Mapping the Born Rule to the Fractal Geometry of Quantum Paths

Ervin Goldfain

Here we show that there is an approximate mapping between multifractal theory and the Born rule of Quantum Mechanics. The derivation is based on the fractal geometry of quantum mechanical paths, which replicates the geometry of unrestricted random walks in $d \geq 2$ Euclidean dimensions.

Key words: Born rule, multifractals, quantum mechanical paths, fractal dimension, random walks.

The goal of this brief note is to apply the ideas of [1-3] to the mathematical framework of Quantum Mechanics (QM). In particular, starting from the conjecture that effective field theories can be cast in the language of strange attractors and multifractals, we bridge the gap between the Born rule and the closure relationship of multifractal theory.

Let's start by recalling the superposition postulate of QM. It implies that state vectors can be linearly expanded in the eigenstate basis as

$$|\psi\rangle = \sum_i c_i |\phi_i\rangle \qquad (1)$$

where

$$c_i = \langle \phi_i | \psi \rangle = \int \phi_i^*(x) \psi(x) dx \qquad (2)$$

subject to the normalization condition

$$\int |\psi(x)|^2 dx = \sum_i |c_i|^2 = 1 \qquad (3)$$

Echoing (1), the closure relationship of multifractal theory is given by [4-6]

$$\sum_i p_i^q r_i^{\tau(q)} = 1 \tag{4}$$

in which $q \in (-\infty, +\infty)$ represents the order parameter, r_i is the spectrum of local scales characterized by probabilities p_i and scaling exponent $\tau(q)$. The scaling exponent and order parameter are linearly related via

$$\tau(q) = (1-q) D_q \tag{5}$$

where D_q is the generalized dimension (or Rényi entropy) of order q.

The *symbolic* connection between (1) and (4) becomes apparent upon recasting (1) as

$$\sum_i c_i \left(\frac{|\psi_i\rangle}{|\psi\rangle} \right) = 1 \tag{6}$$

under the convention that the quotient of the two complex functions in (6) is defined by the quotient of their real-valued amplitudes as in

$$\frac{|\psi_i\rangle}{|\psi\rangle} = \frac{|a_i e^{j\varphi_i}\rangle}{|a e^{j\varphi}\rangle} \square \frac{a_i}{a} \tag{6}$$

A glance at (1), (4)–(6) reveals the following *symbolic mapping* between the complex entities of expansion (1) and the real valued parameters of (4), namely,

$$c_i \Leftrightarrow p_i^q \tag{7a}$$

$$\frac{a_i}{a} \Leftrightarrow r_i^{\tau(q)} \tag{7b}$$

It is known that quantum mechanical paths in $d \geq 2$ dimensions are described as fractal trajectories matching the geometry of unrestricted random walks with Hausdorff dimension $D_0 = 2$ [7-9]. Since q spans a wide range of real numbers, it is not unreasonable to assume that $D_0 \approx D_q$ for q in proximity to the null value ($0 \approx q \ll \infty$). Under this assumption and by (5) and (7), we derive

$$\boxed{D_0 \approx D_q = 2,\ \tau(q) = 1 \Rightarrow q = \frac{1}{2}} \tag{8}$$

This is our main result. It states that the coefficients c_i of expansion (1) stand for probability amplitudes such that $|c_i|^2 \Leftrightarrow (p_i^{1/2})^2 = p_i$ map to *Born probabilities* fulfilling the normalization condition (3).

<u>References</u>

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