

THE VIBRATION OF THE STRING WITH THE INTERSTITIAL MASSIVE POINT

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May 20, 2021

Abstract

We will consider the string, the left end of which is fixed at the beginning of the coordinate system, the right end is fixed at point l and mass m is fixed between the ends of the string. We determine the vibration of such system. The proposed model can be also related in the modified form to the problem of the Mössbauer effect, or recoilless nuclear resonance fluorescence, which is the resonant and recoil-free emission and absorption of gamma radiation by atomic nuclei bound in a solid. (Mössbauer,1958).

1 Introduction

We will consider the string, the left end of which is fixed at the beginning of the coordinate system, the right end is fixed at point l and mass m is fixed interstitially between the ends of the string. The vibration motion of the string and the massive point with mass m is the problem of the mathematical physics in case that the tension is linearly dependent on elongation.

The differential equation of motion of string elements can be derived by the well known way. We suppose that the string tension force acting on the element dl of the string is given by the law (Tikhonov et al., 1977):

$$T(x, t) = ES \left(\frac{\partial u}{\partial x} \right), \quad (1)$$

where E is the modulus of elasticity, S is the cross section of the string. We easily derive that

$$T(x + dx) - T(x) = ES \left(\frac{\partial u}{\partial x} \right) (x + dx) - ES \left(\frac{\partial u}{\partial x} \right) (x) \approx ESu_{xx}dx. \quad (2)$$

The mass dm of the element dl is ρSdx , where ρ is the mass density of the string matter and the dynamical equilibrium gives

$$\rho Sdxu_{tt} = ESu_{xx}dx. \quad (3)$$

Or, after minimal modification we get

$$\frac{1}{c^2}u_{tt} - u_{xx} = 0; \quad c = \left(\frac{E}{\rho} \right)^{1/2}. \quad (4)$$

The last procedure was performed evidently in order to get the wave equation.

2 The string motion with the interstitial massive point

Now, let us consider the string with the point-like mass at coordinate s in the interval $(0, l)$. Then, the left part string motion of the string be $u_1(x, t)$ and the right side of the string motion is $u_2(x, t)$. The corresponding equation of motion of both part of the string are as follows:

$$(u_1)_{tt} = c^2(u_1)_{xx}; \quad (0 < x < s), \quad (5a)$$

$$(u_2)_{tt} = c^2(u_2)_{xx}; \quad (s < x < l). \quad (5b)$$

The boundary and interstitial conditions are with $S = 1$:

$$u_1(x = 0) = 0, \quad u_2(x = l) = 0. \quad (6)$$

$$u_1(x = s) = u_2(x = s). \quad (7)$$

The dynamical equation involving interstitial point is with $E = \rho c^2$:

$$\rho c^2(u_1)_x(s) - \rho c^2(u_2)_x(s) = m(u_1)_{tt}(s) - m(u_2)_{tt}(s). \quad (8)$$

Let us look for the solution of the last equation in the form

$$u_1(x, t) = C_1 \sin \frac{\omega x}{c} \sin \omega t \quad (9)$$

$$u_2(x, t) = C_2 \sin \frac{\omega(l-x)}{c} \sin \omega t. \quad (10)$$

We see that the suggested solution is in harmony with the boundary conditions:

$$u_1(x = 0) = 0, \quad u_2(x = l) = 0. \quad (11)$$

After insertion of $u_1(x, t), u_2(x, t)$ from (9-10) into eqs. (7-8), we get the system of equations

$$C_1 \sin \frac{\omega s}{c} = C_2 \sin \frac{\omega(l-s)}{c} \quad (12)$$

and

$$\begin{aligned} C_1 \rho c \omega \cos \frac{\omega s}{c} - C_2 \rho c \omega \cos \frac{\omega(l-s)}{c} = \\ C_1 m \omega^2 \sin \frac{\omega s}{c} - C_2 m \omega^2 \sin \frac{\omega(l-s)}{c}. \end{aligned} \quad (13)$$

In order to get the regular solution, the determinant of the system must be zero. Or,

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 0, \quad (14)$$

where

$$A = \sin \frac{\omega s}{c} \quad (15)$$

$$B = -\sin \frac{\omega(l-s)}{c} \quad (16)$$

$$C = \rho c \omega \cos \frac{\omega s}{c} - m \omega^2 \sin \frac{\omega s}{c} \quad (17)$$

$$D = -\rho c \omega \cos \frac{\omega(l-s)}{c} + m \omega^2 \sin \frac{\omega(l-s)}{c}. \quad (18)$$

It follows from eq. (14)

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD - BC = 0. \quad (19)$$

Or,

$$\begin{aligned} \sin \frac{\omega s}{c} \cos \frac{\omega(l-s)}{c} \left[-\rho c \omega + m \omega^2 \tan \frac{\omega(l-s)}{c} \right] + \\ \sin \frac{\omega(l-s)}{c} \cos \frac{\omega s}{c} \left[\rho c \omega - m \omega^2 \tan \frac{\omega s}{c} \right] = 0, \end{aligned} \quad (20)$$

So, we see, that the determination of the frequency ω involves the transcendent equation. The solution can be performed graphically, or by computer. Such problem is the integral part of the university mathematical methods (Arfken, 1967).

Nevertheless, it is evident that the trivial solution is for $\omega = 0$ and for

$$\omega s = \pi n c, \quad n = 0, 1, 2, \dots; \quad \omega(l-s) = \pi k c, \quad k = 0, 1, 2, \dots, \quad (21)$$

which implies that the correspondence between l and k is only for

$$\frac{l}{s} = \frac{(k+n)}{n}. \quad (22)$$

The determination of the ω from the transcendent equations

$$\left[-\rho c \omega + m \omega^2 \tan \frac{\omega(l-s)}{c} \right] = 0, \quad \left[\rho c \omega - m \omega^2 \tan \frac{\omega s}{c} \right] = 0 \quad (23)$$

is difficult and it can be solved by the appropriate mathematical methods (Arfken, 1967).

3 Discussion

The article was inspired by the author's university elaborate (Pardy, 1965), in which the interaction of light with the crystal defect was calculated. At this elaborate the crystal was replaced by the Euler-Bernoulli linear chain (Landau, et al., 1965) with some defects.

The proposed model can be also related in the modified form to the problem of the Mössbauer effect, or recoilless nuclear resonance fluorescence, which is the resonant and recoil-free emission and absorption of gamma radiation by atomic nuclei bound in a solid. (Mössbauer, 1958). In this effect, a narrow resonance for nuclear gamma emission and absorption results from the momentum of recoil transferred to a surrounding crystal lattice and not to the emitting or absorbing nucleus alone. No gamma energy is lost. Emission and absorption occur at the same energy, resulting in strong, resonant absorption.

The generalization of our continual model can be performed in such a way that we replace one massive point m by the massive points $m_1, m_2, m_3, \dots, m_k$ at coordinates $s_1, s_2, s_3, \dots, s_k$ and solve the adequate system of differential equations.

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