\varepsilon, -2\varepsilon\}$, and, since $W_{G,T} = \{1, -1\}$, we can assume that $\mathfrak{L}_{G,T,V} = \{\frac{\varepsilon}{\langle \mu, \varepsilon \rangle} \mid \mu \in \Delta_{V,T} \cap \mathbb{Z}_+ \varepsilon\}$. Since $\mathrm{char}\, k = 0$, we have, up to an isomorphism, $V = F_{d_1,2} \oplus \ldots \oplus F_{d_s,2}$. Let d_{odd} (respectively, d_{even}) be the greatest positive odd (respectively, even) number among $d_1, \ldots, d_s$, if such numbers exist, and 0 otherwise. Since $\Delta_{F_{d,2},T} = \{d\varepsilon, (d-2)\varepsilon, \ldots, (2-d)\varepsilon, -d\varepsilon\}$ and all the weights in $\Delta_{F_{d,2},T}$ have mumtiplicity 1, we obtain that the number of strata in $\mathcal{N}_{G,V}$ that are different from $\{0\}$ is equal to $\left[\frac{d+1}{2}\right] + \frac{d}{2}$, and the dimension of the stratum $\mathcal{N}[l]$, defined by the element $l = \frac{\varepsilon}{\langle m\varepsilon, \varepsilon \rangle} \in \mathfrak{L}_{G,T,V}$, is equal to $1 + \sum_{d_i \geqslant m}]m, d_i[$, where $]a, b[$ with $a, b \in \mathbb{Z}_+$, $a \leqslant b$, denotes the number of integers in the interval $[a, b]$, that have the same parity as b.

For example, when $s = 5$ and $(d_1, d_2, d_2, d_4, d_5) = (2, 3, 3, 4, 5)$, we have $d = 5$, $d = 4$ and $\mathfrak{L}_{G,T,V}$ consists of $\left[\frac{d+1}{2}\right] + \frac{d}{2} = 5$ elements $l_1, \ldots, l_5$. Thus, there are exactly five strata that are different from $\{0\}$. The weights from $\Delta_{V,T}$ are depicted by circles in Fig. 7; the numbers at these circles are the multiplicities of these weights. The roots are marked by arrows. One can choose l_i so that the point $\{l_i = 1\}$ is such as shown in Fig. 7. The dimension of the stratum $\mathcal{H}[l_i]$ is greater by 1 than the sum of multiplicities of the circles lying to the left of the circle $\{l_i = 1\}$; i.e., it is equal to 11, 8, 6, 3, and 2 for $i = 1, \ldots, 5$, respectively. This, in particular, yields $\dim \mathcal{N}_{G,V} = 11$.
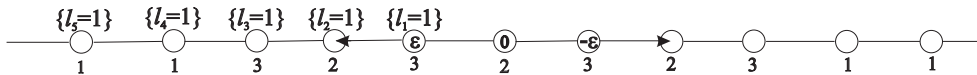

Fig. 7

Example 2 (the case of rank 2)**.** Let $\dim T = 2$. By Corollary 2 to Theorem 2, we assume that G is not a torus. The description of the straight lines $\{l = 1\}$ for $l \in \mathfrak{L}_{G,T,V}$ is equivalent, up to the action of the Weyl group, to the description of all possible straight lines L on the plane $\mathfrak{t}(\mathbb{Q})$ that share the following properties:

(i) L does not pass through 0;

(ii) L passes at least through one weight;

(iii) the end of the perpendicular dropped from 0 onto L, is contained in a segment representing a convex hull of the weights lying on L.

If L is such a line, then there are two possibilities: either L is parallel to no root, or it is parallel to a certain root α. In the first case, l is a stratifying element by Corollary 1 to Theorem 2. In the second case, it may happen that $\mathfrak{M}_{G[l],T[l],V[l]} \neq \varnothing$. Namely, since $\Delta_{V,T}$ is invariant with respect to $W_{G,T}$ and, in particular, with respect to the reflection about a straight line orthogonal to α, it follows from (21) that this occurs precisely when L contains exactly two different single weights (in Fig. 8, the weights are depicted by dark, and the roots, by light circles).
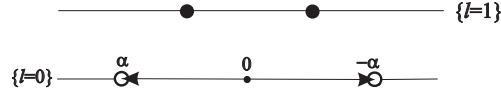

Fig. 8

In this case, $\mathfrak{M}_{G[l],T[l],V[l]}$ consists of a single element l'. Since, obviously, $\mathfrak{M}_{G[l][l'],T[l][l'],V[l][l']} = \varnothing$, in this case the tree Γ_l with an appropriate sign allocation has the form shown in Fig. 9.

Fig. 9

Hence, by Theorem 2, l is not a stratifying element.

Example 3 (dependence on the norm; null-cones with a finite number of orbits). Let $G = \mathrm{GL}_2$ and $V = F_{1,2} \oplus \wedge^2 F_{1,2}$. There are exactly three nonzero G-orbits in V: $\mathcal{O}_1 := G \cdot x_1$, $\mathcal{O}_2 := G \cdot (x_1 \wedge x_2)$, and $\mathcal{O}_3 := G \cdot (x_1 + x_1 \wedge x_2)$. Let T be the diagonal maximal torus in G and let $\varepsilon_i \in \mathrm{Hom}(T, k^\times)$ be the weight of the variable x_i. The weights ε_1 and ε_2 form a basis in the lattice $\mathrm{Hom}(T, k^\times)$ and thereby in $\mathfrak{t}(\mathbb{Q})$. The Weyl group $W_{G,T}$ has order 2 and acts by permutations of ε_1 and ε_2. The sequence of vectors x_1, x_2, $x_1 \wedge x_2$ is a weight basis in V. Therefore, the system of weights has the form $\Delta_{V,T} = \{\varepsilon_1, \varepsilon_2, \varepsilon_1 + \varepsilon_2\}$ and the multiplicity of every weight is 1. The system of roots has the form $\Phi_{\mathfrak{g},T} = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_1\}$. Positive definite $W_{G,T}$-invariant inner products on $\mathfrak{t}(\mathbb{Q})$ are classified by the pairs of rational numbers $(a, b) \in \mathbb{Q}^2$, satisfying the conditions $a > 0$ and $a^2 > b^2$: to a pair (a, b) is assigned the inner multiplication for which $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ is the Gram matrix of the basis $\varepsilon_1, \varepsilon_2$.

Thus, in this case, the weights and the roots are depicted by dark and light circles, respectively (Fig. 10); the left part of Fig. 10 illustrates the case when $b > 0$, while the right part, the case when $b \leqslant 0$:
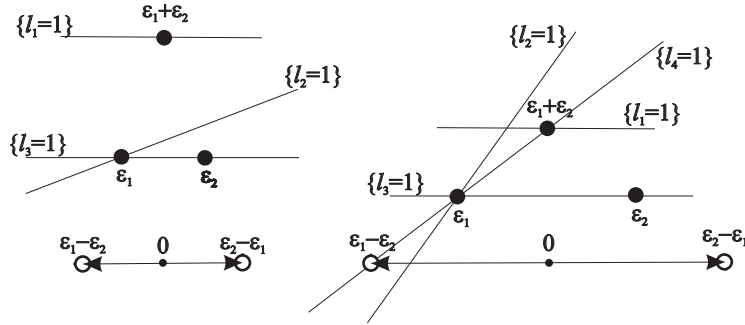
17

Fig. 10

Figure 10 shows that the set $\mathfrak{L}_{G,T,V}$ consists of three elements l_1, l_2, and l_3 for $b > 0$ and of four elements l_1, l_2, l_3, and l_4 for $b \leqslant 0$, and that, subjecting these elements, if necessary, to transformations from the Weyl group, one can assume that the straight lines $\{l_i = 1\}$ are such as they are depicted in Fig. 10. According to Example 2, if $b > 0$, then there are exactly two strata in the null-cone $\mathcal{N}_{G,V}$ that are different from $\{0\}$, namely, $\mathcal{H}[l_1]$ and $\mathcal{H}[l_2]$; if $b \leqslant 0$, then there are exactly three such strata, namely, $\mathcal{H}[l_1]$, $\mathcal{H}[l_2]$, and $\mathcal{H}[l_4]$. It is clear that, if $b > 0$, then $\mathcal{H}[l_1] = \mathcal{O}_2$, and $\mathcal{H}[l_2] = \mathcal{O}_1 \cup \mathcal{O}_3$, whereas, if $b \leqslant 0$, then $\mathcal{H}[l_1] = \mathcal{O}_2$, $\mathcal{H}[l_2] = \mathcal{O}_1$ and $\mathcal{H}[l_4] = \mathcal{O}_3$.

Thus, in the general case, the strata depend on the norm chosen and, if the null-cone contains only a finite number of orbits, then some strata may even not be orbits (see [He$_2$, Remark 4.10]).

Example 4 (ternary forms)**.** Let $G = \mathrm{SL}_3$ and $V = F_{d,3}$, $d \geqslant 1$. Let T be the diagonal maximal torus in G and $\varepsilon_i \in \mathrm{Hom}(T, k^\times)$ be the weight of the variable x_i. The space $\mathfrak{t}(\mathbb{Q})$ is two-dimensional. The weights ε_1, ε_2, and ε_3 generate the lattice $\mathrm{Hom}(T, k^\times)$, and the sum of these weights is equal to 0. The system of roots $\Phi_{T,\mathfrak{g}}$ is the set $\{\varepsilon_i - \varepsilon_j \mid i \neq j\}$. The set $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is preserved by the Weyl group $W_{G,T}$, and any permutation of the vectors $\varepsilon_1, \varepsilon_2$, and ε_3 is performed by a certain element from $W_{G,T}$. Therefore, $\| \varepsilon_1 \| = \| \varepsilon_2 \| = \| \varepsilon_3 \|$, and $\angle\{\varepsilon_i, \varepsilon_j\} = 2\pi/3$ for $i \neq j$, which determines $\langle \, , \, \rangle$ uniquely up to proportionality. If $c_1, c_2, c_3 \in \mathbb{Q}$, then we denote the vector $c_1\varepsilon_1 + c_2\varepsilon_2 + c_3\varepsilon_3 \in \mathfrak{t}(\mathbb{Q})$ by $(c_1 c_2 c_3)$. Then, the system of weights is expressed as $\Delta_{V,T} = \{(c_1 c_2 c_3) \mid c_1 + c_2 + c_3 = d, \ c_i \in \mathbb{N}\}$. The multiplicity of the weight $(c_1 c_2 c_3)$ is equal to 1, and the set of monomials $\{x_1^{c_1} x_2^{c_2} x_3^{c_3}\}$ is linearly independent and spans $V_{(c_1 c_2 c_3)}$. Thus, the system of weights is given by the set of vertices of regular triangles with the side $\sqrt{3} \, \| \varepsilon_i \|$ that form a partition of the regular triangle with vertices $(d00)$, $(0d0)$, $(00d)$. In Fig. 11 on the next page, the weights are depicted by dark circles and the roots, by light circles.

The straight lines $\{l = 1\}$ for $l \in \mathfrak{L}_{G,T,V}$ and the stratifying elements in $\mathfrak{L}_{G,T,V}$ are described according to Example 2. There is a unique, up to the action of the Weyl group, case when l is not a stratifying element: in this case, the straight line $L_0 := \{l = 1\}$ passes through the weights $(1(d-1)0)$ and $(0(d-1)1)$.
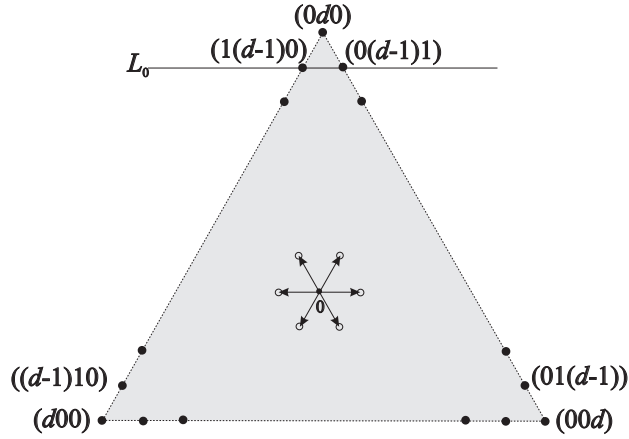
18

Fig.11

For example, in the case of ternary quartics (i.e., when $d = 4$), one can easily verify that $\mathfrak{L}_{G,T,V}$ contains exactly 12 elements $l_1, \ldots, l_{12}$ and that, subjecting them, if necessary, to transformations from the Weyl group, one can assume that the straight lines $\{l_i = 1\}$ are such as they are depicted in Fig. 12.
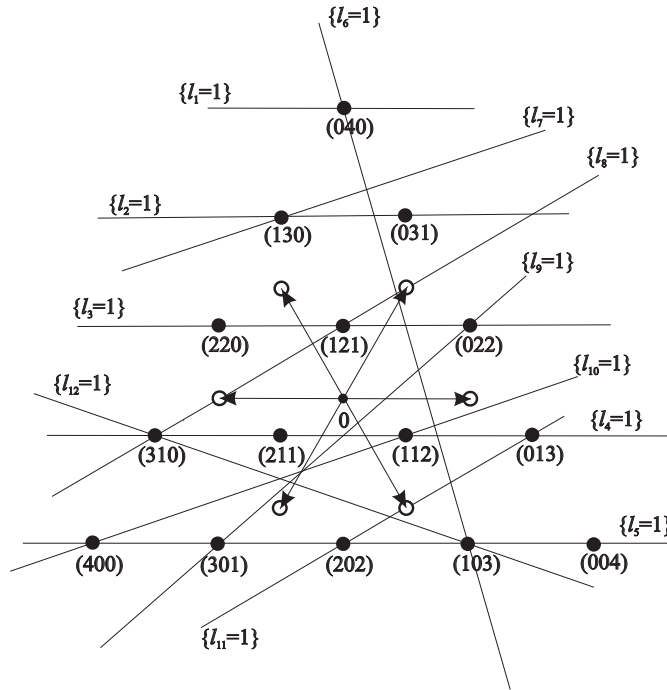

Fig.12

Here, only l_2 is not a stratifying element. Thus, there are exactly 11 strata in this case that are different from $\{0\}$, and these strata are given by $\mathcal{H}[l_i]$, $1 \leqslant i \leqslant 12$, $i \neq 2$. Formula (11) gives their dimensions: $\dim \mathcal{H}[l_i] = 3, 8, 11, 7, 9, 5, 9, 10, 10, 7$, and 8 for $i = 1, 3, 4, 5, 6, 7, 8, 9, 10, 11$, and 12, respectively. Hence, $\dim \mathcal{N}_{G,V} = 11$ and $\mathcal{N}_{G,V}$ has a unique irreducible component of maximal dimension (one can show that $\mathcal{N}_{G,V}$ has another irreducible component of smaller dimension [Hi]).

19

Example 5 (nilpotent elements in reductive Lie algebras)**.** If $V = \mathfrak{g}$, then $\mathcal{N}_{G,V}$ is the variety of all nilpotent elements of the Lie algebra $\mathfrak{g}$. It is well known [He$_1$], [Sl] that, in this case, the strata of the null-cone $\mathcal{N}_{G,V}$ that are different from $\{0\}$ are exactly the nonzero G-orbits contained in $\mathcal{N}_{G,V}$, i.e., the conjugacy classes of nonzero nilpotent elements of the Lie algebra $\mathfrak{g}$. Hence, in this case our algorithm turns into the classification algorithm for the conjugacy classes of nilpotent elements in reductive Lie algebras $\mathfrak{g}$, which is reduced to only simple geometric–combinatorial operations with the system of roots $\Phi_{\mathfrak{g},T}$. For the first time, such a classification was obtained in [Dy] by a different method, which was later improved in [BC]. This classification is a particular case of the classification of the orbits of nullforms for the so-called θ-representations (see [Vi$_3$]). To obtain the latter classification, Dynkin's method was improved and extended in [Vi$_1$], [Vi$_2$], [GV] to the case of θ-representations. A partial computer implementation of this method was obtained in [Li$_2$].

We will illustrate the solution of this problem by means of our algorithm through the example of the simple exceptional Lie algebra $\mathfrak{g}$ of type G_2. In this case, the weights are depicted by dark circles in Fig. 13.
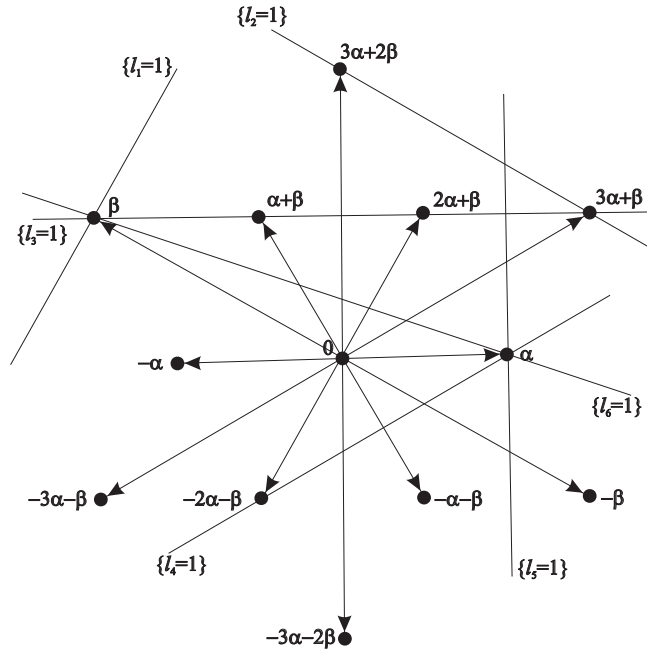

Fig. 13

Here, the roots are nonzero weights. Using Fig. 13, one can easily verify that the set $\mathfrak{L}_{G,T,V}$ contains exactly six elements $l_1, \ldots, l_6$ and that, subjecting them, if necessary, to transformations from the Weyl group, one can assume that the straight lines $\{l_i = 1\}$ are as is shown in this figure. Of these elements, only l_6 is not orthogonal to any root; for all the other elements l_i, the line $\{l_i = 1\}$ contains exactly two roots precisely for $i = 2\ 4$. Hence, the total number of strata different from $\{0\}$ is four (therefore, there are precisely four nonzero conjugacy classes of nonzero nilpotent elements), namely, $\mathcal{H}[l_i]$, where $i = 1, 3, 5$

and 6.

The representatives of these four classes are obtained by means of what has been said in Subsection 2.8. Namely, it follows from Proposition 2 that, if $x_\gamma \in \mathfrak{g}_\gamma$ is an element of the Chevalley basis of the algebra $\mathfrak{g}$, then the representatives of the classes $\mathcal{H}[l_1]$, $\mathcal{H}[l_3]$, $\mathcal{H}[l_5]$, and $\mathcal{H}[l_6]$ are x_β, $t_\beta x_\beta + t_{\alpha+\beta} x_{\alpha+\beta} + t_{2\alpha+\beta} x_{2\alpha+\beta} + x_{3\alpha+\beta}$, x_α, and $x_\alpha + t_\beta x_\beta$, respectively, where t_β, $t_{\alpha+\beta}$, $t_{2\alpha+\beta}$ are elements from the field k that are algebraically independent over $\mathbb{Q}$.

A simple additional argument shows that these representatives can be simplified. Namely, according to (10) and (7), if a $G[l_i]$-orbit of a nonzero vector $v \in V[l_i]$ is closed, then this vector lies in the class $\mathcal{H}[l_i]$. The group $G[l_i]$ is a one-dimensional torus $T[l_i]$ for $i = 6$ and is locally isomorphic to SL_2 for $i \neq 6$. Therefore, in the first case, the closedness of $G[l_i] \cdot v$ is equivalent to the fact that zero lies in the convex hull of the weights of the $T[l_6]$-module $V[l_6]$ that occur in the decomposition of v into a sum of weight vectors. This fact and Fig. 13 imply that $e_\alpha + e_\beta$ is a representative of the class $\mathcal{H}[l_6]$. In the second case, one can use the fact that, if the weights of certain weight vectors possess the following properties:

(a) all these weights are different;

(b) zero the an inner point of their convex hull; and

(c) the difference of any two of these weights is not a root,

then the orbit of the sum of these weight vectors is closed (see [DK, Prop. 1.2], [PV, 6.19]). This is applicable to the case of $i = 3$. Figure 13 shows that the roots of the group $G[l_3]$ with respect to the torus $T[l_3]$ are $\pm\alpha$, and the weights of $V[l_3]$ are the orthogonal projections of β, $\alpha + \beta$, $2\alpha + \beta$, and $3\alpha + \beta$ onto the straight line $\{l_3 = 0\}$. Hence, the $G[l_3]$-orbit of the vector $e_\beta + e_{3\alpha+\beta}$ is closed. Therefore, $e_\beta + e_{3\alpha+\beta}$ is a representative of the conjugacy class $\mathcal{H}[l_3]$.

In conclusion, note that, for θ-representations, our algorithm turns into a classification algorithm for the orbits of nullforms. This claim results from the following proposition.

Proposition 4. *For every θ-representation, the strata of the null-cone are orbits.*

Proof. The existence of a homogeneous version of the Morozov theorem [Vi$_4$, Theorem 1] allows one to apply the arguments similar to those by which this fact was proved in [Sl, Prop. 1] for the adjoint representation (in turn, these arguments were inspired by [Ka$_2$, §1, Lemma 1.4]). $\qquad\square$

References

[Ad]   O. M. Adamovich, *Equidimensional representations of simple algebraic groups*, Transl., Ser. 2, Amer. Math. Soc. **128** (1986), 25–29.

[An]   L. V. Antonyan, *The classification of four-vectors of the eight-dimensional space*, Tr. Semin. Vektorn. Tenzorn. Anal. Prilozh. Geom. Mekh. Fiz. **20** (1981), 144–161 (Russian). Zbl 0467.15018.

[AE]   L. V. Antonyan, A. G. Elashvili, *Classification of spinors of dimension sixteen*, Tr. Tbilis. Mat. Inst. Razmadze **70** (1982), 5–23 (Russian). Zbl 0519.17006.

[BC]   P. Bala, R. W. Carter, *Classes of unipotent elements in simple algebraic groups,* I *and* II, Math. Proc. Camb. Phil. Soc. **79** (1976), 401–425 and **80** (1976), 1–18.

[Bo]   F. A. Bogomolov, *Holomorphic tensors and vector bundles on projective varieties*, Math. USSR, Izv. **13** (1979), 499–555.

[Dy]   E. B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Amer. Math. Soc. Transl., Ser. 2 **6** (1957), 111–244.

[DK]   J. Dadok, V. Kac, *Polar groups*, J. Algebra **92** (1985), 504–524.

[GV]   V. Gatti, E. Viniberghi, *Spinors of a* 13-*dimensional space*, Adv. Math. **30** (1978), 137–155.

[He$_1$]   W. H. Hesselink, *Uniform instability in reductive groups*, J. reine angew. Math. **303/304** (1978), 74–96.

[He$_2$]   W. H. Hesselink, *Desingularizations of varieties of nullforms*, Invent. math. **55** (1979), 141–163.

[Hi]   D. Hilbert, *Über die vollen Invariantensysteme*, Math. Ann. **42** (1893), 313–373.

[Hu]   J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics, Vol. 9, 1972, Springer-Verlag, New York, Heidelberg.

[Ka$_1$]   V. G. Kac, *On the question of describing the orbit space of linear algebraic groups*, Usp. Mat. Nauk **30** (1975), no. 6(186), 173–174 (Russian). Zbl 0391.20035.

[Ka$_2$]   V. Kac, *Some remarks on nilpotent orbits*, J. Algebra **64** (1980), 190–213.

[Ke]   G. R. Kempf, *Instability in invariant theory*, Ann. of Math. **108** (1978), 299–316.

[Ki]   F. C. Kirwan, *Cohomology of Quotients in Symplectic and Algebraic Geometry*, Math. Notes **31**, Princeton Univ. Press, Princeton, 1984.

[LB]   L. Le Bruyn, *Nilpotent representations*, J. Algebra **197** (1997), 153–177.

[Li$_1$]   P. Littelmann, *Koreguläre und äquidimensionale Darstellungen halbeinfacher Liegruppen*, J. Algebra **123** (1989), no. 1, 193–222.

[Li$_2$]   P. Littelmann, *An effective method to classify nilpotent orbits*, in: Progress in Math., Vol. 143, 1996, Birkhäuser Basel, 255–269.

[Mu]   D. Mumford, *Geometric Invariant Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 34, Berlin, Springer-Verlag, 1965.

[Ne]   L. Ness, *A stratification of the null-cone via moment map*, Amer. J. Math. **106** (1984), 1281–1329.

[Po$_1$]   V. L. Popov, *Representations with a free module of covariants*, Funct. Anal. Appl. **10** (1976), no. 3, 242–244.

[Po$_2$]   V. L. Popov, *Classification of spinors of dimension fourteen*, Trans. Mosc. Math. Soc. **1** (1980), 181–232.

[Po$_3$]   V. L. Popov, *Sections in invariant theory*, in: *Proceedings of the Sophus Lie Memorial Conference* Oslo, 1992, Scandinavian University Press (1994), 315–362.

[PV]   V. L. Popov, E. B. Vinberg, *Invariant Theory*, in: *Algebraic Geometry*, IV, Encyclopaedia of Mathematical Sciences, Vol. 55, Springer-Verlag, Berlin, 1994, pp. 123–284.

[Ro]   G. Rousseau, *Immeubles sphériques et theorie des invariants*, C. R. Acad. Sc. Paris **286** (1978), A 247–250.

[Sc$_1$]   G. W. Schwarz, *Representations of simple Lie groups with a free module of covariants*, Invent. math. **50** (1978), 1–12.

[Sc$_2$]   G. W. Schwarz, *Lifting smooth homotopies of orbit spaces*, Inst. Hautes Etudes Sci. Publ. Math. **51** (1980), 37–135.

[Sl]   P. Slodowy, *Die Theorie der optimalen Einparameteruntergruppen für instabile Vektoren*, in: *Algebraic Transformation Groups and Invariant Theory*, Edited by H. Kraft, P. Slodowy, T. A. Springer, DMV Seminar, Vol. 13, 1989, Birkhäuser Basel, pp. 115–130.

[Vi$_1$]   E. B. Vinberg, *On the linear groups associated to periodic automorphisms of semisimple algebraic groups*, Sov. Math., Dokl. **16** (1975), 406–409.

[Vi$_2$]   E. B. Vinberg, *On the classification of the nilpotent elements of graded Lie algebras*, Sov. Math., Dokl. **16** (1976), 1517–1520.

[Vi$_3$]   E. B. Vinberg, *The Weyl group of a graded Lie algebra*, Math. USSR, Izv. **10** (1977), 463–495.

[Vi$_4$]   E. B. Vinberg, *Classification of homogeneous nilpotent elements of a semisimple graded Lie algebra*, Sel. Math. Sov. **6** (1987), 15–35.

[VE]   E. B. Vinberg, A. G. Elashvili, *A classification of three-vectors of a nine-dimensional space*, Sel. Math. Sov. **7** (1988), no. 1, 63–98.