# An algebraic analysis framework for quantum calculus

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**ABSTRACT**

An algebraic analysis framework for quantum calculus is proposed. The quantum derivative operator $D_{\tau,\sigma}$ is based on two commuting bijections $\tau$ and $\sigma$ defined on an arbitrary set $M$ equipped with a tension structure determined by a single tension function $\theta$, i.e. a 1-dimensional case is analyzed here. The well known cases, i.e. $h$- and $q$-calculi together with their symmetric versions, can be obtained owing to special choice of mappings $\tau$ and $\sigma$.

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## 1 Introduction

The term "algebraic analysis" is used by many authors to indicate an algebraic approach to analytic problems and, in fact, it is used in many different senses. In the present paper this term we use in the sense of D. Przeworska-Rolewicz [8] since our main interest here is the calculus of right invertible operators. The examples of such operators can be the usual derivative $\frac{d}{dx}$ as well as the divided difference operators studied in quantum calculus [5]. In Section 4 we interpret the quantum $h$- and $q$-definite integrals within the algebraic analysis framework. Then, in Section 6 some more general setting is proposed. Namely, we analyze linear operators defined on function (commutative) algebras and satisfying certain product rules being the modified versions of the Leibniz rule. Such operators have many properties which are quite analogous with the corresponding ones for differential operators. In parallel, there is a natural possibility to define some kind of algebraic integration associated with right invertible operators. The algebraic concept of definite integration, with respect to a given right invertible linear operator, has been defined by using initial operators within the algebraic analysis framework [8].

For the reader's convenience, below we present some basics of algebraic analysis and quantum calculus.

## 2 Calculus of right invertible operators

Quantum calculus is in fact a sort of a discrete calculus, in which some discrete versions of differentiation and integration are studied. In the present paper we are going to compare the proposal of quantum calculus integration with the corresponding general idea of integration based on the calculus of right invertible operators. For the reader's convenience we give a short survey of the basic concepts concerning the right invertible operators but the comprehensive treatment of the topic one can find in Reference [8].

Let $X$ be a linear space over $\mathbb{R}$ and $L(X)$ be the family of all linear operators in $X$ with the domains being linear subspaces of $X$. Then, for any $A \in L(X)$, let $\mathcal{D}_A$ denote the domain of $A$ and let $L_0(X) = \{A \in L(X) : \mathcal{D}_A = X\}$. By the space of constants of an operator $D \in L(X)$ we shall mean the set $Z_D = ker D$. A linear operator $D \in L(X)$ is said to be right invertible if $DR = I$, for some linear operator $R \in L_0(X)$ called a right inverse of $D$ and $I = id_X$. The family of all right invertible operators in $X$ will be denoted by $\mathcal{R}(X)$. In turn, by $\mathcal{R}_D = \{R_\gamma\}_{\gamma\in\Gamma}$ we shall denote the family of all right inverses of a given $D \in \mathcal{R}(X)$. If $R \in \mathcal{R}_D$ is a given right inverse of $D \in \mathcal{R}(X)$, the family $\mathcal{R}_D$ is characterized by

$$\mathcal{R}_D = \{R + (I - RD)A : A \in L_0(X)\}. \tag{2.1}$$

Consider a family of right invertible operators $D_i \in \mathcal{R}(X)$ and a corresponding family of their right inverses $R_i \in \mathcal{R}_D$, for $i = 1, \dots, n$ and some $n \in \mathbb{N}$. Then, the composition $D = D_1 \dots D_n$ is right invertible, i.e. $D \in \mathcal{R}(X)$, and one of its right inverses $R \in \mathcal{R}_D$ is given by

$$R = R_n \dots R_1 \ . \tag{2.2}$$

For any $x, y \in X$, we say that $y$ is a primitive element of $x$ whenever $Dy = x$. Thus, the element $Rx$ is a primitive element of $x$, for any $x \in X$ and $R \in \mathcal{R}_D$. The set

$$\mathcal{I}(x) = \{y \in X : Dy = x\} \tag{2.3}$$

is called the indefinite integral of a given $x \in X$. One can easily check, that

$$\mathcal{R}_D x = \{Rx + (I - RD)Ax : A \in L_0(X)\} = \mathcal{R}_D x + Z_D = Rx + Z_D, \tag{2.4}$$

for any $R \in \mathcal{R}_D$ and any non-zero element $x \in X$. Hence, we obtain

$$\mathcal{I}(x) = \mathcal{R}_D x + Z_D = Rx + Z_D, \tag{2.5}$$

for any $x \in X$ and $R \in \mathcal{R}_D$.

Any projection operator $F \in L_0(X)$ onto $Z_D$, i.e. $F^2 = F$ and $ImF = Z_D$, is said to be an initial operator induced by $D \in \mathcal{R}(X)$ and the family of all such operators we denote by $\mathcal{F}_D$. For an initial operator $F$ and $x \in X$, the element $Fx \in Z_D$ is called the initial value of $x$. Additionally, we say that an initial operator $F \in \mathcal{F}_D$ corresponds to $R \in \mathcal{R}_D$ if $FR = 0$ or equivalently if

$$F = I - RD, \tag{2.6}$$

on the domain of $D$. The two families $\mathcal{R}_D$ and $\mathcal{F}_D$ uniquely determine each other. Indeed, for any $R \in \mathcal{R}_D$ we define $F = I - RD \in \mathcal{F}_D$. On the other hand, for any $F \in \mathcal{F}_D$, we define $R = R_1 - FR_1$, where $R_1 \in \mathcal{R}_D$ can be any since the result is independent of the choice of $R_1$. Thus, for any $\gamma \in \Gamma$ we have $F_\gamma = I - R_\gamma D$ and consequently $\mathcal{F}_D = \{F_\gamma\}_{\gamma\in\Gamma}$ .

By a simple calculation one can verify that $F_\alpha F_\beta = F_\beta$ and $F_\beta R_\alpha = R_\alpha - R_\beta$, for any $\alpha, \beta \in \Gamma$. Hence, for any indices $\alpha, \beta, \gamma \in \Gamma$, there is

$$F_\beta R_\gamma - F_\alpha R_\gamma = F_\beta R_\alpha, \tag{2.7}$$

which means that in fact the left side of equation (2.7) is independent of $\gamma$. The last property allows one to define the following definite integration operator

$$\mathcal{I}_\alpha^\beta = F_\beta R_\gamma - F_\alpha R_\gamma, \tag{2.8}$$

for any $\alpha, \beta, \gamma \in \Gamma$. Amongst many properties of the operator $\mathcal{I}_\alpha^\beta$ we can mention the most intuitive one, namely

$$\mathcal{I}_\alpha^\beta D = F_\beta - F_\alpha \ . \tag{2.9}$$

Hence, for any $x \in X$ and its arbitrary primitive element $y \in X$, i.e. $Dy = x$, we obtain

$$\mathcal{I}_\alpha^\beta x = F_\beta y - F_\alpha y \ , \tag{2.10}$$

which is called the definite integral of $x$.

To intuitively demonstrate the basic concepts of algebraic analysis, we end this section with two important examples. In the first example we take the usual derivative operator $D = \frac{d}{dx}$ while in the second example we consider $D = D_h$ being the following difference operator defined by

$$D_h f(x) = \frac{f(x+h) - f(x)}{h} \ , \tag{2.11}$$

and giving rise to h-quantum calculus.

**Example 1.1** Assume the linear space $X = C^0(\mathbb{R})$ (all continuous real functions) and $D = \frac{d}{dx}$. Then we recognize the domain $\mathcal{D}_D = C^1(\mathbb{R})$ (all real functions having continuous derivative) and the set (linear subspace) of all constants of $D$ is $Z_D = \{f \in X \ : \ f$ is a constant function$\}$. Since $Z_D$ is a 1-dimensional linear space over $\mathbb{R}$, we shall assume the identification $Z_D \equiv \mathbb{R}$. Thus, the initial operators $F$ in this example are projections of $X$ onto $\mathbb{R}$. To see why the name "initial operator" is intuitively consonant, it is enough to notice that $F_a : X \ni f \mapsto f(a) \in Z_D \equiv \mathbb{R}$, $a \in \mathbb{R}$, is a projection operator associating with any $f$ its value at $a$. Obviously, $\{F_a : a \in \mathbb{R}\} \subset \mathcal{F}_D$ but $\{F_a : a \in \mathbb{R}\} \neq \mathcal{F}_D$. The reason is that any convex combination of initial operators is again an initial operator [8]. For example, one can easily check that $F_{ab} = \frac{1}{2}(F_a + F_b) \in \mathcal{F}_D$ and $F_{ab} \neq F_c$, if $a \neq b$, for any $a, b, c \in \mathbb{R}$. Therefore, although $\mathcal{F}_D$ can be viewed as an indexed family, i.e. $\mathcal{F}_D = \{F_\gamma\}_{\gamma \in \Gamma}$, we cannot naturally identify $\Gamma$ with $\mathbb{R}$. As an example of a right inverse of $D = \frac{d}{dx}$ we can take $R : X \to C^1(\mathbb{R})$, such that $R_a f(x) = \int\limits_a^x f(t)dt$, for a fixed $a \in \mathbb{R}$. Let us notice that $F_a$ is the initial operator corresponding to $R_a$, for any $a \in \mathbb{R}$.

**Example 1.2** Let $X = \mathbb{R}^{\mathbb{R}}$ be the linear space of all real functions and consider $D$ defined by formula (2.11), for a fixed $h > 0$. Evidently, the linear space $Z_D$ consists of all $h$-periodic functions. Then, the operator $R$ defined by

$$Rf(x) = \begin{cases} -h \sum\limits_{m=0}^{-\left\lfloor \frac{x}{h} \right\rfloor - 1} f(x + mh) & x \in (-\infty, 0) \\ 0 \qquad\qquad \text{for} & x \in [0, h) \\ h \sum\limits_{m=1}^{\left\lfloor \frac{x}{h} \right\rfloor} f(x - mh) & x \in [h, \infty) \end{cases} \tag{2.12}$$

fulfils the condition $DR = I$ and therefore it is a right inverse of $D$. In the above formula the floor brackets $\lfloor \cdot \rfloor$ stand for the integer value function of its argument. Then, let us define the operator $F$ by formula

$$Ff(x) = f(x - \left\lfloor \frac{x}{h} \right\rfloor h), \tag{2.13}$$

for any $x \in \mathbb{R}$. Since $Ff(x + h) = Ff(x)$, for any $x \in \mathbb{R}$, the function $Ff$ is $h$-periodic, i.e. $Ff \in Z_D$. On the other hand, for any function $f \in Z_D$, there is $Ff = f$. Hence, the operator $F$ is a projection of $X$ onto $Z_D$ and therefore it is an initial operator induced by $D = D_h$. Moreover, one can check the property (2.6) which means that $F$ is the initial operator corresponding to $R$ given by (2.12). By using formula (2.1), the family $\mathcal{R}_D$ is fully determined by the above operator $R$, then with the help of (2.6) we obtain the family $\mathcal{F}_D$.

# 3 Quantum h- and q-calculus

In this section we briefly recall the main elements of quantum calculus but more detailed study of the topic, motivation and many properties reflecting the analogies with the usual differential calculus the reader will find in [5]. For the history of q-calculus, its relation to other mathematical and physical areas and the imposing list of references we recommend [1].

Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function and consider the well known difference quotient

$$\frac{f(x) - f(x_0)}{x - x_0} \,, \tag{3.1}$$

for some $x \neq x_0$. The limit of the last expression when $x \to x_0$, if it exists, defines the derivative of $f$ at $x_0$. Now, if we take $x = x_0 + h$ for a fixed $h \neq 0$ or $x = qx_0$ for a fixed $q \neq 1$ and do not take the corresponding limit, we enter the so-called quantum $h$- or $q$-calculus. For any $f : \mathbb{R} \to \mathbb{R}$ one defines its $h$-differential $d_h f$

$$d_h f(x) = f(x + h) - f(x), \tag{3.2}$$

and its $q$-differential $\delta_q f$

$$\delta_q f(x) = f(qx) - f(x). \tag{3.3}$$

In particular, for the identity mapping $e$ defined on $\mathbb{R}$, i.e. $e(x) \equiv x$, we have $d_h e(x) = h$ and $\delta_q e(x) = (q-1)x$. Quite commonly the simplified notation is used, i.e. $d_h e(x) \equiv d_h x$ and $\delta_q e(x) = \delta_q x$. In applications, the two versions of quantum calculus (i.e. $h$- or $q$-calculus) are considered separately, which allows one to denote both differentials by the same symbol, i.e. one can write $d_h$ or $d_q$ (instead $\delta_q$) and recognize them from context. The above two symbols $d_h$, $\delta_q$ can be viewed as the linear operators $d_h : f \mapsto d_h f$ and $\delta_q : f \mapsto \delta_q f$ defined on some $\mathbb{R}$-algebra $\mathcal{A}$ of real functions. However, the algebra $\mathcal{A}$ should be invariant w.r.t. the h- or q-shifts, i.e. functions $x \mapsto f(x + h)$ or $x \mapsto f(qx)$ should be the elements of $\mathcal{A}$, for any $f \in \mathcal{A}$.

One can easily verify the following Leibniz product rules

$$d_h(fg)(x) = d_h(f)(x)g(x + h) + f(x)d_h(g)(x), \tag{3.4}$$

and similarly

$$\delta_q(fg)(x) = \delta_q(f)(x)g(qx) + f(x)\delta_q(g)(x). \tag{3.5}$$

The above Leibniz formulae define the corresponding classes ($\mathcal{A}$-modules) of linear difference-like operators, defined on some $\mathbb{R}$-algebra $\mathcal{A}$ of functions.

Evidently, the above product rules are also fulfilled by operators $D_h$ and $\Delta_q$, called the quantum derivatives and defined as

$$D_h(f)(x) = \frac{d_h f(x)}{d_h e(x)} \equiv \frac{d_h f(x)}{d_h x} \tag{3.6}$$

and similarly

$$\Delta_q(f)(x) = \frac{\delta_q f(x)}{\delta_q e(x)} \equiv \frac{\delta_q f(x)}{\delta_q x} \,. \tag{3.7}$$

**Remark:** Since $\delta_q e(0) = 0$, the expression $\Delta_q(f)(x)$ is not defined at $x = 0$ unless $f'(0)$ does exist. Therefore, the $q$-calculus can be developed in algebras $\mathcal{A}$ of functions defined on $\mathbb{R} \setminus \{0\}$ or in algebras of functions defined on $\mathbb{R}$ and differentiable at $x = 0$.

In $h$-calculus, an $h$-antiderivative of a function $f : \mathbb{R} \to \mathbb{R}$ is defined to be any function $g : \mathbb{R} \to \mathbb{R}$ such that $D_h g(x) = f(x)$, for any $x \in \mathbb{R}$. The family of all $h$-antiderivatives of a given function $f$ is called the indefinite $h$-integral and is denoted by

$$\int f(x) d_h x \,. \tag{3.8}$$

Then, the definite $h$-integral is defined by formula

$$\int\limits_a^b f(x) d_h x = \begin{cases} h\left(f(a) + f(a+h) + \ldots + f(b-h)\right) & if \quad a < b \\ 0 & if \quad a = b \,, \\ -h\left(f(b) + f(b+h) + \ldots + f(a-h)\right) & if \quad a > b \end{cases} \tag{3.9}$$

for any $a, b \in \mathbb{R}$, such that $a$ and $b$ differ by an integer multiple of $h$.

Concerning $q$-calculus, in this paper we shall assume $q \in (0,1) \cup (1,\infty)$. This restriction follows from the physical motivation that the two quantum parameters are usually related by $q = e^h$. The last exponential relation transforms the real line $\mathbb{R}$ onto $\mathbb{R}_+ = (0,\infty)$. Consequently, the $h$-calculus for functions defined on $\mathbb{R}$ and the $q$-calculus for functions defined on $\mathbb{R}_+$ can be unified within a more general framework (generalized quantum calculus). A $q$-antiderivative of a function $f$ is said to be any function $g$ such that $\Delta_q g(x) = f(x)$. A special $q$-antiderivative, the so-called Jackson integral of $f$, is formally derived in [5] as the geometric series expansion

$$g(x) = (1-q)x \sum_{m=0}^{\infty} q^m f(xq^m). \tag{3.10}$$

Then, formula (3.10) is used to define

$$\int\limits_0^b f(x) d_q x = (1-q)b \sum_{m=0}^{\infty} q^m f(bq^m), \tag{3.11}$$

and finally define the definite $q$-integral [4]

$$\int\limits_a^b f(x) d_q x = \int\limits_0^b f(x) d_q x - \int\limits_0^a f(x) d_q x \,, \tag{3.12}$$

for $0 < a < b$. Since formula (3.10) has been derived formally, one needs to examine the conditions when it converges to a $q$-antiderivative. Within the algebraic analysis framework, used in this paper, we construct many $q$-antiderivatives which are finite sums and no condition has to be examined to justify their convergency. However, the above Jackson integral can be recovered in this approach provided the corresponding infinite expansion is convergent.

At the end of this section let us briefly discuss the lack of symmetry one can notice concerning the product rules (3.4), (3.5).

On the strength of formulae (3.4), (3.5), for any $a, b \in \mathbb{R}$ such that $a+b=1$, one can write the following combinations

$$d_h(fg)(x) = d_h(f)(x)(ag(x) + bg(x+h)) + (bf(x) + af(x+h))d_h(g)(x), \tag{3.13}$$

and analogously

$$\delta_q(fg)(x) = \delta_q(f)(x)(ag(x) + bg(xq)) + (bf(x) + af(xq))\delta_q(g)(x). \tag{3.14}$$

If $a \neq b$, the above combined formulae (3.13), (3.14) are equivalent with (3.4) and (3.5), respectively, i.e. the corresponding classes of operators defined coincide. On the other hand, for $a = b = \frac{1}{2}$ the corresponding symmetric product rule defines a larger class of operators, in general. However, for some algebras the symmetric rule can be equivalent with its all non-symmetric counterparts. A non-trivial example of an algebra, for which the symmetric product rule implies all the other ones, is the algebra of polynomials $\mathcal{A} = \mathbb{R}[x]$. Indeed, assume $h = 1$, $a = b = \frac{1}{2}$ and consider the symmetric product rule

$$D(fg)(x) = D(f)(x) \cdot \frac{g(x) + g(x+1)}{2} + \frac{f(x) + f(x+1)}{2} \cdot D(g)(x)\ , \tag{3.15}$$

for any $f, g \in \mathbb{R}[x]$. One can prove that $D = D(1_{\mathcal{A}}) \cdot d_1$, where $d_1 f(x) = f(x+1) - f(x)$. Hence, any operator $D$ is proportional to the usual difference operator $d_1$ and consequently it fulfills (non-symmetric) formula (3.4). In turn, an algebra for which the symmetric product rule (3.15) is weaker than any non-symmetric one is for example the $\mathbb{R}$-algebra (of real functions) $\mathcal{A} = gen(\{e, z\})$ generated by the identity $e(x) = x$ and the integer valued function $z(x) = \lfloor x \rfloor$.

# 4 An algebraic analysis approach to quantum integration

In this section we present an approach to quantum integration within the algebraic analysis framework [8].

Let us construct the following right inverses $R_{hs}$, $s \in \mathbb{R}$. Namely, for $h < 0$

$$R_{hs}f(x) = \begin{cases} \displaystyle\sum_{m=1}^{-\lfloor \frac{x-s}{-h} \rfloor} hf(x - mh) & x \in (-\infty, s) \\ 0 \qquad\qquad\qquad \text{for} & x \in [s, s-h) \\ \displaystyle -\sum_{m=0}^{\lfloor \frac{x-s}{-h} \rfloor - 1} hf(x + mh) & x \in [s-h, \infty) \end{cases} \tag{4.1}$$

and for $h > 0$

$$R_{hs}f(x) = \begin{cases} \displaystyle -\sum_{m=0}^{-\lfloor \frac{x-s}{h} \rfloor - 1} hf(x + mh) & x \in (-\infty, s) \\ 0 \qquad\qquad\qquad \text{for} & x \in [s, s+h) \\ \displaystyle\sum_{m=1}^{\lfloor \frac{x-s}{h} \rfloor} hf(x - mh) & x \in [s+h, \infty) \end{cases} \tag{4.2}$$

By a straightforward calculation one can show that $D_h R_{hs} = I$, for any $h \neq 0$ and $s \in \mathbb{R}$.

Then, on the strength of formula (2.5), the indefinite $h$-integral of a function $f \in \mathbb{R}^{\mathbb{R}}$ can be written as

$$\int f(x) d_h x = R_{hs} f(x) + Z_{D_h} \equiv R_{h0} f(x) + Z_{D_h}, \tag{4.3}$$

where $s \in \mathbb{R}$ is an arbitrarily fixed index, e.g. $s = 0$, and the notation (3.8) was used.

Define the operators $F_{hs}$ by

$$F_{hs} f(x) = f(x - \left\lfloor \frac{x-s}{|h|} \right\rfloor \cdot |h|), \tag{4.4}$$

for any $s \in \mathbb{R}$.

One can verify that $F_{hs}$ are the initial operators induced by $D_h$, for any $s \in \mathbb{R}$. Indeed, the function $F_{hs} f$ is $h$-periodic, since

$$F_{hs} f(x+h) = f(x + h - \left\lfloor \frac{x+h-s}{|h|} \right\rfloor \cdot |h|) =$$

$$= f(x + h - \left( \left\lfloor \frac{x-s}{|h|} \right\rfloor + \frac{h}{|h|} \right) \cdot |h|) = F_{hs} f(x).$$

Moreover, for any $h$-periodic function $f$, we have the evident identity $F_{hs} f = f$, which proves that $F_{hs}$ is a surjective projection onto $Z_{D_h}$. Therefore $F_{hs}$ is an initial operator induced by $D_h$, for any $s \in \mathbb{R}$. One can also verify that the initial operators $F_{hs}$ correspond to (4.1) and (4.2), respectively.

In turn, the definite $h$-integrals are defined in a general manner by using formula (2.8). Within this approach we obtain a large class of definite $h$-integrals, with the integration limits being arbitrary (indices of) initial operators. Below we consider definite $h$-integrals determined by the (particular) initial operators $F_{hs}$, for $s \in \mathbb{R}$. As we shall see, these integrals can be used to obtain the ordinary $h$-definite integrals defined by formula (3.9). Namely, by formula (2.8), for any $a, b \in \mathbb{R}$ and the corresponding (particular) initial operators $F_{ha}$, $F_{hb}$ we obtain

$$\mathcal{I}_a^b = F_{hb} R_{h0} - F_{ha} R_{h0}. \tag{4.5}$$

The concrete right inverse $R_{h0}$, for $s = 0$, is used above only for simplicity since the result is independent of this choice. Assume $h > 0$ and calculate the definite integral of a function $f$ at $x$, i.e.

$$\mathcal{I}_a^b f(x) = R_{h0} f(x - \left\lfloor \frac{x-b}{h} \right\rfloor \cdot h) - R_{h0} f(x - \left\lfloor \frac{x-a}{h} \right\rfloor \cdot h). \tag{4.6}$$

Then, the ordinary $h$-definite integrals, defined intuitively in [5], are obtained here as the value of $\mathcal{I}_a^b f(x)$ at any point $x = a + kh$, $k \in \mathbb{Z}$. Indeed, assume $0 < k \in \mathbb{Z}$ and $b = a + kh$. Then, for $x = a$ we obtain

$$\mathcal{I}_a^b f(a) = R_{h0} f(a + kh) - R_{h0} f(a) =$$

$$= \sum_{j=0}^{k-1} (R_{h0} f(a + (j+1)h) - R_{h0} f(a + jh)) = \sum_{j=0}^{k-1} h D_h R_{h0} f(a + jh) =$$

$$= \sum_{j=0}^{k-1} h f(a + jh) = h(f(a) + f(a+h) + \ldots + f(b-h)).$$

If $a = b$, the result is obviously $\mathcal{I}_a^b f(a) = 0$. In turn, for $a > b$ and $a = b + kh$, for some $0 < k \in \mathbb{Z}$, we have

$$\mathcal{I}_a^b f(b) = R_{h0} f(b) - R_{h0} f(b + kh) =$$

$$= \sum_{j=0}^{k-1} (R_{h0} f(b + jh) - R_{h0} f(b + (j+1)h)) = - \sum_{j=0}^{k-1} h D_h R_{h0} f(b + jh) =$$

$$- \sum_{j=0}^{k-1} h f(b + jh) = -h(f(b) + f(b+h) + \ldots + f(a-h)).$$

The above calculation demonstrates how the ordinary $h$-definite integrals, defined by (3.9), emerge from the algebraic analysis approach used here.

Directly from definition of the initial operator concept and from (4.5), we conclude that $\mathcal{I}_a^b f \in Z_D$, i.e. it is an $h$-periodic function and $\mathcal{I}_a^b f(x) = \mathcal{I}_a^b f(a)$, for any $x \in a + h\mathbb{Z}$. Let us emphasize the conceptual difference between definitions (3.9) and (4.5). In $h$-calculus, by formula (3.9) one defines the definite integral to be a scalar-valued linear functional, while in the algebraic analysis approach the corresponding definite integral value is an $h$-periodic function (non-constant, in general). The above two formulations of definite integrals are equivalent for functions defined on the domain $a + h\mathbb{Z}$, for some fixed $a, h \in \mathbb{R}$.

**Remark:** Imagine that an action functional of a physical system is defined as an $h$-integral of some lagrangian. Consequently, such an action is $h$-periodic and its $h$-periodicity can be viewed as a physical symmetry giving rise to a corresponding conservation law.

Concerning $q$-calculus, we shall work here with functions $f$ defined on the domain $(0, +\infty)$ and $q \in (0, 1) \cup (1, +\infty)$. By analogy with the above right inverse operators $R_{hs}$ we first construct the operators $\rho_{qs}$, where $s \in (0, +\infty)$, being the (particular) right inverses of $\delta_q$. Then, we define the corresponding (particular) right inverses $P_{qs}$ of $\Delta_q$. Namely, for $q \in (0, 1)$ we have

$$\rho_{qs} f(x) = \begin{cases} \sum_{m=1}^{-\lfloor log_q \frac{s}{x} \rfloor} f(xq^{-m}) & x \in (0, s) \\ 0 & \text{for} \quad x \in [s, sq^{-1}) \ , \\ - \sum_{m=0}^{\lfloor log_q \frac{s}{x} \rfloor - 1} f(xq^m) & x \in [sq^{-1}, \infty) \end{cases} \tag{4.7}$$

and for $q \in (1, \infty)$ we have

$$\rho_{qs} f(x) = \begin{cases} -\sum_{m=0}^{-\lfloor log_q \frac{x}{s} \rfloor - 1} f(xq^m) & x \in (0, s) \\ 0 \qquad\qquad \text{for} & x \in [s, sq) \\ \sum_{m=1}^{\lfloor log_q \frac{x}{s} \rfloor} f(xq^{-m}) & x \in [sq, \infty) \end{cases} . \tag{4.8}$$

One can easily verify that $\delta_q \rho_{qs} = I$, for any $s \in (0, +\infty)$. Now, to find the right inverses $P_{qs}$ of the divided difference operator $\Delta_q$, defined by (3.7), we can write

$$\Delta_q = T_q^{-1} \circ \delta_q, \tag{4.9}$$

where $T_q$ is the invertible operator defined as

$$T_q f(x) = (q-1) x f(x), \tag{4.10}$$

and apply formula (2.2), i.e. $P_{qs} = \rho_{qs} \circ T_q$. For $q \in (0, 1)$ the result is

$$P_{qs} f(x) = \begin{cases} \sum_{m=1}^{-\lfloor log_q \frac{s}{x} \rfloor} (q-1) x q^{-m} f(xq^{-m}) & x \in (0, s) \\ 0 \qquad\qquad \text{for} & x \in [s, sq^{-1}) \\ -\sum_{m=0}^{\lfloor log_q \frac{s}{x} \rfloor - 1} (q-1) x q^m f(xq^m) & x \in [sq^{-1}, \infty) \end{cases} , \tag{4.11}$$

and for $q \in (1, \infty)$ there is

$$P_{qs} f(x) = \begin{cases} -\sum_{m=0}^{-\lfloor log_q \frac{x}{s} \rfloor - 1} (q-1) x q^m f(xq^m) & x \in (0, s) \\ 0 \qquad\qquad \text{for} & x \in [s, sq) \\ \sum_{m=1}^{\lfloor log_q \frac{x}{s} \rfloor} (q-1) x q^{-m} f(xq^{-m}) & x \in [sq, \infty) \end{cases} . \tag{4.12}$$

Although a single right inverse operator can generate all the other ones by formula (2.1), the right inverses $P_{qs}$ can be used to reach certain $q$-antiderivative, the so-called Jackson integral, being an infinite series, derived formally in [5]. From this approach it becomes clear that Jackson integral is not the only $q$-antiderivative existing and even though it is divergent for certain function $f$, we can still work with other $q$-antiderivatives of $f$, well defined by the finite sums, which are never threatened by a divergency problem.

Namely, in the lower part of formula (4.11) we put $s \to 0$ and obtain Jackson integral, being the series

$$\int f(x) d_q x = (1-q) x \sum_{m=0}^{\infty} q^m f(xq^m), \tag{4.13}$$

for $x \in (0, +\infty)$.

As a next step we formulate definite integrals in terms of algebraic analysis and compare them with definite $q$-integrals originally defined in $q$-calculus. In analogy to formula (4.4) let us consider the operators $G_a$ defined by

$$G_a f(x) = f(xq^{-\lfloor log_q \frac{x}{a} \rfloor}), \tag{4.14}$$

for any function $f : (0, \infty) \to \mathbb{R}$ and $a \in (0, \infty)$. Evidently, for $a \in (0, \infty)$, operators $G_a$ are surjective onto the family of all q-periodic functions defined on $(0, \infty)$. One can also verify the property $G_a^2 = G_a$, for any $a \in (0, \infty)$. Therefore the operators $G_a$ are the initial operators induced by the operator $\Delta_q$, for any $a \in (0, \infty)$.

Now, according to formula (2.8), we obtain a $q$-definite integral determined by the initial operators $G_a$ and $G_b$

$$\mathcal{I}_a^b = G_b P_{qs} - G_a P_{qs}, \tag{4.15}$$

for any $a, b, s \in (0, +\infty)$ (the above result is independent of $s$).

In order to interpret formula (3.12) within this framework, for any $a, b \in (0, +\infty)$, we should take $q \in (0, 1)$ and sufficiently big positive $s$ for which $a, b \in [q^{-s+1}, +\infty)$, since the last interval corresponds with $(0, +\infty)$ when $s \to +\infty$. Assume $a < b = aq^k$, for some $0 > k \in \mathbb{Z}$ and calculate

$$\mathcal{I}_a^b f(a) = G_b P_{qs} f(a) - G_a P_{qs} f(a) = P_{qs} f(aq^{-\lfloor log_q \frac{a}{b} \rfloor}) - P_{qs} f(aq^{-\lfloor log_q \frac{a}{a} \rfloor}) =$$

$$= P_{qs} f(aq^k) - P_{qs} f(a) = (1-q)aq^k \sum_{m=0}^{\lfloor log_q \frac{1}{aq^k} \rfloor - 1 + s} q^m f(aq^k q^m) -$$

$$-(1-q)aq^k \sum_{m=0}^{\lfloor log_q \frac{1}{a} \rfloor - 1 + s} q^m f(aq^m) = (1-q)a \sum_{m=k}^{-1} q^m f(aq^m).$$

On the other hand, from formula (3.12) we obtain

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x =$$

$$= (1-q)b \sum_{m=0}^{\infty} q^m f(bq^m) - (1-q)a \sum_{m=0}^{\infty} q^m f(aq^m) =$$

$$= (1-q)a \sum_{m=k}^{-1} q^m f(aq^m),$$

which coincides with the previous result. Let us notice that formula (3.12) defines $q$-definite integral provided the Jackson $q$-antiderivative is a convergent series. A simple example of a function $f$, for which such a formulation of a definite $q$-integral cannot be applied is $f(x) = \frac{1}{x}$, for which Jackson $q$-antiderivative is evidently divergent. But fortunately, according to formula (2.8), a definite integral depends only on the initial operators and is completely independent of a particular choice of a right inverse used in the calculation. Therefore, divergency of Jackson integral merely means that this particular $q$-antiderivative cannot be used in the calculation of a given $q$-definite integral.

Let us end this section with the example of a definite $q$-integral for the above mentioned function $f(x) = \frac{1}{x}$, where we assume $q \in (0,1)$ and $0 < a < b = aq^k$, for some negative $k \in \mathbb{Z}$. We obtain

$$\int_a^b \frac{1}{x}\, d_q x = (1-q)a \sum_{m=k}^{-1} q^m \frac{1}{aq^m} = (1-q)\cdot(-k) = (1-q) log_q \frac{a}{b}\ .$$

An interesting observation is that the above definite $q$-integral depends only on the ratio of its limits $a$ and $b$.

# 5 Tension spaces

The usual quantum calculus, i.e. $h$- or $q$-calculus [5], is based on very special difference and divided difference operators. As one can easily notice, formulae (3.2), (3.3) can be realized for functions defined on an arbitrary set $M$ while there arises a problem with formulae (3.6), (3.7) since the differences appeared in the corresponding denominators are undefined unless $M$ is equipped with the usual algebraic structure. In order to avoid that problem we propose here to study more general formulation of quantum calculus in a tension space $(M, \theta)$.

Let $M \neq \emptyset$ and assume the following definition.

**Definition 5.1.** *By a tension function on $M$ we understand any function $\theta : M \times M \to \mathbb{R}$ such that*

$$\theta(p_1, p_2) + \theta(p_2, p_3) = \theta(p_1, p_3), \tag{5.1}$$

*for any $p_1, p_2, p_3 \in M$.*

Directly from the above definition, we can prove that any tension function is skew symmetric, i.e. for any $p_1, p_2 \in M$ there is

$$\theta(p_1, p_2) = -\,\theta(p_2, p_1). \tag{5.2}$$

**Definition 5.2.** *By a tension space we shall mean a pair $(M, \theta)$, where $M \neq \emptyset$ and $\theta$ is a tension function on $M$.*

In this paper we shall assume that $(M, \theta)$ is a nontrivial tension space, i.e. there exist points $p, q \in M$ for which

$$\theta(p, q) \neq 0. \tag{5.3}$$

**Remark:** One can easily check that a linear combination of tension functions on $M$ is a tension function again. Consequently, any family $\{\theta^j\}_{j\in J}$ of tension functions on $M$ generates the linear space $L = Lin(\{\theta^j\}_{j\in J})$, the so-called tension structure on $M$. Then, by a (multidimensional) tension space we can understand the pair $(M, L)$. However, in this paper we consider only a tension space $(M, \theta)$ defined by a single tension function $\theta$.

With a tension function $\theta$ we shall associate the equivalence relation in $M$ defined by the formula

$$p \sim q \quad \text{iff} \quad \theta(p, q) = 0. \tag{5.4}$$

Then the equivalence classes of this relation are the following

$$[p\,] = \{q \in M : \ \theta(p, q) = 0\}. \tag{5.5}$$

One can easily check that the function $\hat{\theta}$ given by

$$\hat{\theta}([p\,], [q\,]) = \theta(p, q), \tag{5.6}$$

for $p, q \in M$, is a well defined tension function on the quotient set $\hat{M} \equiv M/\sim$. Thus we have constructed the "effective" tension space $(\hat{M}, \hat{\theta})$.

On the quotient set $\hat{M} = M/\sim$ we have the natural linear ordering relation

$$[p\,] \preceq [q\,] \quad \text{iff} \quad \hat{\theta}([p\,], [q\,]) \leq 0. \tag{5.7}$$

We shall also write $[p\,] \prec [q\,]$ whenever $[p\,] \preceq [q\,]$ and simultaneously $[p\,] \neq [q\,]$.

Then, there is a natural metric $g_\theta$ defined on $\hat{M}$ by

$$g_\theta([p\,], [q]) = |\,\hat{\theta}([p\,], [q])|\,, \tag{5.8}$$

for any $p, q \in M$.

In the sequel we will often use mappings $\theta_q : M \to \mathbb{R}$ defined by

$$\theta_q(p) = \theta(p, q), \tag{5.9}$$

for any $p, q \in M$. One can easily verify that $\theta_{q_1} = \theta_{q_2}$, whenever $q_1 \sim q_2$. Intuitively, the mapping $\theta_q$ we can interpret as a potential function defined on $M$, associating a scalar potential $\theta_q(p)$ with any point $p \in M$ and such that $\theta_q(q) = 0$ at $q \in M$.

**Definition 5.3.** *A mapping $\tau : M \to M$ is said to be rightward $\theta$-directed if*

$$[p\,] \prec [\tau(p)\,]\,, \tag{5.10}$$

*and it is said to be leftward $\theta$-directed if*

$$[\tau(p)\,] \prec [p\,]\,, \tag{5.11}$$

*for any $p \in M$. We say that $\tau$ is a $\theta$-directed mapping if it is either rightward or leftward $\theta$-directed mapping.*

Assume the notation: $\tau^0 = id_M$ and $\tau^n = \tau \circ \tau^{n-1}$, for any $n \in \mathbb{N}$.

**Proposition 5.4.** *For any $\theta$-directed mapping $\tau : M \to M$ and any $n \in \mathbb{N}$, the composition $\tau^n$ has no fixed points, i.e.*

$$\tau^n(p) \neq p\,, \tag{5.12}$$

*for $p \in M$.*

**Proof:** Let $\tau$ be a rightward $\theta$-directed mapping. Then we have inequalities $\theta(\tau(p), p) > 0$, ... , $\theta(\tau^n(p), \tau^{n-1}(p)) > 0$, for any $n \in \mathbb{N}$ and $p \in M$. Consequently,

$$\theta(\tau^n(p), p) = \theta(\tau^n(p), \tau^{n-1}(p)) + \ldots + \theta(\tau(p), p) > 0\,.$$

Analogously, for a leftward $\theta$-directed mapping we show that $\theta(\tau^n(p), p) < 0$, for any $n \in \mathbb{N}$ and $p \in M$. □

Let us notice that condition (5.12) is not a consequence of the weaker assumption that $\theta(\tau(p), p) \neq 0$, for any $p \in M$. In that case there would be $\tau(p) \neq p$ but not necessarily $\tau^n(p) \neq p$, for any $n \in \mathbb{N}$ and $p \in M$.

**Definition 5.5.** *We say that $\theta$ is homogeneous with respect to $\tau$ (shortly, $\tau$-homogeneous) if there exists $t \in \mathbb{R}$, the so-called $\tau$-homogeneity coefficient, such that*

$$\theta(\tau(p_1), \tau(p_2)) = t \cdot \theta(p_1, p_2), \tag{5.13}$$

*for any $p_1, p_2 \in M$.*

**Proposition 5.6.** *Let $\tau : M \to M$ be a $\theta$-directed mapping and $\theta$ be a $\tau$-homogeneous tension function. Then, for the $\tau$-homogeneity coefficient we get $t > 0$.*

**Proof:** Suppose that $t$ is a $\tau$-homogeneity coefficient for some $\tau$-homogeneous tension function $\theta$ and assume that $\tau$ is a $\theta$-directed mapping. Then $\theta(\tau^2(p), \tau(p))$ and $\theta(\tau(p), p)$ are of common sign and $\theta(\tau^2(p), \tau(p)) = t \cdot \theta(\tau(p), p)$. Directly from Definition (5.3) we get $t \neq 0$. Hence we conclude that $t > 0$. □

**Proposition 5.7.** *Let $\theta$ be $\tau$-homogeneous and $\sim$ be the equivalence relation defined by (5.4). Then, we have the implication*

$$p \sim q \quad \Rightarrow \quad \tau(p) \sim \tau(q) \tag{5.14}$$

*for any $p, q \in M$, or equivalently*

$$\tau([p]) \subset [\tau(p)], \tag{5.15}$$

*for any $p \in M$.*

**Proof:** Suppose that $p \sim q$, i.e. $\theta(p,q) = 0$. Then we have $\theta(\tau(p), \tau(q)) = = t \cdot \theta(p,q) = 0$. $\square$

In general, the inclusion (5.15) cannot be inverted, which can be confirmed by the following

**Example:** Assume $M = \mathbb{R} \times [0, +\infty)$, $\theta((x_1, y_1), (x_2, y_2)) = x_1 - x_2$ and $\tau(x, y) = (x + 1, y+1)$. Then we obtain $\tau([(x, y)]) = \{x+1\} \times [1, +\infty)$ and $[\tau(x, y)] = \{x+1\} \times [0, +\infty)$, i.e. $\tau([(x, y)]) \subsetneq [\tau(x, y)]$.

**Proposition 5.8.** *Let $\theta$ be $\tau$-homogeneous with the $\tau$-homogeneity coefficient $t \neq 0$. Then we have*

$$p \nsim q \quad \Rightarrow \quad \tau(p) \nsim \tau(q)\,, \tag{5.16}$$

*for any $p, q \in M$.*

**Proof:** $\theta(\tau(p), \tau(q)) = t \cdot \theta(p, q) \neq 0$, whenever $p \nsim q$. $\square$

**Corollary 5.9.** *Let $\theta$ be $\tau$-homogeneous with a $\tau$-homogeneity coefficient $t$. Assume that $p_0 \nsim q_0$ and $\tau(p_0) \sim \tau(q_0)$, for some $p_0, q_0 \in M$. Then $t = 0$ and consequently $\tau(p) \sim \tau(q)$, or equivalently $[\tau(p)\,] = \tau(M)$, for any $p, q \in M$.*

# 6 Quantum $(\tau, \sigma)$-calculus

Let $\sigma, \tau : M \to M$ be two commuting bijections and assume $\mathcal{A} \subset \mathbb{R}^M$ to be a $\sigma^*$, $\tau^*$-invariant $\mathbb{R}$-algebra, i.e. $\sigma^*\mathcal{A}, \tau^*\mathcal{A} \subset \mathcal{A}$.

**Definition 6.1.** *By the $(\tau, \sigma)$-quantum differential we mean the mapping $d_{\tau,\sigma} : \mathcal{A} \to \mathcal{A}$ given by*

$$d_{\tau,\sigma} f(p) = f(\tau(p)) - f(\sigma(p)), \tag{6.1}$$

*for $p \in M$.*

One can easily check that the quantum differential $d_{\tau,\sigma}$ is a linear operator and it fulfills the following Leibniz product rule

$$d_{\tau,\sigma}(f \cdot g)(p) = d_{\tau,\sigma} f(p) \cdot g(\tau(p)) + f(\sigma(p)) \cdot d_{\tau,\sigma} g(p)\,, \tag{6.2}$$

for any functions $f, g \in \mathcal{A}$ and $p \in M$.

**Definition 6.2.** *By a $(\tau, \sigma)$-quantum derivation we shall mean any linear operator $\delta : \mathcal{A} \to \mathcal{A}$ that fulfills formula (6.2).*

Since the elements $f, g \in \mathcal{A}$ commute, the following combinations are also fulfilled

$$\delta(f \cdot g)(p) = [af(\sigma(p)) + bf(\tau(p))] \cdot \delta g(p) + \delta f(p) \cdot [bg(\sigma(p)) + ag(\tau(p))]\,, \tag{6.3}$$

where $a, b \in \mathbb{R}$ are coefficients such that $a + b = 1$. If $a \neq b$, formula (6.3) is equivalent with (6.2). In turn, when $a = b = \frac{1}{2}$, formula (6.3) becomes symmetric

$$\delta(f \cdot g)(p) = H(f)(p) \cdot \delta g(p) + \delta f(p) \cdot H(g)(p)\,, \tag{6.4}$$

where $H(f)(p) = \frac{f(\sigma(p))+f(\tau(p))}{2}$. In general, formula (6.4) is weaker than (6.2) but there exist algebras $\mathcal{A}$ for which both formulae are equivalent, i.e. they define the same $\mathcal{A}$-module of linear operators (e.g. $\mathcal{A} = \mathbb{R}[x]$, compare the corresponding comment in Section 3).

**Remark:** The mapping $H : \mathcal{A} \to \mathcal{A}$, defined above, is linear and preserving the unity $1_{\mathcal{A}}$ but in general it is not an algebra homomorphism. The last defect is precisely the reason why operators defined by (6.4) are not differential operators.

Now, we assume

$$[\sigma(p)]_{\tau,\sigma} \prec [\tau(p)]_{\tau,\sigma}\,, \tag{6.5}$$

for any $p \in M$, and define the quantum $(\tau,\sigma)$-derivative operator in a tension space $(M,\theta)$.

**Definition 6.3.** *By the $(\tau,\sigma)$-quantum derivative we shall mean the mapping $D_{\tau,\sigma} : \mathcal{A} \to \mathcal{A}$ given by*

$$D_{\tau,\sigma} f(p) = \frac{d_{\tau,\sigma} f(p)}{\theta(\tau(p),\sigma(p))} \equiv \frac{d_{\tau,\sigma} f(p)}{d_{\tau,\sigma}\theta_q(p)}\,, \tag{6.6}$$

*for any $f \in \mathcal{A}$, independently of $q \in M$ .*

The assumption (6.5) prevents formula (6.6) from zero-valued denominator. However, owing to the evident symmetry $D_{\tau,\sigma} = D_{\sigma,\tau}$, all properties associated with the operator $D_{\tau,\sigma}$ remain unchanged if the direction of (6.5) is reversed. Equivalently, relation (6.5) can be formulated as

$$[p\,]_{\tau,\sigma} \prec [\tau\sigma^{-1}(p)]_{\tau,\sigma}\,, \tag{6.7}$$

for any $p \in M$. By Definition (5.3) it means that $\tau\sigma^{-1}$ is a rightward $\theta$-directed bijection. Indeed, it is enough to replace $p$ by $\sigma^{-1}(p)$ in formula (6.5) and obtain (6.7).

Evidently, the quantum derivative $D_{\tau,\sigma}$ fulfills the product rule (6.2).

In order to formulate the idea of quantum integration (or the Taylor interpolation polynomial) we shall need the right inverse operators defined for the above quantum differential (6.1) and quantum derivative (6.6).

The following definition will play an important role in our further analysis.

**Definition 6.4.** *We say that a family of subsets $M_k \subset M$, $k \in \mathbb{Z}$, is a $(\tau,\sigma)$-partition of $M \neq \emptyset$ if*

*1)* $\bigcup_{k\in\mathbb{Z}} M_k = M$,

*2)* $M_{k_1} \cap M_{k_2} = \emptyset$, *for any* $k_1 \neq k_2$,

*3)* $\tau\sigma^{-1} : M_k \to M_{k+1}$ *is a bijective mapping, for any* $k \in \mathbb{Z}$.

To shorten our notation, the circle symbol "$\circ$" is omitted for the composition of mappings above and later on.

**Proposition 6.5.** *If $M_k \subset M$, $k \in \mathbb{Z}$, is a $(\tau,\sigma)$-partition of $M \neq \emptyset$, then $M_0 \neq \emptyset$ and the composed mapping $(\tau\sigma^{-1})^m$, for any $m \in \mathbb{Z}$, has no fixed points.*

**Proof:** Suppose $M_0 = \emptyset$. Then, by condition (3) we get $M_k = \emptyset$, for all $k \in \mathbb{Z}$, which contradicts condition (1). In turn, let $(\tau\sigma^{-1})^m(p) = p$ for some $p \in M_k$ and $m \neq 0$. Then by condition (3) we obtain $p = (\tau\sigma^{-1})^m(p) \in M_k \cap M_{k+m}$ which contradicts condition (2). □

With a given $(\tau,\sigma)$-partition of $M$ we associate the following integer-valued function $\lfloor\cdot\rfloor_{\tau,\sigma} : M \to \mathbb{Z}$, defined by

$$\lfloor p \rfloor_{\tau,\sigma} = k \quad \text{iff} \quad p \in M_k, \tag{6.8}$$

for any $k \in \mathbb{Z}$. We shall omit the indices and write $\lfloor\cdot\rfloor$ whenever $\tau$ and $\sigma$ are fixed. Automatically, for any $p \in M$, from the above formula we conclude

$$p \in M_{\lfloor p \rfloor}\ . \tag{6.9}$$

**Proposition 6.6.** *For any $p \in M$ there is*

$$\lfloor \tau\sigma^{-1}(p) \rfloor = \lfloor p \rfloor + 1. \tag{6.10}$$

**Proof.** Let $\lfloor p \rfloor = k$, i.e. $p \in M_k$ for some $k \in \mathbb{Z}$. Then, $\tau\sigma^{-1}(p) \in M_{k+1}$ and consequently $\lfloor \tau\sigma^{-1}(p) \rfloor = \lfloor p \rfloor + 1$. □

**Remark:** Since $\sigma$ is a bijection, we can always replace $p$ by $\sigma(p)$ and repeat formula (6.10) in the following equivalent form

$$\lfloor \tau(p) \rfloor = \lfloor \sigma(p) \rfloor + 1. \tag{6.11}$$

**Definition 6.7.** *By a $(\tau,\sigma)$-partition function (partition function, for short) of $M$ we mean any integer valued function $\lambda : M \to \mathbb{Z}$ such that*

$$\lambda(\tau\sigma^{-1}(p)) = \lambda(p) + 1\,, \tag{6.12}$$

*for any $p \in M$.*

One can easily prove the following

**Proposition 6.8.** *For any $(\tau,\sigma)$-partition function $\lambda$ of $M$, the family of sets*

$$M_k = \lambda^{-1}(k) \subset M\,, \tag{6.13}$$

*where $k \in \mathbb{Z}$, is a $(\tau,\sigma)$-partition of $M$.*

In the sequel, we say that the $(\tau,\sigma)$-partition of $M$ given by formula (6.13) is determined by $\lambda$. Naturally, for a given $(\tau,\sigma)$-partition of $M$ determined by $\lambda$ we have

$$\lfloor p \rfloor = \lambda(p)\,, \tag{6.14}$$

for any $p \in M$. With any $(\tau,\sigma)$-partition of $M$ we associate the following

**Proposition 6.9.** *A right inverse of the* $(\tau,\sigma)$*-differential* $d_{\tau,\sigma}$ *is given by the formula*

$$r_{\tau,\sigma}f(p)=\begin{cases} -\sum\limits_{m=0}^{-\lfloor p\rfloor-1} f(\tau^m\sigma^{-m-1}(p)) & if \quad \lfloor p\rfloor\le -1\\ 0 & if \quad \lfloor p\,\rfloor=0\\ \sum\limits_{m=1}^{\lfloor p\,\rfloor} f(\tau^{-m}\sigma^{m-1}(p)) & if \quad \lfloor p\,\rfloor\ge 1\,. \end{cases} \tag{6.15}$$

**Proof:** For $\lfloor \sigma(p)\,\rfloor = k\le -2$ there is $\lfloor \tau(p)\,\rfloor = k+1\le -1$. Then

$$d_{\tau,\sigma}r_{\tau,\sigma}f(p)=r_{\tau,\sigma}f(\tau(p))-r_{\tau,\sigma}f(\sigma(p))=-\sum_{m=0}^{-k-2} f(\tau^{m+1}\sigma^{-(m+1)}(p))+$$

$$+\sum_{m=0}^{-k-1} f(\tau^m\sigma^{-m}(p))=-\sum_{m=1}^{-k-1} f(\tau^m\sigma^{-m}(p))+\sum_{m=0}^{-k-1} f(\tau^m\sigma^{-m}(p))=f(p)\,.$$

For $\lfloor \sigma(p)\,\rfloor=-1$ there is $\lfloor \tau(p)\,\rfloor=0$. Then

$$d_{\tau,\sigma}r_{\tau,\sigma}f(p)=0-r_{\tau,\sigma}f(\sigma(p))=\sum_{m=0}^{-(-1)-1} f(\tau^m\sigma^{-m}(p))=f(p).$$

For $[\sigma(p)\,]_{\tau,\sigma}=k\ge 1$ there is also $[\tau(p)\,]_{\tau,\sigma}=k+1\ge 1$. Then

$$d_{\tau,\sigma}r_{\tau,\sigma}f(p)=r_{\tau,\sigma}f(\tau(p))-r_{\tau,\sigma}f(\sigma(p))=\sum_{m=1}^{k+1} f(\tau^{-(m-1)}\sigma^{m-1}(p))-$$

$$-\sum_{m=1}^{k} f(\tau^{-m}\sigma^{m}(p))=\sum_{m=0}^{k} f(\tau^{-m}\sigma^{m}(p))-\sum_{m=1}^{k} f(\tau^{-m}\sigma^{m}(p))=f(p). \quad \square$$

Next, by using formula (2.2) we can find the right inverse $R_{\tau,\sigma}$ of the $(\tau,\sigma)$-derivative $D_{\tau,\sigma}$.

**Proposition 6.10.** *A right inverse* $R_{\tau,\sigma}$ *of the* $(\tau,\sigma)$*-derivative* $D_{\tau,\sigma}$ *is given by*

$$R_{\tau,\sigma}f(p)=\begin{cases} -\sum\limits_{m=0}^{-\lfloor p\,\rfloor-1} \theta(\tau^{m+1}\sigma^{-m-1}(p),\tau^m\sigma^{-m}(p))f(\tau^m\sigma^{-m-1}(p)) & \text{if} \quad \lfloor p\,\rfloor\le -1\\ 0 & \text{if} \quad \lfloor p\,\rfloor=0\\ \sum\limits_{m=1}^{\lfloor p\,\rfloor} \theta(\tau^{-m+1}\sigma^{m-1}(p),\tau^{-m}\sigma^{m}(p))f(\tau^{-m}\sigma^{m-1}(p)) & \text{if} \quad \lfloor p\,\rfloor\ge 1\,. \end{cases} \tag{6.16}$$

**Proof:** Let us define the operator $T_{\tau,\sigma}$ by formula

$$T_{\tau,\sigma}f(p)=\theta(\tau(p),\sigma(p))\cdot f(p). \tag{6.17}$$

Thus we write $D_{\tau,\sigma} = T_{\tau,\sigma}^{-1} \circ d_{\tau,\sigma}$ and using formula (2.2) we obtain

$$R_{\tau,\sigma} = r_{\tau,\sigma} \circ T_{\tau,\sigma}. \tag{6.18}$$

Finally, we apply (6.15) and after some calculations obtain formula (6.16). □

**Remark:** Let us notice that the tension function $\theta$ makes no explicit contribution on the construction of the right inverse $r_{\tau,\sigma}$. The only connection between $r_{\tau,\sigma}$ and $\theta$ is through formula (6.5) which means that $\tau\sigma^{-1}$ is a $\theta$-directed mapping. On the other hand, by formula (6.18), the right inverse $R_{\tau,\sigma}$ depends on $\theta$ explicitly.

Now, let us determine the initial operator $F_{\tau,\sigma}$ induced by $D_{\tau,\sigma}$ and corresponding with $R_{\tau,\sigma}$. Since

$$F_{\tau,\sigma} = I - R_{\tau,\sigma} D_{\tau,\sigma} = I - r_{\tau,\sigma} d_{\tau,\sigma}\,, \tag{6.19}$$

it becomes simultaneously the initial operator for $d_{\tau,\sigma}$ corresponding with $r_{\tau,\sigma}$.

**Proposition 6.11.** *The initial operator $F_{\tau,\sigma}$ induced by $D_{\tau,\sigma}$ and corresponding with $R_{\tau,\sigma}$ is given by the formula*

$$F_{\tau,\sigma} f(p) = f((\tau\sigma^{-1})^{-\lfloor p \rfloor}(p)). \tag{6.20}$$

**Proof:** For $\lfloor p \rfloor \leq -1$, we have

$$r_{\tau,\sigma} d_{\tau,\sigma} f(p) = -\sum_{m=0}^{-\lfloor p \rfloor -1} f(\tau\tau^m\sigma^{-m-1}(p)) + \sum_{m=0}^{-\lfloor p \rfloor -1} f(\sigma\tau^m\sigma^{-m-1}(p)) =$$

$$= -\sum_{m=1}^{-\lfloor p \rfloor} f(\tau^m\sigma^{-m}(p)) + \sum_{m=0}^{-\lfloor p \rfloor -1} f(\tau^m\sigma^{-m}(p)) = f(p) - f((\tau\sigma^{-1})^{-\lfloor p \rfloor}(p))$$

If $\lfloor p \rfloor = 0$, there is $r_{\tau,\sigma} d_{\tau,\sigma} f(p) = 0$. For $\lfloor p \rfloor \geq 1$, we have

$$r_{\tau,\sigma} d_{\tau,\sigma} f(p) = \sum_{m=1}^{\lfloor p \rfloor} f(\tau\tau^{-m}\sigma^{m-1}(p)) - \sum_{m=1}^{\lfloor p \rfloor} f(\sigma\tau^{-m}\sigma^{m-1}(p)) =$$

$$= \sum_{m=0}^{\lfloor p \rfloor} f(\tau^{-m}\sigma^{m}(p)) - \sum_{m=1}^{\lfloor p \rfloor} f(\tau^{-m}\sigma^{m}(p)) = f(p) - f((\tau\sigma^{-1})^{-\lfloor p \rfloor}(p)). \qquad \square$$

If a $(\tau,\sigma)$-partition of $M$ is determined by a partition function $\lambda$, we shall index the right inverses or initial operators by $\lambda$, i.e. we shall write $r_\lambda \equiv r_{\tau,\sigma}$, $R_\lambda \equiv R_{\tau,\sigma}$ and $F_\lambda \equiv F_{\tau,\sigma}$.

If $\lambda_1$ and $\lambda_2$ are two $(\tau,\sigma)$-partition functions of $M$ and $R$ is an arbitrary right inverse of the $(\tau,\sigma)$-quantum derivative $D_{\tau,\sigma}$, according to formula (2.8) the corresponding definite $(\tau,\sigma)$-integrals are given by

$$\mathcal{I}_{\lambda_1}^{\lambda_2} = F_{\lambda_2} R - F_{\lambda_1} R\,. \tag{6.21}$$

**Example:** Let $(M,\theta)$ be a tension space, $D_{\tau,\sigma}$ be a quantum $(\tau,\sigma)$-derivation of an algebra $\mathcal{A}\subset\mathbb{R}^M$ and $\eta$ be another tension function on $M$ such that the bijective mapping $\tau\sigma^{-1}$ is $\eta$-directed. Additionally, assume that $\eta$ is $\tau$- and $\sigma$-homogeneous with both homogeneity coefficients equal 1. Then, for any point $s\in M$, the function $\lambda_s$ defined by

$$\lambda_s(p)=\left\lfloor\frac{\eta(p,s)}{\eta(\tau\sigma^{-1}(s),s)}\right\rfloor \tag{6.22}$$

is a $(\tau,\sigma)$-partition function. In particular, when $M=\mathbb{R}$, $\tau(x)=x+h$, $\sigma(x)=x$, $\eta(x,y)=x-y$, for $x,y,h,s\in\mathbb{R}$, $h>0$, we get the partition function $\lambda_s(x)=\lfloor\frac{x-s}{h}\rfloor$ used in $h$-calculus (see Section 3). Hence we obtain the right inverse operators as well as the initial operators $F_{\lambda_s}$ corresponding with $\lambda_s$. Consequently, the $(\tau,\sigma)$-definite integral, for $a,b\in\mathbb{R}$, is given as

$$\mathcal{I}_a^b=F_{\lambda_b}R-F_{\lambda_a}R\,, \tag{6.23}$$

where $R$ is an arbitrary right inverse of $D_{\tau,\sigma}$.

At the end, let us make a comment about higher order $(\tau,\sigma)$-difference-like operators. Let $M\neq\emptyset$ and $\mathcal{A}_n\subset\mathbb{R}^{M^n}$ be a sequence of $\mathbb{R}$-algebras, for $n\in\mathbb{N}$, and let $\mathcal{A}=\mathcal{A}_1$. Assume $(p_1,\ldots,p_n)\in M^n$ and define $\mu_{p_1,\ldots,p_n}=\{f\in\mathcal{A}_n : f(p_1,\ldots,p_n)=0\}$, the ideal of $\mathcal{A}_n$, for any $n\in\mathbb{N}$.

**Definition 6.12.** *A linear mapping $\Lambda:\mathcal{A}_1\to\mathcal{A}_n$, for a fixed $n\in\mathbb{N}$, is said to be of pre-order $n$ if $\Lambda(\mu_{p_1}\cdot\ldots\cdot\mu_{p_n})\subset\mu_{p_1,\ldots,p_n}$.*

For example, let us explicitely formulate the rule fulfilled by an operator $\Lambda$ of pre-order $n=1$. From the above definition we obtain

$$\Lambda((f_1-f_1(p_1))((f_2-f_2(p_2))\in\mu_{p_1,p_2}, \tag{6.24}$$

which means that

$$\Lambda((f_1-f_1(p_1))((f_2-f_2(p_2))(p_1,p_2)=0. \tag{6.25}$$

Formula (6.25) can be written equivalently as

$$\begin{aligned}\Lambda(f_1f_2)(p_1,p_2)-f_1(p_1)\Lambda(f_2)(p_1,p_2)-f_2(p_2)\Lambda(f_1)(p_1,p_2)+\\ +f_1(p_1)f_2(p_2)\Lambda(1)(p_1,p_2)=0.\end{aligned} \tag{6.26}$$

Now, let us define $\delta:\mathcal{A}\to\mathcal{A}$ by formula

$$\delta(f)(p)=\Lambda(f)(\tau(p),\sigma(p))\,, \tag{6.27}$$

for any $p\in M$. Directly from formula (6.26) we obtain

$$\begin{aligned}\delta(f_1f_2)(p)-f_2(\tau(p))\delta(f_1)(p)-f_1(\sigma(p))\delta(f_2)(p)+\\ +f_1(\sigma(p))f_2(\tau(p))\delta(1)(p)=0.\end{aligned} \tag{6.28}$$

**Definition 6.13.** *By a quantum $(\tau,\sigma)$-difference-like operator of order 1 we shall mean any operator $\delta_{\tau,\sigma}$ that fulfills formula (6.28). In the case $\delta_{\tau,\sigma}(1)=0$, an operator $\delta_{\tau,\sigma}$ is said to be a quantum $(\tau,\sigma)$-derivative (compare with formula (6.2)).*

In the particular case $\tau=\sigma=id_M$ the above $(\tau,\sigma)$-differential operator $\delta_{\tau,\sigma}$ becomes a usual differential operator of order 1 of algebra $\mathcal{A}$.

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