THE n-ARY ADDING MACHINE AND SOLVABLE GROUPS

JOSIMAR DA SILVA ROCHA AND SAID NAJATI SIDKI

ABSTRACT. We describe under a various conditions abelian subgroups of the automorphism group $\mathrm{Aut}(T_n)$ of the regular n-ary tree T_n, which are normalized by the n-ary adding machine $\tau = (e, ..., e, \tau)\sigma_\tau$ where σ_τ is the n-cycle $(0, 1, ..., n-1)$. As an application, for $n = p$ a prime number, and for $n = p^2$ when $p = 2$, we prove that every finitely generated soluble subgroup of $\mathrm{Aut}(T_n)$, containing τ is an extension of a torsion-free metabelian group by a finite group.

CONTENTS

Date: August 2011.
Key words and phrases. Adding machine, Tree automorphisms, Automata, Solvable Groups.

1. Introduction

Adding machines have played an important role in dynamical systems, and in the theory of groups acting on trees : see [1, 2, 5, 4, 10].

An element α in the automorphism group $\mathcal{A}_n = \mathrm{Aut}(T_n)$ of the n-ary tree T_n, is represented as $\alpha = \alpha|_\phi = (\alpha|_0, ..., \alpha|_{n-1})\,\sigma_\alpha$ where ϕ is the empty sequence from the free monoid $\mathcal{M}$ generated by $Y = \{0, 1, .., n-1\}$, where $\alpha|_i \in \mathcal{A}_n$ $(i \in Y)$-called 1st level states of α- and where σ_α (the activity of α) is a permutation in the symmetric group Σ_n on Y extended 'rigidly' to act on the tree. In applying the same representation to $\alpha|_0$ we produce $\alpha|_{0i}$ where $i \in Y$ and in general we produce $\{\alpha|_u \mid u \in \mathcal{M}\ \}$ the set of *states* of α. Following this notation, the n-ary adding machine is represented as $\tau = (e, ..., e.\tau)\sigma_\tau$ where e is the identity automorphism an σ_τ is the regular permutation $\sigma = (0, 1, ..., n-1)$. In this sense the adding machine may be viewed as an infinite variant of the regular permutation which often appears in geometric and combinatorial contexts.

A characteristic feature of τ is that its n-th power τ^n is the diagonal automorphism of the tree $(\tau, ..., \tau)$. This fact implies that the centralizer of the cyclic group $\langle\tau\rangle$ in $\mathcal{A}_n$ is equal to its topological closure $\overline{<\tau>}$ in $\mathcal{A}_n$ seen as a topological group with respect to the the natural topology induced by the tree.

A large variety of subgroups of $\mathcal{A}_n$ which contain τ have been constructed, including finitely generated groups which are torsion-free and just non-solvable, yet without free subgroups of rank 2 [3, 6], and generalizations thereof [9], as well as constructions of free groups of rank 2 [11]. Yet solvable groups which contain τ are expected to have restricted structure [2]. For nilpotent groups we show

Proposition. *Let G be a nilpotent subgroup of $\mathcal{A}_n$ which contains the n-adic adding machine τ. Then G is a subgroup of $\overline{<\tau>}$*

Let $\mathbb{Z}_n$ be the ring of n-adic integers and $U(\mathbb{Z}_n)$ its subgroup of units. The normalizer of $\overline{<\tau>}$ in $\mathcal{A}_n$ is isomorphic to the holomorph of $\mathbb{Z}_n$, the semi-direct product $\mathbb{Z}_n \rtimes U(\mathbb{Z}_n)$, and is therefore metabelian.

The main examples of finitely generated solvable groups containing τ are conjugate to subgroups of those belonging to the sequence of groups

$$\Gamma_0 = N_{\mathcal{A}_n}\overline{<\tau>}, \Gamma_1 = (\times_n \Gamma_0) \rtimes G_1, ..., \Gamma_{i+1} = (\times_n \Gamma_i) \rtimes G_{i+1}, ...$$

where $\times_n \Gamma_i$ is a direct product of n copies of Γ_i (seen as a subgroup of the 1st level stabilizer of the tree) and where G_i is a solvable subgroup of Σ_n in its canonical action on the tree, containing the cycle σ_τ. We note that for all i, the groups Γ_i are metabelian by 'finite solvable subgroups of Σ_n'. It was shown by the second author that for $n = 2$, the answer conforms precisely to this model [7].

The description for degrees $n \geq 2$ requires a classification of solvable subgroups of Σ_n which contain the cycle $\sigma = (0, 1, ..., n-1)$[8]. This is an open problem, even for metabelian groups. On the other hand, the answer for primitive solvable subgroups of Σ_n is simple and classical. For then, n is a prime number p or $n = 4$. In case $n = p$, the solvable subgroups G_i can all be taken to be the normalizer $F = N_{\Sigma_n}(\langle\sigma\rangle)$ of order $p(p-1)$ and in case $n = 4$, the G_i's can all be taken to be Σ_4.

Given this background, the main theorem of this paper is

Theorem A. *Let $n = p$, a prime number, or $n = 4$. Then any finitely generated solvable subgroup of $\mathcal{A}_n$,which contains the n-ary machine τ is conjugate to a subgroup of Γ_i for some i.*

The result follows first from general analysis of the conditions $[\beta, \beta^{\tau^x}] = e$ (for some $\beta \in \mathcal{A}_n$ and all $x \in \mathbb{Z}$), their impact on the 1st level states of the subgroup $\langle\beta, \tau\rangle$ and then how these in turn translate successively to conditions on states at lower levels. It is somewhat surprising that the process converges to a clear global description for trees of degrees p and 4.

If σ_β is a power of σ_τ, or if it is a transposition, we prove

Theorem B. *Let B be an abelian subgroup of $\mathcal{A}_n$ normalized by τ, let $\beta = (\beta|_0, \beta|_1, \cdots, \beta|_{n-1})\sigma_\beta \in B$ and define the subgroup $H = \langle\beta|_i \ (i \in Y), \tau\rangle$.*
(I) Suppose $\sigma_\beta = (\sigma_\tau)^s$ for some integer s and set $m = \frac{n}{\gcd(n,s)}$. Then, H is metabelian-by-finite. Indeed,on defining the subgroup

$$K = \left\langle [\beta|_i, \tau^k],\ \beta|_i\beta|_{\overline{i+s}}\beta|_{\overline{i+2s}}\cdots\beta|_{\overline{i+(m-1)s}} \mid k \in \mathbb{Z},\ i \in Y \right\rangle$$

(the bar notation means 'modulo m') then K is a normal subgroup of H and $O = K\langle\tau\rangle$ is a metabelian normal subgroup of H where $\frac{H}{O}$ is a homomorphic image of a subgroup of the wreath product $C_m \wr C_n$ of the cyclic groups C_m, C_n.
(II) Let n be an even number. Then H is a metabelian group if $s = \frac{n}{2}$ or σ_β is a transposition.

Let P be a subgroup of Σ_n. The *layer closure* of P in $\mathcal{A}_n$ is the group $L(P)$ formed by elements of $\mathcal{A}_n$ all of whose states lie in P. The following result is yet another characterization of the adding machine.

Theorem C. *Let n be an odd number, $\sigma = (0, \cdots, n-1) \in \Sigma_n$ and let $L = L(\langle\sigma\rangle)$, the layer closure of $\langle\sigma\rangle$ in A_n. Let s be an integer relatively prime to n and let $\beta = (\beta|_0, \beta|_1, \cdots, \beta|_{n-1})\sigma^s \in L$ be such* that $[\beta, \beta^{\tau^x}] = e$ *for all $x \in Z$. Then β is a conjugate of τ in L.*

2. Preliminaries

We start by introducing definitions and notation. The n-ary tree T_n can be identified with the free monoid $\mathcal{M} =< 0, 1, .., n-1 >^*$ of finite sequences from $Y = \{0, 1, ..., n-1\}$, ordered by $v \leq u$ provided u is an initial subword of v.

The identity element of $\mathcal{M}$ is the empty sequence ϕ. The level function for T_n, denoted by $|m|$ is the length of $m \in \mathcal{M}$; the root vertex ϕ has level 0.

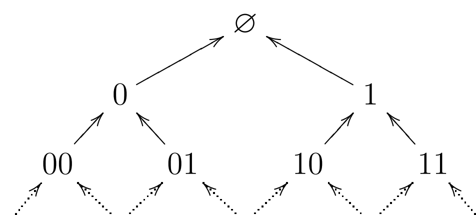

Figure 1. The Binary Tree

The action $\rho : i \to j$ of a permutation $\rho \in \Sigma_n$ will be from the right and written as $(i)\,\rho = j$. If i, j are integers $(i)\,\rho = j$ is to be understood as $(\overline{i})\,\rho = \overline{j}$ where $\overline{i}, \overline{j}$ are their respective representatives in Y modulo n. Permutations σ in Σ_n are extended 'rigidly' to automorphisms of $\mathcal{A}_n$ by

$$(y.u)\rho = (y)\rho.u, \ \forall\, y \in Y, \ \forall\, u \in \mathcal{M}.$$

An automorphism $\alpha \in \mathcal{A}_n$ induces a permutation σ_α on the set Y. Consequently, α affords the representation $\alpha = \alpha'\sigma_\alpha$ where α' fixes Y point-wise and for each $i \in Y$, α' induces $\alpha|_i$ on the subtree whose vertices form the set $i \cdot \mathcal{M}$. If j is an integer the $\alpha|_j$ will be understood as $\alpha|_{\overline{j}}$ where $\overline{j}$ is the representative of j in Y modulo n.

Given i in Y, we use the canonical isomorphism $i \cdot u \mapsto u$ between $i \cdot \mathcal{M}$ and the tree T_n, and thus identify $\alpha|_i$ with an automorphism of T_n; therefore, $\alpha' \in \mathcal{F}(Y, \mathcal{A}_n)$, the set for functions from Y into $\mathcal{A}_n$, or what is the same, the 1st level stabilizer $Stab(1)$ of the tree. This provides us with the factorization $\mathcal{A}_n = \mathcal{F}(Y, \mathcal{A}_n) \cdot \Sigma_n$.

Let $\alpha, \beta, \gamma \in \mathcal{A}_n$. Then following formulas hold

$$\sigma_{\alpha^{-1}} = (\sigma_\alpha)^{-1}, \ \sigma_\alpha \sigma_\beta = \sigma_{\alpha\beta}, \tag{1}$$

$$(\alpha^{-1})|_u = \alpha|_{(u)^{\alpha^{-1}}}, \tag{2}$$

$$(\alpha\beta)|_u = (\alpha|_u)(\gamma|_u) \text{ where } \gamma|_u = \beta|_{(u)^\alpha} \tag{3}$$

$$\gamma = \alpha^{-1}\beta\alpha \Leftrightarrow \sigma_\gamma = \sigma_\alpha^{-1}\sigma_\beta\sigma_\alpha, \tag{4}$$

$$\gamma|_{(i)\sigma_\alpha} = \alpha|_i^{-1}\beta|_i\alpha|_{(i)\sigma_\beta}, \forall i \in Y. \tag{5}$$

$$\theta = [\beta, \alpha] = \beta^{-1}\beta^\alpha \Rightarrow \sigma_\theta = [\sigma_\beta, \sigma_\alpha], \tag{6}$$

$$\theta|_{(i)\sigma_{\alpha\beta}} = \left(\beta|_{(i)\sigma_\alpha}\right)^{-1}(\alpha|_i)^{-1}(\beta|_i)\left(\alpha|_{(i)\sigma_\beta}\right), \forall i \in Y. \tag{7}$$

$$(\alpha^m)|_i = (\alpha|_i)\left(\alpha|_{(i)\sigma_\alpha}\right)\left(\alpha|_{(i)\sigma_\alpha^2}\right)\cdots\left(\alpha|_{(i)\sigma_{\alpha^{m-1}}}\right) \tag{8}$$

$$(\beta^\alpha)|_u = \left(\beta|_{(u)\alpha^{-1}}\right)^{\alpha|_{(u)\alpha^{-1}}}, \text{where } \beta \in Stab(k) \text{ and } |u| \leq k. \tag{9}$$

An automorphism $\alpha \in \mathcal{A}_n$ corresponds to an input-output automaton over the alphabet Y and with the set of states $\mathrm{Q}(\alpha) = \{\alpha|_u \mid u \in \mathcal{M}\}$. The automaton α transforms the letters as follows: if the automaton is in state $\alpha|_u$ and reads a letter $i \in Y$ then it outputs the letter $j = (i)\,\alpha|_u$ and the state changes to $\alpha|_{ui}$; these operations can be best described by the labeled edge $\alpha|_u \xrightarrow{i|j} \alpha|_{ui}$. Following the terminology of automata theory, every automorphism $\alpha|_u$ is called the *state* of α at u.

The tree T_n is a topological space which is the direct limit of its truncations at the n-th levels. Thus the group $\mathcal{A}_n$ is the inverse limit of the permutation groups it induces on the n-th level vertices. This transforms $\mathcal{A}_n$ into a topological group. An infinite product of elements $\mathcal{A}_n$ is a well-defined element of $\mathcal{A}_n$ provided for any given level l, only finitely many of the elements in the product have non-trivial action on vertices at level l. The topological closure of a subgroup H in $\mathcal{A}_n$ will be indicated by $\overline{H}$. We note that if H is abelian then

$$\overline{H} = \{h^\xi | h \in H, \xi \in \mathbb{Z}_n\}.$$

One of the characterizing aspects of the n-ary adding machine is

$$C_{\mathcal{A}_n}(\tau) = \overline{\langle\tau\rangle} = \{\tau^\xi \mid \xi \in \mathbb{Z}_n\}.$$

Let $v = yu$ where $y \in Y, u \in \mathcal{M}$. The image of v under the action of α is

$$(v)\alpha = (yu)\alpha = (y)\,\sigma_\alpha.(u)\alpha|_y.$$

The action extends to infinite sequences (or boundary points of the tree) in the same manner. A boundary point of the tree $c = c_0c_1c_2\ldots$where $c_1 \in Y$ corresponds also to the n-adic integer $\xi = \sum\{c_i n^i | i \geq 0\} \in \mathbb{Z}$, by which, the action of the tree automorphism α can thus be translated to an action on the ring of n-adic integers. We will indicate c_0 by $\overline{\xi}$ which is ξ modulo n. In the case of the automorphism $\tau = (e, e, ..., e, \tau)\sigma$, the action of τ on c is

$$(c)\,\tau = \begin{cases} (c_0+1)\,c_1c_2\ldots & \text{if } 0 \leq c_0 \leq n-2, \\ 0(c_1c_2,\ldots)^{\tau}, & \text{if } c_0 = n-1, \end{cases}$$

which translates to the n-ary addition

$$\xi^{\tau} = 1 + \xi.$$

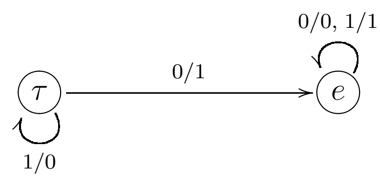

FIGURE 2. The binary adding machine

3. THE HOLOMORPH OF THE n-ADIC INTEGERS

The holomorph of $\mathbb{Z}_n$ is the extension $\mathbb{Z}_n$ by the its group of units $U(\mathbb{Z}_n)$ in its natural action on $\mathbb{Z}_n$. An element ξ is a unit in $\mathbb{Z}_n$ if and only if $\overline{\xi}$ is a unit in $\mathbb{Z}$ modulo n. The subgroup of $U(\mathbb{Z}_n)$ consisting of elements ξ with $\overline{\xi} = 1$ is denoted by by $\mathbb{Z}_n^1$. This subgroup has the transversal $\{j \mid 1 \leq j \leq n-1, \gcd(j, n) = 1\}$ in $\mathbb{Z}_n$ and therefore has index $[U(\mathbb{Z}_n) : \mathbb{Z}_n^1] = \varphi(n)$. We will represent the normalizer of $\overline{\langle\tau\rangle}$ in the group of automorphisms of the tree as the holomorph of $\mathbb{Z}_n$.

Given $\alpha \in \mathcal{A}_n$ we denote the diagonal automorphism $(\alpha, ..., \alpha)$ by $\alpha^{(1)}$ and denote inductively $\alpha^{(i+1)} = \left(\alpha^{(i)}\right)^{(1)}$.

3.1. **Powers of τ.** Let $\xi = \sum_{i\geq 0} a_i n^i \in \mathbb{Z}_n$. Then $a_0 = \overline{\xi}$ and $\sum_{i\geq 1} a_i n^{i-1} = \frac{\xi - \overline{\xi}}{n}$.

Lemma 1. *Let $\xi \in \mathbb{Z}_n$. Then*

$$\tau^{\xi} = (\tau^{\frac{\xi-a_0}{n}}, \cdots, \tau^{\frac{\xi-a_0}{n}}, \underbrace{\tau^{\frac{\xi-a_0}{n}+1}, \cdots, \tau^{\frac{\xi-a_0}{n}+1}}_{a_0 \text{ terms}})\sigma_{\tau}^{a_0}.$$

Proof. For j an integer with $1 \leq j \leq n-1$, we have

$$\tau^j = \left(e, ..., e, \underbrace{\tau, \cdots, \tau}_{j \text{ terms}}\right)\sigma_{\tau}^{j}$$

and $\tau^n = (\tau, ..., \tau) = \tau^{(1)}$.

Given $\xi = \sum_{i\geq 0} a_i n^i$, then

$$\tau^{\xi} = \tau^{a_0}\tau^{na_1}...\tau^{n^i a_i}...$$

Therefore,

$$\tau^{a_0} = (e, \cdots, e, \underbrace{\tau, \cdots, \tau}_{a_0 \text{ terms}})\sigma_{\tau}^{a_0}, \tag{10}$$

$$\tau^{a_j n^j} = \tau^{(a_j n^{j-1})n} = \left(\tau^{a_j n^{j-1}}\right)^{(1)}, \tag{11}$$

$$\tau^{\xi} = (\tau^{\frac{\xi - a_0}{n}}, \cdots, \tau^{\frac{\xi - a_0}{n}}, \underbrace{\tau^{\frac{\xi - a_0}{n}+1}, \cdots, \tau^{\frac{\xi - a_0}{n}+1}}_{a_0 \text{ terms}})\sigma_{\tau}^{a_0} \tag{12}$$

$$= (\tau^{\frac{\xi - \overline{\xi}}{n}}, \cdots, \tau^{\frac{\xi - \overline{\xi}}{n}}, \underbrace{\tau^{\frac{\xi - \overline{\xi}}{n}+1}, \cdots, \tau^{\frac{\xi - \overline{\xi}}{n}+1}}_{\overline{\xi} \text{ terms}})\sigma_{\tau}^{\overline{\xi}}. \tag{13}$$

□

In the description of τ^{ξ}, the interval $[0, ..., n-1]$ divides into two subintervals and therefore we introduce the step function $\delta : \frac{\mathbb{Z}}{n\mathbb{Z}} \times \frac{\mathbb{Z}}{n\mathbb{Z}} \to \{0, 1\}$ given by

$$\delta(i, j) = \frac{i + j - \overline{i + j}}{n} = \begin{cases} 0, & \text{if } 0 \leq i \leq n - j \\ 1, & \text{otherwise} \end{cases}.$$

which we will call the *Polarizer Function.* With this,

$$\tau^{\xi} = \left(\tau^{\frac{\xi - \overline{\xi}}{n} + \delta(i,\xi)}\right)_{0\leq i\leq n-1} \sigma_{\tau}^{\overline{\xi}}.$$

The function δ extends to $\mathbb{Z}_n \times \mathbb{Z}_n$, simply by defining $\delta(\eta, \kappa) = \delta(i, k)$ where $i = \overline{\eta}, k = \overline{\kappa}$. Note that

$$\sum_{i=0}^{n-1} \delta(i, j) = j.$$

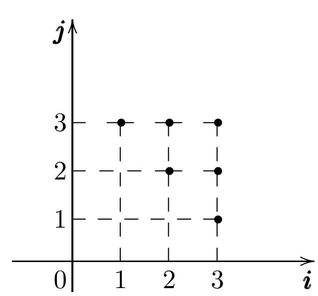

FIGURE 3. Polarizer Function for $n = 4$.

3.2. **Centralizer of τ.**

Lemma 2. $C_{\mathcal{A}_n}(\tau) = \overline{\langle \tau \rangle}$.

Proof. Let $\alpha \in \mathcal{A}_n$ commute with τ. Then, $[\sigma_\alpha, \sigma_\tau] = e$ and therefore $\sigma_\alpha = (\sigma_\tau)^{s_0}$ for some integer $0 \leq s_0 \leq n-1$. Therefore, $\beta = \alpha\tau^{-s_0} = (\beta|_0, ..., \beta|_{n-1})$ commutes with τ and $\sigma_\beta = e$. Hence,

$$\begin{aligned} \theta|_{(i)\sigma_{\alpha\beta}} &= \left(\beta|_{(i)\sigma_\tau}\right)^{-1}(\tau|_i)^{-1}\beta|_i\tau|_i = e \text{ (by (7))} \\ (\tau|_i)^{-1}\beta|_i\tau|_i &= \beta|_{i+1}, \\ \beta|_i &= \beta|_0 \text{ for all } (0 \leq i \leq n-1) \text{ and } [\beta|_0, \tau] = e. \end{aligned}$$

Therefore $\beta = (\beta|_0)^{(1)}$ and $\beta|_0$ replaces α in previous argument. Hence, there exists an integer $0 \leq s_1 \leq n-1$ such that $\gamma = \beta|_0\tau^{-s_1} = (\gamma|_0)^{(1)}$. From which we conclude

$$\begin{aligned} \alpha &= \beta\tau^{s_0} = (\beta|_0)^{(1)}\tau^{s_0} \\ &= \left((\gamma|_0)^{(1)}\tau^{s_1}, .., (\gamma|_0)^{(1)}\tau^{s_1}\right)\tau^{s_0} \\ &= (\gamma|_0)^{(2)}\tau^{ns_1}\tau^{s_0} = (\gamma|_0)^{(2)}\tau^{ns_1+s_0}. \end{aligned}$$

Inductively then, we obtain the desired form $\alpha = \tau^\xi$ where $\xi = s_0 + ns_1 + ...$ □

A characterization of nilpotent groups which contain τ follows.

Proposition 1. *Let G be a nilpotent subgroup of $\mathcal{A}_n$ which contains the n-adic adding machine. Then G is a subgroup of* $\overline{<\tau>}$

Proof. Suppose G is a nilpotent group of class $k > 1$ which contains τ Then, the center $Z(G)$ is contained in $\overline{\langle \tau \rangle}$. Let i be the maximum index such that $Z_i(G) \leq \overline{\langle \tau \rangle}$; therefore $i < k$. Let $\alpha \in Z_{i+1}(G) \backslash Z_i(G)$.; then $[\tau, \alpha] = \tau^\xi$ and $\xi \neq 0$. Now, $[\tau, \alpha, \alpha] = \left[\tau^\xi, \alpha\right] = e$. Yet $\left[\tau^\xi, \alpha\right] = [\tau, \alpha]^\xi = \tau^{\xi^2} = e$ and so, $\xi = 0$ and $[\tau, \alpha] = e$; a contradiction. □

3.3. **Normalizer of the topological closure $\overline{\langle \tau \rangle}$.**

Lemma 3. *The group $\Gamma_0 = N_{\mathcal{A}_n}\left(\overline{\langle \tau \rangle}\right)$ is metabelian. Indeed, the derived subgroup Γ_0' is contained in $\overline{\langle \tau \rangle}$.*

Proof. Let $\alpha, \beta \in \Gamma_0$, then $\tau^\alpha = \tau^\xi$ and $\tau^\beta = \tau^\eta$.for some $\eta, \xi \in U(\mathbb{Z}_n)$. Therefore,

$$\tau^\alpha = \tau^\xi, \tau = (\tau^\xi)^{\alpha^{-1}} = (\tau^{\alpha^{-1}})^\xi,$$
$$\tau^{\alpha^{-1}} = \tau^{\xi^{-1}}.$$

Likewise, $\tau^{\beta^{-1}} = \tau^{\eta^{-1}}$. Thus, $\tau^{[\alpha,\beta]} = \tau$ and $\Gamma_0' \leq C_{\mathcal{A}_n}(\tau) = \overline{\langle\tau\rangle}$ follows. $\square$

We present a property of the polarizer function δ which we use in the sequel.

Lemma 4. *For all $i, j \in \mathbb{Z}$, $\xi \in \mathbb{Z}_n$ we have*

$$\frac{j\xi - \overline{j\xi}}{n} - j\left(\frac{\xi - \overline{\xi}}{n}\right) + \delta(i, j\xi) = \sum_{k=0}^{j-1} \delta(i + k\xi, \xi).$$

Proof. Since

$$\begin{aligned} (\tau^\xi)^j|_i &= (\tau^\xi)|_i.(\tau^\xi)|_{\overline{i+\xi}} \cdots (\tau^\xi)|_{\overline{i+(j-1)\xi}}, \\ (\tau^\xi)|_i &= \tau^{\frac{\xi-\overline{\xi}}{n}+\delta(i,\xi)} \end{aligned}$$

it follows that

$$\tau^{\frac{j\xi-\overline{j\xi}}{n}+\delta(i,j\xi)} = \tau^{j\left(\frac{\xi-\overline{\xi}}{n}\right)+\sum_{k=0}^{j-1}\delta(i+k\xi,\xi)}$$

and the assertion follows. $\square$

Proposition 2. *Suppose $\alpha \in \mathcal{A}_n$ satisfy $\tau^\alpha = \tau^\xi$ for some $\xi \in U(\mathbb{Z}_n)$. Then:*

(i)

$$\alpha|_i = \alpha|_0 \tau^{\mu_i}, (1 \leq i \leq n-1)\};$$

where

$$\mu_i = i\frac{(\xi - \overline{\xi})}{n} + \sum_{k=0}^{i-1} \delta((v(\alpha) + k)\xi, \xi)$$

and $0 \leq v(\alpha) \leq n-1$ is such that

$$(0)\,\sigma_\alpha = \overline{v(\alpha)\xi};$$

(ii) (recursion) $\tau^{\alpha|_0} = \tau^\xi$;

(iii)

$$(j)\sigma_\alpha = \overline{(v(\alpha) + j)\xi}, (0 \leq j \leq n-1)\}.$$

If $\xi \in \mathbb{Z}_n^1$ then $v(\alpha) = 0$, $(j)\sigma_\alpha = \overline{j\xi} = j, \mu_i = i\frac{\xi-1}{n}$.

Proof. Since $\sigma_\tau^{\sigma_\alpha} = \sigma_\tau^\xi$, we have

$$((0)\,\sigma_\alpha, (1)\,\sigma_\alpha, \cdots, (n-1)\sigma_\alpha) = (0, \overline{\xi}, \overline{2\xi}, \cdots, \overline{(n-1)\xi}).$$

Therefore, there exists $v(\alpha) \in Y$ such that $(0)\,\sigma_\alpha = \overline{v(\alpha)\xi}$ and so,

$$(j)\sigma_\alpha = \overline{(v(\alpha) + j)\xi},\ \forall j \in Y.$$

Now, $\tau^\alpha = \tau^\xi$ is equivalent to

$$\left(\begin{array}{l}\sigma_\tau^{\sigma_\alpha}=\sigma_\tau^\xi \quad \text{and} \quad \alpha|_{(i)\sigma_\tau^s}=\left((\tau^s)|_i\right)^{-1}\alpha|_i(\tau^{\xi s})|_{(i)\sigma_\alpha}, \\ \forall i\in Y, \forall s\in\mathbb{Z}, \text{ by...}\end{array}\right).$$

The latter conditions are equivalent to

$$\left(\begin{array}{l}\alpha|_0=\alpha|_{(0)\sigma_\tau^n}=\left((\tau^n)|_0\right)^{-1}\alpha|_0(\tau^{\xi n})|_{(0)\sigma_\alpha} \\ \quad\text{and} \quad \alpha|_i=\alpha|_{(0)\sigma_\tau^i}=\left((\tau^i)|_0\right)^{-1}\alpha|_0(\tau^{\xi i})|_{(0)\sigma_\alpha}\ \forall i\in Y-\{0\}\end{array}\right)$$

and these to

$$\left(\begin{array}{c}\tau^{\alpha|_0}=\tau^\xi \text{ and } \alpha|_i=\alpha|_0\tau^{\frac{\xi i-\overline{\xi i}}{n}+\delta(v(\alpha)\xi,\xi i)}=\alpha|_0\tau^{\mu_i} \\ \text{where } \mu_i=i\left(\frac{\xi-\overline{\xi}}{n}\right)+\sum_{k=0}^{i-1}\delta((v(\alpha)+k)\xi,\xi)\ \forall i\in Y-\{0\}\end{array}\right).$$

The rest of the assertion follows directly. □

Corollary 1. *Let $\xi\in U(\mathbb{Z}_n)$. Then $\alpha=(\alpha)^{(1)}(e,\tau^{\mu_1},...,\tau^{\mu_{n-1}})$ conjugates τ to τ^ξ. In particular, if $\xi\in\mathbb{Z}_n^1$, then*

$$\alpha=(\alpha)^{(1)}(e,\tau^{\frac{\xi-1}{n}},\tau^{2\frac{\xi-1}{n}},\cdots,\tau^{(n-1)\frac{\xi-1}{n}})$$

which will be denoted by λ_ξ.

Although we have computed above an automorphism which inverts τ, we give below a simpler one. Define the permutation

$$\varepsilon=(0,n-1)(1,n-2)...\left(\left[\frac{n-2}{2}\right],\left[\frac{n+1}{2}\right]\right).$$

Then ε inverts $\sigma_\tau=(0,1,...,n-1)$ and

$$\iota=\iota^{(1)}\varepsilon$$

inverts τ.

Define

$$\begin{array}{rcl}\Lambda &=& \{\lambda_\xi \mid \xi\in\mathbb{Z}_n^1\}, \\ \Psi &=& \{\lambda_\xi\tau^t \mid \xi\in\mathbb{Z}_n^1, t\in\mathbb{Z}_n\}\end{array}$$

and call Λ the *monic normalizer* of $\overline{\langle\tau\rangle}$.

Proposition 3. *(i) Λ is an abelian group isomorphic to $\mathbb{Z}_n^1$;*
(ii) $\Psi=\Lambda\ltimes\overline{\langle\tau\rangle}\cong\mathbb{Z}_n^1\ltimes\mathbb{Z}_n$;
(iii) the derived subgroup $\Psi'=\overline{\langle\tau^n\rangle}$.

Proof. (i) Let $\xi,\theta\in\mathbb{Z}_n^1$. Then, as $\lambda_\xi,\lambda_\theta$ and $\lambda_{\xi\theta}$ are inactive, its follows that

$$(\lambda_\xi\lambda_\theta\lambda_{\xi\theta}^{-1})|_i=(\lambda_\xi)|_i(\lambda_\theta)|_i\left((\lambda_{\xi\theta})|_i\right)^{-1}$$

$$=\lambda_\xi\tau^{i\frac{\xi-1}{n}}\lambda_\theta\tau^{i\frac{\theta-1}{n}}\left(\lambda_{\xi\theta}\tau^{i\frac{\xi\theta-1}{n}}\right)^{-1}=\lambda_\xi\lambda_\theta\lambda_\theta^{-1}\tau^{i\frac{\xi-1}{n}}\lambda_\theta\tau^{i\frac{\theta-1}{n}}\tau^{-i\frac{\xi\theta-1}{n}}\lambda_{\xi\theta}^{-1}$$

$$=\lambda_\xi\lambda_\theta\left(\tau^{i\theta\frac{\xi-1}{n}}\tau^{i\frac{\theta-1}{n}}\tau^{-i\frac{\xi\theta-1}{n}}\right)\lambda_{\xi\theta}^{-1}=\lambda_\xi\lambda_\theta\lambda_{\xi\theta}^{-1},\forall i\in\{0,\cdots,n-1\}.$$

Therefore, $\lambda_\xi\lambda_\theta=\lambda_{\xi\theta}$. In addition, $\lambda_\xi=e$ if and only if $\xi=1$.

(ii) This part is clear.

(iii) Let $\theta = 1 + n\theta', \eta \in \mathbb{Z}_n$. Calculate

$$\begin{array}{rcl} [\tau^{\eta}, \lambda_{\theta}] & = & \tau^{-\eta}\lambda_{\theta^{-1}}\tau^{\eta}\lambda_{\theta} = \\ \tau^{-\eta}\tau^{\eta\theta} & = & \tau^{\eta(\theta-1)} = (\tau^{n})^{\eta\theta'}. \end{array}$$

□

We prove below the existence of conjugates τ^{α} of τ in $N_{\mathcal{A}_n}\left(\overline{\langle\tau\rangle}\right)$ yet ouside $\overline{\langle\tau\rangle}$. This fact provides us with the first important type of metabelian groups $\overline{\langle\tau\rangle}\,\langle\tau^{\alpha}\rangle$ containing τ.

Proposition 4. *Suppose $\alpha = (\alpha|_0, \alpha|_1, \cdots, \alpha|_{n-1}) \in \mathcal{A}_n$ satisfies $\tau^{\alpha} = \lambda_{\xi}\tau^{\rho}$ for some $\xi \in \mathbb{Z}_n^1$, and $\rho = 1 + \kappa n \in \mathbb{Z}_n^1$. Then*

$$\left\{ \begin{array}{l} \alpha|_{i+1} = (\alpha|_0)\,\lambda_{\xi^{i+1}}\tau^{\frac{1}{n}\left[\rho\frac{\xi^{i+1}-1}{\xi-1}-(i+1)\right]} \;\; (0 \le i \le n-2)\,, \\ \tau^{\alpha|_0} = \lambda_{\xi^n}\tau^{\frac{1}{n}\left[\rho\frac{\xi^{n}-1}{\xi-1}\right]}. \end{array} \right. .$$

The converse is true for $n \ge 3$ and for $n = 2$ provided $4|\xi - 1$.

Proof. From $\tau^{\alpha} = \lambda_{\xi}\tau^{1+\kappa n}$, we obtain using (4) and (5),

$$\left\{ \begin{array}{l} \lambda_{\xi}\tau^{i\frac{\xi-1}{n}+\kappa} = \alpha|_i^{-1}\alpha_{i+1}, \;\; \text{if } i \in Y - \{n-1\} \\ \lambda_{\xi}\tau^{(n-1)\frac{\xi-1}{n}+\kappa+1} = \alpha|_{n-1}^{-1}\tau\alpha|_0. \end{array} \right.$$

Therefore,

$$\begin{array}{l} \alpha|_{i+1} = \alpha|_0\lambda_{\xi}\tau^{\kappa}\lambda_{\xi}\tau^{\frac{\xi-1}{n}+\kappa}\cdots\lambda_{\xi}\tau^{i\frac{\xi-1}{n}+\kappa}, \;\; \text{for } i = 0, 1, \cdots, n-2, \\ \alpha|_0 = \tau^{-1}\alpha|_{n-1}\lambda_{\xi}\tau^{(n-1)\frac{\xi-1}{n}+\kappa+1}. \end{array}$$

The first equations can be expresses as

$$\begin{array}{rcl} \alpha|_{i+1} & = & \alpha|_0\lambda_{\xi^{i+1}}\tau^{\kappa\sum_{j=0}^{i}\xi^j + \frac{\xi-1}{n}\xi^i\sum_{j=1}^{i}j(\xi^{-1})^j} \\ & = & \alpha|_0\lambda_{\xi^{i+1}}\tau^{\frac{1}{n}\left[(1+\kappa n)\frac{\xi^{i+1}-1}{\xi-1}-(i+1)\right]} \end{array}$$

and the last as

$$\begin{array}{rcl} \alpha|_0 & = & \tau^{-1}\alpha|_0\lambda_{\xi^n}\tau^{\frac{\xi}{n}\left[(1+\kappa n)\frac{\xi^{n-1}-1}{\xi-1}-(n-1)\right]}\tau^{(n-1)\frac{\xi-1}{n}+\kappa+1} \\ & = & \lambda_{\xi^n}\tau^{\frac{1}{n}\left[(1+\kappa n)\frac{\xi^{n}-1}{\xi-1}\right]}. \end{array}$$

If $n \ge 3$ then $\tau^{\alpha|_0} = \lambda_{\xi^n}\tau^{\frac{1}{n}\left[(1+\kappa n)\frac{\xi^{n}-1}{\xi-1}\right]}$ satisfies the same conditions as those for α; namely, both $\xi^n, \rho' = \frac{1}{n}\left[(1+\kappa n)\frac{\xi^n-1}{\xi-1}\right] \in \mathbb{Z}_n^1$. If $n = 2$

then $\xi = 1 + 2\xi'$, $\rho' = \frac{1}{2}\left[(1+2\kappa)\frac{\xi^2-1}{\xi-1}\right] = (1+2\kappa)(1+\xi')$ and so, $\rho' \in \mathbb{Z}_2^1$ implies $\xi = 1 + 4\xi''$. $\square$

4. Abelian groups B normalized by τ

Let B be an abelian subgroup of $\mathcal{A}_n$ normalized by τ. For a fixed $\beta \in B$, we define group,

$$H = \langle \beta|_i \ (i \in Y), \tau \rangle ,$$

and its subgroups

$$\begin{aligned} N &= \left\langle [\beta|_i, \tau^{k_i}] \mid k_i \in \mathbb{Z}, i \in Y \right\rangle \\ M &= N \langle \tau \rangle . \end{aligned}$$

Furthermore, when $\sigma_\beta = (\sigma_\tau)^s$ *for some integer* s *we set* $m = \frac{n}{\gcd(n,s)}$, *and define*

$$\begin{aligned} K &= \left\langle N, \ \beta|_i \beta|_{\overline{i+s}} \beta|_{\overline{i+2s}} \cdots \beta|_{\overline{i+(m-1)s}} \mid i \in Y \right\rangle , \\ O &= K \langle \tau \rangle . \end{aligned}$$

First, we show that when n is a power of a prime number p^k, the activity range of β is narrowed down to a Sylow p-subgroup of Σ_n.

Proposition 5. *Let* $n = p^k$, $\sigma = (0, 1, ..., n-1)$ *and* P *be a Sylow* p*-subgroup* P *of* Σ_n *which contains* σ. *Then*
(i) P *is a wreath product of cyclic groups of order* p *iterated* k *times, and its normalizer is* $N_{\Sigma_n}(P) = P\langle c \rangle$ *where* c *is cyclic of order* $p-1$*;*
(ii) P *is the unique Sylow* p*-subgroup* P *of* Σ_n *which contains* σ*;*
(iii) if W *is an abelian subgroup of* Σ_n *normalized by* σ *then* W *is contained in* P*;*
(iv) the abelian group B *is a subgroup of the layer closure* $L = L\left(N_{\Sigma_p}(P)\right)$.

Proof. (i) The structure of P is well-known. The center of P is $Z = \left\langle z = \sigma_\tau^{p^{k-1}} \right\rangle$ and $C_{\Sigma_n}(z) = P$. Therefore, $N_{\Sigma_n}(P) = N_{\Sigma_n}(Z) = P\langle c \rangle$ where c is cyclic of order $p-1$.

(ii) If $\sigma \in P^g$ for some $g \in \Sigma_n$ then $z^g \in C_{\Sigma_n}(\sigma) = \langle \sigma \rangle$ and therefore $\langle z^g \rangle = \langle z \rangle$, $P^g = P$. Thus, P is the unique Sylow p-subgroup of Σ_n to contain σ.

(iii) Let W be an abelian subgroup of Σ_n normalized by σ. Let $V = W < \sigma >$ and V_0 be the stabilizer of 0 in V. Then, $V = V_0 \langle \sigma \rangle$, $V_0 \cap \langle \sigma \rangle = \{e\}$. Suppose that there exists a prime q different from p which divides the order of W and let Q be the unique Sylow q-subgroup of W. Then Q is the unique Sylow q-subgroup of V and

$Q \leq V_0$. Therefore $Q = \{e\}$, W a p-group and as $\sigma \in W$ we have $W \leq P$..

(iv) As the normal closure of $\langle \sigma_\beta \rangle$ under $\langle \sigma_\tau \rangle$ is abelian, it follows that $\sigma_\beta \in P$. Furthermore, as $\left\langle [\beta|_u, \tau^k] \mid k \in \mathbb{Z} \right\rangle$ is an abelian group normalized by τ, it follows that $[\sigma_{\beta|_u}, \sigma] \in P$ and therefore $\sigma^{\sigma_{\beta|_u}} \in P$. Thus, $\sigma_{\beta|_u} \in N_{\Sigma_n}(P)$ and hence, $\beta \in L$. □

Lemma 5. *Let $\gamma \in \mathcal{A}_n$. Conditions (i), (ii) below are equivalent:*
(i) $[\gamma, \gamma^{\tau^k}] = e$ for all $k \in \mathbb{Z}$;
(ii) $[\tau^k, \gamma, \gamma] = e$ for all $k \in \mathbb{Z}$.
Condition (i) implies
(iii) $\left\langle [\gamma, \tau^k] \mid k \in \mathbb{Z} \right\rangle$ is a commutative group.
Condition (iii) implies
$\left\langle [\gamma|_u, \tau^k] \mid k \in \mathbb{Z} \right\rangle$ is a commutative group for all indices u.

Proof. First,

$$\begin{aligned} [\gamma, \gamma^{\tau^k}] &= \gamma^{-1} \left(\tau^{-k}\gamma^{-1}\tau^k\right) \gamma \left(\tau^{-k}\gamma\tau^k\right) \\ &= \gamma^{-1} \left(\tau^{-k}\gamma^{-1}\tau^k\gamma\right) \gamma \left(\gamma^{-1}\tau^{-k}\gamma\tau^k\right) \\ &= [\tau^k, \gamma]^\gamma [\gamma, \tau^k] \end{aligned}$$

and so,

$$[\gamma, \gamma^{\tau^k}] = e \Leftrightarrow [\gamma, \tau^k]^\gamma = [\gamma, \tau^k].$$

Furthermore, since

$$[\gamma, \tau^{k_1}]^{\tau^{k_2}} = [\gamma, \tau^{k_2}]^{-1} [\gamma, \tau^{k_1+k_2}] \tag{14}$$

or all integers k_1, k_2, condition (ii) implies

$$\begin{aligned} [\gamma, \tau^{k_1}]^{[\gamma, \tau^{k_2}]} &= [\gamma, \tau^{k_1}]^{\gamma^{-1}\tau^{-k_2}\gamma\tau^{k_2}} = [\gamma, \tau^{k_1}]^{\tau^{-k_2}\gamma\tau^{k_2}} \\ &= \left([\gamma, \tau^{-k_2}]^{-1} [\gamma, \tau^{k_1-k_2}]\right)^{\gamma\tau^{k_2}} = \left([\gamma, \tau^{-k_2}]^{-1} [\gamma, \tau^{k_1-k_2}]\right)^{\tau^{k_2}} \\ &= [\gamma, \tau^{k_1}]. \end{aligned}$$

Finally, we note that by (6) and (7),

$$\begin{aligned} ([\gamma, \tau^{nk}])|_{(i)\sigma_\gamma} &= (\gamma^{-1})|_{(i)\sigma_\gamma} (\tau^{-nk})|_i \,(\gamma|_i)\, (\tau^{nk})|_{(i)\sigma_\gamma} \\ &= \left(\gamma|_i^{-1}\right) \tau^{-k} \,(\gamma|_i)\, \tau^k \\ &= [\gamma|_i, \tau^k]. \end{aligned}$$

Since $[\gamma, \tau^{kn}]$ is inactive for all $k \in \mathbb{Z}$, we obtain $\{[\gamma|_i, \tau^k] \mid k \in \mathbb{Z}\}$ is a commutative set for all i. The rest of the assertion follows by induction on the tree level. □

Obviously, $\left\langle [\beta, \tau^k] \mid k \in \mathbb{Z} \right\rangle$ is normalized by τ and if condition (i) holds then this subgroup is an abelian normal subgroup of $\langle \beta, \tau \rangle$.

Proposition 6. *Let $l \geq 1$ and suppose $\alpha, \gamma \in \mathrm{Stab}(l)$ such that $[\alpha, \gamma^{\tau^x}] = e$ for all $x \in \mathbb{Z}$. Then*

$$\begin{aligned} [\alpha|_u, \gamma|_v{}^{\tau^x}] &= e \ \forall u, v \in \mathcal{M} \\ \text{such that } |u| &= |v| \ \leq l \text{ and } \forall x \in \mathbb{Z}. \end{aligned}$$

Proof. We start with the case $l = 1$. Write $x = r + kn$ where $r = \overline{x}$.
By (4) and (5),

$$\begin{aligned} \left(\gamma^{\tau^x}\right)|_{(i)\tau^x} &= (\tau^x)|_i^{-1} \gamma|_i (\tau^x)_i, \\ \left(\gamma^{\tau^x}\right)|_i &= \tau^{-k-\delta(i-r,r)} \gamma|_{\overline{i-r}} \tau^{k+\delta(i-r,r)}. \end{aligned}$$

As $[\alpha, \gamma^{\tau^x}] = e$ and $\alpha, \gamma^{\tau^x} \in \mathrm{Stab}(1)$,we have, for all $i, j, r \in Y$ and all $k, x \in \mathbb{Z}$,

$$\begin{aligned} [\alpha|_i, (\gamma^{\tau^x})|_i] &= e, \ [\alpha|_i, \gamma|_{\overline{i-r}}^{\tau^{k+\delta(i-r,r)}}] = e, \\ [\alpha|_i, (\gamma|_j)^{\tau^x}] &= e. \end{aligned}$$

The general case $l \geq 1$ follows by induction. □

The following is an application to $\beta \in B$.

Corollary 2. *Let $\sigma_\beta = e$. Then for all $i, j \in Y$ and for all $x \in \mathbb{Z}$*

$$[\beta|_i, \beta|_j^{\tau^x}] = e.$$

We derive further relations in H.

Proposition 7. *Let $\beta \in B$. Then the following relations hold in H for all $v \in \mathbb{Z}$ and for all $i \in Y$:*

(I)

$$\begin{aligned} &\left(\tau^v|_{(i)\sigma_\tau^{-v}}\right)^{-1} \left(\beta|_{(i)\sigma_\tau^{-v}}\right) \left(\tau^v|_{(i)\sigma_\tau^{-v}\sigma_\beta}\right) \left(\beta|_{(i)\sigma_\tau^{-v}\sigma_\beta\sigma_\tau^v}\right) \\ = \ &(\beta|_i) \left(\tau^v|_{(i)\sigma_\beta\sigma_\tau^{-v}}\right)^{-1} \left(\beta|_{(i)\sigma_\beta\sigma_\tau^{-v}}\right) \left(\tau^v|_{(i)\sigma_\beta\sigma_\tau^{-v}\sigma_\beta}\right), \end{aligned}$$

$$[\sigma_\beta, \sigma_\beta^{\sigma_\tau^v}] = e;$$

(II)

$$[\beta|_i, \tau^v]^{\beta|_{(i)\sigma_\beta}} = [\beta|_{(i)\sigma_\beta}, \tau^v];$$

(III)

$$\beta|_{(i)\sigma_\beta} \beta|_{(i)\sigma_\beta^2} \cdots \beta|_{(i)\sigma_\beta^{s_i}} \text{ commutes with } [\beta|_i, \tau^v]$$

where s_i is the size of the orbit of i under the action of $\langle \sigma_\beta \rangle$.

Proof. (I) Clearly $[\beta,\beta^{\tau^v}]=e$ implies $[\sigma_\beta,\sigma_\beta^{\sigma_\tau^v}]=e$. Also, it implies

$$\left(\beta|_{(i)\sigma_{\beta^{\tau^v}}}\right)^{-1}\left(\beta^{\tau^v}|_i\right)^{-1}\beta|_i\left(\beta^{\tau^v}|_{(i)\sigma_\beta}\right)=e,$$

$$\left(\beta^{\tau^v}|_i\left(\beta|_{(i)\sigma_{\beta^{\tau^v}}}\right)=\beta|_i\left(\beta^{\tau^v}|_{(i)\sigma_\beta}\right),\right.$$

$$\begin{aligned}&\left(\tau^v|_{(i)\sigma_{\tau^v}^{-1}}\right)^{-1}\left(\beta|_{(i)\sigma_{\tau^v}^{-1}}\right)\left(\tau^v|_{(i)\sigma_{\tau^v}^{-1}\sigma_\beta}\right)\left(\beta|_{(i)\sigma_{\beta^{\tau^v}}}\right)\\ =\;&\left(\beta|_i\right)\left(\tau^v|_{(i)\sigma_\beta\sigma_{\tau^v}^{-1}}\right)^{-1}\left(\beta|_{(i)\sigma_\beta\sigma_{\tau^v}^{-1}}\right)\left((\tau^v)|_{(i)\sigma_\beta\sigma_{\tau^v}^{-1}\sigma_\beta}\right).\end{aligned}$$

(II) Exchanging v by nv in (I), we obtain:

$$\tau^{-v}\left(\beta|_i\right)\tau^v\left(\beta|_{(i)\sigma_\beta}\right)=\left(\beta|_i\right)\tau^{-v}\left(\beta|_{(i)\sigma_\beta}\right)\tau^v,$$

$$\begin{aligned}&\left(\beta|_{(i)\sigma_\beta}\right)^{-1}\left(\beta|_i^{-1}\tau^{-v}\beta|_i\tau^v\right)\left(\beta|_{(i)\sigma_\beta}\right)\\ =\;&\left(\left(\beta|_{(i)\sigma_\beta}\right)^{-1}\beta|_i^{-1}\right)\beta|_i\tau^{-v}\left(\beta|_{(i)\sigma_\beta}\right)\tau^v.\end{aligned}$$

(III) From (II), we get

$$[\beta|_i,\tau^v]^{\left(\beta|_{(i)\sigma_\beta}\beta|_{(i)\sigma_\beta^2}\cdots\beta|_{(i)\sigma_\beta^{s_i}}\right)}=[\beta|_{(i)\sigma_\beta},\tau^v]^{\left(\beta|_{(i)\sigma_\beta^2}\cdots\beta|_{(i)\sigma_\beta^{s_i}}\right)}=\ldots=[\beta|_i,\tau^v].$$

□

5. The case $\beta\in B$ with $\sigma_\beta\in\langle\sigma_\tau\rangle$

This section is devoted to the proof of the second part *(I)* of Theorem B. We introduce the following combination of step functions

$$\Delta_s(i,t)=\delta(i,t-i)-\delta(i-s,t-i)$$

and call it the *Inductor Function.* Then

Lemma 6. *Let $\beta\in\mathcal{A}_n$ such that $[\beta,\beta^{\tau^x}]=e$ for any $x\in\mathbb{Z}$ and let $\sigma_\beta=\sigma_\tau^s$ for some $s\in Y$. Then,*

$$\begin{aligned}&\tau^{\Delta_s(i,t)}\left(\beta|_{i-s}\right)[\beta|_{i-s},\tau^z]\left(\beta|_t\right)\\ =\;&\left(\beta|_{t-s}\right)\left(\beta|_i\right)[\beta|_i,\tau^z]\tau^{\Delta_s(i+s,t+s)}.\end{aligned}$$

for all $i,t\in\{0,1,\cdots,n-1\}$, $z\in\mathbb{Z}$

Proof. Since $\sigma_\beta=\sigma_\tau^s$, we have $\sigma_{\beta^{\tau^x}}=\sigma_\beta=\sigma_\tau^s$.

From (4), (5), (6) and (7), we obtain

$$\begin{aligned}&\tau^{-\frac{x-\overline{x}}{n}-\delta(j-x,x)}\beta\big|_{j-x}\tau^{\frac{x-\overline{x}}{n}+\delta(j-x+s,x)}\beta\big|_{j+s}\\ =\;&\beta\big|_j\tau^{-\frac{x-\overline{x}}{n}-\delta(j+s-x,x)}\beta\big|_{j+s-x}\tau^{\frac{x-\overline{x}}{n}+\delta(j+2s-x,x)}\end{aligned}\tag{15}$$

Setting $k=\dfrac{x-\overline{x}}{n}$ and $r=\overline{x}$ and using (15), we have

$$\tau^{-k-\delta(j-r,r)}\beta\big|_{j-r}\tau^{k+\delta(j+s-r,r)}\beta\big|_{j+s} = \beta\big|_j\tau^{-k-\delta(j+s-r,r)}\beta\big|_{j+s-r}\tau^{k+\delta(j+2s-r,r)}, \tag{16}$$

for all $r, j \in Y$ and all $k \in \mathbb{Z}$.

Also on setting $t = \overline{j+s}, i = \overline{j+s-r}$ and $z = k + \delta(j+s-r, r) = k + \delta(i, t-i)$ and using (16), we obtain

$$\begin{aligned} & \tau^{-z+\delta(i,t-i)-\delta(i-s,t-i)}\beta\big|_{i-s}\tau^{z}\beta\big|_{t} \\ = \; & \beta\big|_{t-s}\tau^{-z}\beta\big|_{i}\tau^{z-\delta(i,t-i)+\delta(i+s,t-i)}, \end{aligned}$$

for all $t, i \in \{0, 1, \cdots, n-1\}$ and all $z \in \mathbb{Z}$.

Thus

$$\begin{aligned} & \tau^{\delta(i,t-i)-\delta(i-s,t-i)}\beta\big|_{i-s}[\beta\big|_{i-s}, \tau^{z}]\beta\big|_{t} \\ = \; & \beta\big|_{t-s}\beta\big|_{i}[\beta\big|_{i}, \tau^{z}]\tau^{-\delta(i,t-i)+\delta(i+s,t-i)} \end{aligned}$$

for all $t, i \in \{0, 1, \cdots, n-1\}$ and all $z \in \mathbb{Z}$. □

We develop below some properties of the Δ_s function to be used in further results.

Proposition 8. *The inductor function satisfies*

(i) $\Delta_s(i,t) = \delta(i,-s) - \delta(t,-s) = \begin{cases} 0, & \text{if } \overline{t}, \overline{i} \geq \overline{s} \text{ or } \overline{t}, \overline{i} < \overline{s} \\ 1, & \text{if } \overline{t} < \overline{s} \leq \overline{i} \\ -1, & \text{if } \overline{i} < \overline{s} \leq \overline{t} \end{cases}$,

(ii) $\Delta_s(i,t) = -\Delta_s(t,i)$,

(iii) $\Delta_s(i+s, t+s) = -\Delta_{-s}(i,t)$,

(iv) $\Delta_s(i,t) = \Delta_s(i,z) + \Delta_s(z,t)$,

(v) $\displaystyle\sum_{k=0}^{\frac{n}{(s,n)}-1} \Delta_s(i+ks, t+ks) = 0$,

(vi) $\displaystyle\sum_{k=0}^{n-1} \Delta_s(k,t) = \begin{cases} n - \overline{s}, & \text{if } \overline{t} < \overline{s} \\ -\overline{s} & \text{if } \overline{t} \geq \overline{s} \end{cases}$

for all $i, t, z \in \mathbb{Z}$.

Proof.

(i) Using the definition $\delta(i,j)=\frac{\overline{i}+\overline{j}-\overline{i+j}}{n}$ we have

$$\begin{aligned}\Delta_s(i,t) &= \frac{\overline{i}+\overline{t-i}-\overline{t}}{n}-\frac{\overline{i-s}+\overline{t-i}-\overline{t-s}}{n}\\ &= \frac{\overline{i}+\overline{-s}-\overline{i-s}}{n}-\frac{\overline{t}+\overline{-s}-\overline{t-s}}{n}\\ &= \delta(i,-s)-\delta(t,-s)\\ &= \begin{cases} 0, & \text{if } \overline{t},\overline{i}\geq\overline{s} \text{ or } \overline{t},\overline{i}<\overline{s}\\ 1, & \text{if } \overline{t}<\overline{s}\leq\overline{i}\\ -1, & \text{if } \overline{i}<\overline{s}\leq\overline{t}\end{cases}.\end{aligned}$$

(ii) Follows from (i).

(iii)

$$\begin{aligned}\Delta_s(i+s,t+s) &= \delta(i+s,t-i)-\delta(i,t-i)\\ &= -\left(\delta(i,t-i)-\delta(i+s,t-i)\right)\\ &= -\Delta_{-s}(i,t).\end{aligned}$$

(iv) Follows from (i).

(v) Follows from the definition of the Polarizer function and

$$\sum_{k=0}^{\frac{n}{(n,s)}-1}\delta(i+ks,t-i)=\sum_{k=0}^{\frac{n}{(n,s)}-1}\delta(i+(k-1)s,t-i).$$

(vi)

$$\begin{aligned}\sum_{k=0}^{n-1}\Delta_s(k,t) &= \textstyle\sum_{k=0}^{\overline{s}-1}\Delta_s(k,t)+\sum_{k=\overline{s}}^{n-1}\Delta_s(k,t)\\ &\overset{(i)}{=} \begin{cases} n-\overline{s}, & \text{if } \overline{t}<\overline{s}\\ -\overline{s}, & \text{if } \overline{t}\geq\overline{s}\end{cases}.\end{aligned}$$

□

With the use of the inductor function notation we obtain

Proposition 9. *The following relations are verified in H, for all $x,z\in\mathbb{Z}$ and for all $i,t\in Y$:*

(I) $\tau^{\Delta_s(i,t)}\beta|_{\overline{i-s}}\beta|_t=\beta|_{\overline{t-s}}\beta|_i\tau^{\Delta_s(i+s,t+s)}$;

(II) $[\beta|_{\overline{i-s}},\tau^z]^{\beta|_t\tau^{-\Delta_s(i+s,t+s)}}=[\beta|_i,\tau^z]$;

(III) $[[\beta|_i,\tau^z],[\beta|_t,\tau^x]]=e$.

Proof. Returning to Lemma 6, we have

$$\begin{aligned}&\tau^{\Delta_s(i,t)}\left(\beta|_{i-s}\right)[\beta|_{i-s},\tau^z]\left(\beta|_t\right)\\ =\ &\left(\beta|_{t-s}\right)\left(\beta|_i\right)[\beta|_i,\tau^z]\tau^{\Delta_s(i+s,t+s)}.\end{aligned}$$

Consequently,

$$\tau^{\Delta_s(i,t)}\beta|_{\overline{i-s}}\beta|_t = \beta|_{\overline{t-s}}\beta|_i\tau^{\Delta_s(i+s,t+s)} \tag{17}$$

and

$$[\beta|_{\overline{i-s}}, \tau^z]^{\beta|_t\tau^{-\Delta_s(i+s,t+s)}} = [\beta|_i, \tau^z], \tag{18}$$

for all $t, i \in Y$ and all $z \in \mathbb{Z}$.

From (18) and (14), $N = \left\langle [\beta|_i, \tau^{k_i}] \mid k_i \in \mathbb{Z}, i \in Y \right\rangle$ is a normal subgroup of H. Moreover, applying alternately the above equations, we obtain

$$\begin{aligned}
[\beta|_i, \tau^z]^{[\beta|_t,\tau^k]} &= [\beta|_i, \tau^z]^{\beta|_t^{-1}\tau^{-k}\beta|_t\tau^k} \\
&= [\beta|_i, \tau^z]^{\left(\tau^{-\Delta_s(i+s,t+s)}\tau^{\Delta_s(i+s,t+s)}\beta|_t^{-1}\tau^{-k}\beta|_t\tau^k\right)} \\
&\overset{(14)}{=} \left([\beta|_i, \tau^{-\Delta_s(i+s,t+s)}]^{-1}.[\beta|_i, \tau^{z-\Delta_s(i+s,t+s)}]\right)^{\left(\tau^{\Delta_s(i+s,t+s)}\beta|_t^{-1}\tau^{-k}\beta|_t\tau^k\right)} \\
&\overset{(18)}{=} \left([\beta|_{\overline{i-s}}, \tau^{-\Delta_s(i+s,t+s)}]^{-1}.[\beta|_{\overline{i-s}}, \tau^{z-\Delta_s(i+s,t+s)}]\right)^{\tau^{-k}\beta|_t\tau^k} \\
&\overset{(14)}{=} \begin{pmatrix} \left([\beta|_{\overline{i-s}}, \tau^{-k}]^{-1}.[\beta|_{i-s}, \tau^{-k-\Delta_s(i+s,t+s)}]\right)^{-1} \\ \left([\beta|_{\overline{i-s}}, \tau^{-k}]^{-1}.[\beta|_{\overline{i-s}}, \tau^{-k+z-\Delta_s(i+s,t+s)}]\right) \end{pmatrix}^{\beta|_t\tau^k} \\
&= \left([\beta|_{\overline{i-s}}, \tau^{-k-\Delta_s(i+s,t+s)}]^{-1}.[\beta|_{\overline{i-s}}, \tau^{-k+z-\Delta_s(i+s,t+s)}]\right)^{\beta|_t\tau^k} \\
&\overset{(18)}{=} \left([\beta|_i, \tau^{-k-\Delta_s(i+s,t+s)}]^{-1}.[\beta|_i, \tau^{-k+z-\Delta_s(i+s,t+s)}]\right)^{\tau^{k+\Delta_s(i+s,t+s)}} \\
&\overset{(14)}{=} [\beta|_i, \tau^z].
\end{aligned}$$

□

Corollary 3. *Let $\beta \in A_n$ such that $[\beta, \beta^{\tau^x}] = e$ for every $x \in \mathbb{Z}$ with $\sigma_\beta = \sigma_\tau^s$ for some $s \in \{0, 1, \cdots, n-1\}$. Then*

$$M = \left\langle [\beta|_i, \tau^{k_i}], \tau \mid k_i \in \mathbb{Z}, 0 \leq i \leq n-1 \right\rangle.$$

is a normal metabelian subgroup of H.

Proof. By Proposition 9 $N = \left\langle [\beta|_i, \tau^{k_i}] \mid k_i \in \mathbb{Z}, 0 \leq i \leq n-1 \right\rangle$ is abelian and normal in H. Since $N\tau \in Z(H/N)$, it follows that $M = N\langle\tau\rangle$ is a normal subgroup of H and is clearly metabelian. □

We are ready to prove part *(II) (i) of* Theorem B.

Define the following sequence of subgroups of H,

Theorem 1. *Let $\beta \in \mathcal{A}_n$ be such that $[\beta, \beta^{\tau^x}] = e, \forall x \in \mathbb{Z}$ and $\sigma_\beta = \sigma_\tau^s$ for some $s \in Y$ and $H = \langle \beta|_0, \cdots, \beta|_{n-1}, \tau\rangle$. Then,*

(i) $O = \left\langle [\beta|_i, \tau^x], \beta|_j \beta|_{j+s} \cdots \beta|_{j+(m-1)s}, \tau \mid i, j \in Y, x \in \mathbb{Z}_n \right\rangle$ *is an abelian normal subgroup of* H;

(ii) H/O *is isomorphic to a subgroup of* $C_m \wr C_n$.
In particular, H *is metabelian-by-finite.*

Proof. (i) Recall

$$\begin{aligned} N &= \left\langle [\beta|_i, \tau^{k_i}] \mid k_i \in \mathbb{Z}, i \in Y \right\rangle, \\ K &= N \left\langle \beta|_j \beta|_{\overline{j+s}} \cdots \beta|_{j+(m-1)s} \mid j \in Y \right\rangle \end{aligned}$$

where $m = \frac{n}{\gcd(n,s)}$. Then, by Proposition 9, N is an abelian normal subgroup of H.

By (18), we have

$$\begin{aligned} & [\beta|_i, \tau^z]^{\beta|_j \beta|_{\overline{j+s}} \cdots \beta|_{\overline{j+(m-1)s}}} \\ = \; & [\beta|_{i+s}, \tau^z]^{\tau^{\Delta_t(i+2s, j+s)} \beta|_{\overline{j+s}} \cdots \beta|_{\overline{j+(m-1)s}}} \\ = \; & [\beta|_{i+2s}, \tau^z]^{\tau^{\Delta_s(i+2s, j+s) + \Delta_s(i+3s, j+2s)} \beta|_{\overline{j+2s}} \cdots \beta|_{\overline{j+(m-1)s}}} \\ = \; & [\beta|_i, \tau^z]^{\tau^{\sum_{k=0}^{m-1} \Delta_s(i+(k+1)s, j+ks)}} \\ \overset{\text{Prop.}8(v)}{=} \; & [\beta|_i, \tau^z] \end{aligned}$$

Thus,

$$[[\beta|_i, \tau^z], (\beta^m)|_j] = e, \forall i, j \in Y, \forall z \in \mathbb{Z} \tag{19}$$

Since $\sigma_\beta = \sigma_\tau^s$, we have by Lemma 2

$$[(\beta^m)|_i, (\beta^m)|_j] = e, \forall i, j \in Y. \tag{20}$$

Moreover,

$$(\beta^m)|_i^\tau = (\beta^m)|_i [(\beta^m)|_i, \tau]. \tag{21}$$

Since $[\beta, \beta^{\tau^x}] = e, \forall x \in \mathbb{Z}$, it follows that $[\beta^m, \beta^{\tau^x}] = e, \forall x \in \mathbb{Z}$. Therefore, by (6) and (7),

$$e = (\beta^m)|_{(i)\sigma_{\beta^{\tau^x}}}^{-1} (\beta^{\tau^x})|_i^{-1} (\beta^m)|_i (\beta^{\tau^x})|_{(i)\sigma_{\beta^m}}, \forall x \in \mathbb{Z}, \forall i \in Y.$$

Now, as $\sigma_\beta = \sigma_\tau^s$ and $\sigma_{\beta^m} = e$, we reach

$$(\beta^m)|_{\overline{i+s}} = (\beta^m)|_i^{(\beta^{\tau^x})|_i}, \forall x \in \mathbb{Z}, \forall i \in Y. \tag{22}$$

By (4) and (5), the following

$$(\beta^{\tau^x})_i = (\tau^x)_{(i)\sigma_{\tau^x}^{-1}}^{-1} \beta|_{(i)\sigma_{\tau^x}^{-1}} (\tau^x)|_{(i)\sigma_{\tau^x}^{-1}\sigma_\beta} = (\tau^x)|_{\overline{i-x}}^{-1} \beta|_{\overline{i-x}} (\tau^x)_{\overline{i-x+s}}$$

holds for all $i \in Y$ and all $x \in \mathbb{Z}$.

From which,

$$(\beta^{\tau^x})|_i = \tau^{-\frac{x-\overline{x}}{n} - \delta(i-x,x)} \beta|_{\overline{i-x}} \tau^{\frac{x-\overline{x}}{n} + \delta(i-x+s,x)}, \tag{23}$$

holds for all $i \in Y$ and all $x \in \mathbb{Z}$.

Therefore, by (22) and (23),

$$(\beta^m)|_{\overline{i+s}} = (\beta^m)|_i^{\tau^{-\frac{x-\overline{x}}{n}-\delta(i-x,x)}\beta|_{\overline{i-x}}\tau^{\frac{x-\overline{x}}{n}+\delta(i-x+s,x)}},$$

for all $i \in Y$ and all $x \in \mathbb{Z}$..

On writing $x = kn + \overline{x} = kn + r, r \in \mathbb{Z}$ in the above equation, we obtain

$$(\beta^m)|_{\overline{i+s}} = (\beta^m)|_i^{\tau^{-k-\delta(i-r,r)}\beta|_{\overline{i-r}}\tau^{k+\delta(i-r+s,r)}}$$

$$\Rightarrow (\beta^m)|_{\overline{i+s}}^{\tau^{-k-\delta(i-r+s,r)}} = (\beta^m)|_i^{\beta|_{\overline{i-r}}\tau^{-k-\delta(i-r,r)}[\tau^{-k-\delta(i-r,r)},\beta|_{\overline{i-r}}]}$$

$$\Rightarrow (\beta^m)|_{\overline{i+s}}^{\tau^{-k-\delta(i-r+s,r)}[\beta|_{\overline{i-r}},\tau^{-k-\delta(i-r,r)}]\tau^{k+\delta(i-r,r)}} = (\beta^m)|_i^{\beta|_{\overline{i-r}}}$$

for all $i, r \in Y$ and all $k \in \mathbb{Z}$.

By (19), (21) and using the fact that N is abelian and normal in H, we find

$$(\beta^m)|_{\overline{i+s}}^{\tau^{\delta(i-r,r)-\delta(i-r+s,r)}} = (\beta^m)|_i^{\beta|_{\overline{i-r}}}$$

$$\Rightarrow (\beta^m)|_{\overline{i+s}}^{\tau^{\delta(i-r,i-r+s)}} = (\beta^m)|_i^{\beta|_{\overline{i-r}}}$$

for all $i, r \in Y$.

On setting $j = \overline{i-r}$, we get

$$(\beta^m)|_{\overline{i+s}}^{\tau^{\delta(j,j+s)}} = (\beta^m)|_i^{\beta|_j} \tag{24}$$

for all $i, j \in Y$.

Further, by using equations (19),(20) (21), (24) and

$$(\beta^m)|_i = \beta|_i\beta|_{\overline{i+s}}\cdots\beta|_{\overline{i+(m-1)s}}, \tag{25}$$

we conclude that also K is an abelian normal subgroup of H.

Now, $O = K\langle\tau\rangle$ is metabelian. Moreover it is normal in H, because

$$\tau^{\beta|_i} = \tau\tau^{-1}\tau^{\beta|_i} = \tau[\tau,\beta|_i] \in O$$

for all $i \in Y$.

(ii) Now consider the Fibonacci type group defined by

$$X = \left\langle b_0, \cdots, b_{n-1} \mid b_i b_{\overline{j+s}} = b_j b_{\overline{i+s}}, b_i b_{\overline{i+s}} \cdots b_{\overline{i+(m-1)s}} = e, \forall i, j \in Y \right\rangle$$

Equations (17) and (18) show that $\dfrac{H}{M}$ is a homomorphic image of X. We will prove that G is isomorphic to a subgroup of the wreath product $C_m \wr C_n$.

As a matter of fact the group $C_m \wr C_n$ has the presentation

$$\left\langle u, a \mid u^m = e, a^n = e, u^{a^i} u^{a^j} = u^{a^j} u^{a^i} \right\rangle.$$

On defining $b = a^s u^{-1}$, we have

$$\begin{aligned} u^m = e \quad & (a^{-s}b)^m = e \\ & \Rightarrow (\underbrace{a^{-s}b \cdots a^{-s}b}_{m \text{ terms}})^{a^{-s+i}} = e \\ & \Rightarrow b^{a^i} b^{a^{i+s}} \cdots b^{a^{i+(m-1)s}} = e \end{aligned}$$

and

$$u^{a^i} u^{a^j} = u^{a^j} u^{a^i}$$

implies

$$\begin{aligned} \Rightarrow \quad & (b^{-1}a^s)^{a^i}(b^{-1}a^s)^{a^j} = (b^{-1}a^s)^{a^j}(b^{-1}a^s)^{a^i} \\ \Rightarrow \quad & (a^{-s}b)^{a^j}(a^{-s}b)^{a^i} = (a^{-s}b)^{a^i}(a^{-s}b)^{a^j} \\ \Rightarrow \quad & b^{a^j} a^{-s} b^{a^i} = b^{a^i} a^{-s} b^{a^j} \\ \Rightarrow \quad & b^{a^j} b^{a^{i+s}} = b^{a^i} b^{a^{j+s}}. \end{aligned}$$

Thus, by Tietze transformations $C_m \wr C_n$ has the presentation

$$\left\langle a, b \mid a^n = e, b^{a^j} b^{a^{i+s}} = b^{a^i} b^{a^{j+s}}, b^{a^i} b^{a^{i+s}} \cdots b^{a^{i+(m-1)s}} = e, \forall i, j \in Y \right\rangle$$

Then, on introducing $b_i = b^{a^i}, i = 0, \cdots, n-1$, the above presentation can be expressed as

$$\left\langle a, b_0, \cdots, b_{n-1} \mid a^n = e, b_i = b_0^{a^i},\ b_j b_{\overline{i+s}} = b_i b_{\overline{j+s}},\ b_i b_{\overline{i+s}} \cdots b_{\overline{i+(m-1)s}} = e, \right.$$
$$\left. \forall i, j \in Y \right\rangle.$$

□

The next results will lead to a proof of Theorem C.

Lemma 7. *Let L be the layer closure of $\langle \sigma \rangle$ in $\mathcal{A}_n$ and let $\sigma = (0, 1, ..., n-1)$. Suppose $\beta = (\beta|_0, \beta|_1, \cdots, \beta|_{n-1})\sigma_\beta \in L$ satisfies $[\beta, \beta^{\tau^x}] = e$ for all $x \in \mathbb{Z}$. Write $\sigma_\beta = \sigma^s$ and $\sigma_{\beta|_i} = \sigma$ for all $i \in Y$. Then for all $i, j \in Y$, the following congruence holds*

$$\Delta_s(i,t) + m_{\overline{i-s}} + m_t \equiv m_{\overline{t-s}} + m_i + \Delta_s(i+s, t+s) \bmod n, \tag{26}$$

Proof. Since $\sigma_{\beta|_i} = \sigma^{m_i}$, we conclude by (17),

$$\sigma^{\Delta_s(i,t) + m_{\overline{i-s}} + m_t} = \sigma^{m_{\overline{t-s}} + m_i + \Delta_s(i+s, t+s)}$$

and therefore, $\Delta_s(i,t) + m_{\overline{i-s}} + m_t \equiv m_{\overline{t-s}} + m_i + \Delta_s(i+s, t+s) \bmod n$. □

Lemma 8. *Maintain the notation of the previous lemma and let n be an odd integer. Then,*

$$\sigma_{(\beta^n)|_0} = \sigma_{(\beta|_0\beta|_1\cdots\beta|_{n-1})} = \sigma.$$

Proof. From

$$\Delta_1(i,t) + m_{\overline{i-1}} + m_t \equiv m_{\overline{t-1}} + m_i + \Delta_1(i+1,t+1) \bmod n$$

we conclude

$$\begin{aligned} &\sum_{i=0}^{n-2}\sum_{t=i+1}^{n-1}\big(\Delta_1(i,t) + m_{\overline{i-1}} + m_t\big) \\ \equiv &\sum_{i=0}^{n-2}\sum_{t=i+1}^{n-1}\left(m_{\overline{t-1}} + m_i + \Delta_1(i+1,t+1)\right) \bmod n. \end{aligned}$$

Now,

$$\sum_{i=0}^{n-2}\sum_{t=i+1}^{n-1}\Delta_1(i,t) \overset{\text{Prop.8(i)}}{=} \sum_{t=1}^{n-1}\Delta_1(0,t) \overset{\text{Prop.8(ii)}}{=} \sum_{t=0}^{n-1}\Delta_1(0,t)$$

$$\overset{\text{Prop.8(ii)}}{=} \sum_{t=0}^{n-1} -\Delta_1(t,0) \overset{\text{Prop.8(vi)}}{=} -(n-1),$$

$$\sum_{i=0}^{n-2}\sum_{t=i+1}^{n-1}\Delta_1(i+1,t+1) \overset{\text{Prop.8(i)}}{=} \sum_{i=0}^{n-2}\Delta_1(i+1,0) \overset{\text{Prop.8(ii)}}{=} \sum_{i=0}^{n-1}\Delta_1(i,0)$$

$$\overset{\text{Prop.8(vi)}}{=} (n-1),$$

$$\sum_{i=0}^{n-2}\sum_{t=i+1}^{n-1}\big(m_{\overline{i-1}} + m_t\big) = 2(n-1)m_{n-1} + (n-2)\sum_{k=0}^{n-2} m_k$$

and

$$\sum_{i=0}^{n-2}\sum_{t=i+1}^{n-1}\left(m_{\overline{t-1}} + m_i\right) = n\sum_{k=0}^{n-1} m_k.$$

Since n is odd, we have

$$\sum_{k=0}^{n-1} m_k \equiv 1 \bmod n$$

and therefore, $\sigma_{\beta|_0\cdots\beta|_{n-1}} = \sigma^{\sum_{k=0}^{n-1} m_k} = \sigma$. □

Now we prove Theorem C.

Theorem 2. *Let n be an odd number, $\sigma = (0, \cdots, n-1) \in \Sigma_n$ and let L be the layer closure of $\langle\sigma\rangle$ in A_n. Let s an integer relatively prime to n and $\beta = (\beta|_0, \beta|_1, \cdots, \beta|_{n-1})\sigma^s \in L$ be such that $[\beta, \beta^{\tau^x}] = e$ for all $x \in Z$. Then β is a conjugate of τ in $Aut(T_n)$.*

Proof. We start with the case $s = 1$. The element

$$\alpha(1) = (e, \beta|_0^{-1}, (\beta|_0\beta|_1)^{-1}, \cdots, (\beta|_0 \cdots \beta|_{n-2})^{-1}) \in \mathrm{Stab}_L(1)$$

conjugates β to

$$\beta^{\alpha(1)} = (e, \cdots, e, \beta|_0 \cdots \beta|_{n-1})\sigma.$$

By Lemma8 we find $\sigma_{\beta|_0\beta|_1\cdots\beta|_{n-1}} = \sigma$. Moreover by Proposition 6,

$$[(\beta^n)|_0, (\beta^n)|_0^{\tau^x}] = [\beta|_0\beta|_1 \cdots \beta|_{n-1}, (\beta|_0\beta|_1 \cdots \beta|_{n-1})^{\tau^x}] = e,$$

for all integers x. Therefore $\beta|_0\beta|_1 \cdots \beta|_{n-1}$ satisfies the hypothesis of the theorem. The process can be repeated until we obtain a sequence $(\alpha(k))_{k\in\mathbb{N}}$ such that $\beta^{\alpha(1)\alpha(2)\cdots\alpha(k)\cdots} = \tau$, where $\alpha(k) \in \mathrm{Stab}_L(k)$ satisfies $\alpha(k)|_u = \alpha(k)|_v$ for all $u, v \in \mathcal{M}$ with $|u| = |v| = k-1$.

Now, suppose more generally s is such $\gcd(s, n) = 1$ and let k be a minimum positive integer for which $sk \equiv 1 \bmod (n)$. Then β^k satisfies the hypothesis of the first part and so, there exists $\alpha \in L$ such that $(\beta^k)^\alpha = \tau$. Since k is invertible in $\mathbb{Z}_n$, there exists an automorphism γ of the tree such that $\tau^\gamma = \tau^{k^{-1}}$. Thus, $\beta^{\alpha\gamma^{-1}} = \tau$. □

6. Solvable groups for $n = p$, a prime number.

We will prove in this section the case $n = p$ of Theorem A.

Let B be an abelian subgroup of $Aut(T_p)$ normalized by τ and let $\beta \in B$. By Lemma 5, $\sigma_\beta \in \langle\sigma_\tau\rangle$ and therefore basically we have two cases, $\sigma_\beta = e, \sigma_\tau$.

Proposition 10. *Suppose $\sigma_\beta = \sigma_\tau$. Then, $\sigma_{\beta|_i} \in \langle\sigma_\tau\rangle$ for all $i \in Y$.*

Proof. By theorem 1, O is a normal subgroup of H and $\frac{H}{O}$ is isomorphic to a subgroup of $C_p \wr C_p$.

By Lemma 5, O is a subgroup of $\langle\sigma_\tau\rangle$ modulo $Stab_p(1)$.

Therefore, H is a p-group modulo $Stab_p(1)$ and by Lemma 5, we have $\sigma_{\beta|_i} \in \langle\sigma_\tau\rangle$. □

Theorem 3. *Let p be a prime number and $\beta \in \mathrm{Aut}(T_p)$ such that $\sigma_\beta = \sigma_\tau^s$ for some integer s relatively prime to p. Suppose $[\beta, \beta^{\tau^x}] = e$ for all $x \in \mathbb{Z}$. Then β is conjugate to τ in $\mathrm{Aut}(T_p)$.*

Proof. Suppose $s = 1$. Recall that

$$\alpha(1) = (e, \beta|_0^{-1}, (\beta|_0\beta|_1)^{-1}, \cdots, (\beta|_0 \cdots \beta|_{p-2})^{-1}) \in \mathrm{Stab}_G(1)$$

conjugates β to its normal form

$$\beta^{\alpha(1)} = (e, \cdots, e, \beta|_0 \cdots \beta|_{p-1})\sigma.$$

By Lemma 8 we have $\sigma_{\beta|_0\beta|_1\cdots\beta|_{p-1}} = \sigma_\tau$. Moreover by Proposition 6,

$$[\beta^p|_0, (\beta^p|_0)^{\tau^x}] = [\beta|_0\beta|_1 \cdots \beta|_{p-1}, (\beta|_0\beta|_1 \cdots \beta|_{p-1})^{\tau^x}] = e,$$

for all integers x. Therefore $\beta|_0\beta|_1 \cdots \beta|_{n-1}$ satisfies the condition of the theorem.. This process can be repeated to produce a sequence $(\alpha(k))_{k\in\mathbb{N}}$ such that $\beta^{\alpha(1)\alpha(2)\cdots\alpha(k)\cdots} = \tau$, where $\alpha(k) \in \mathrm{Stab}(k)$ satisfies $\alpha(k)|_u = \alpha(k)|_v$ for all $u, v \in \mathcal{M}$ where $|u| = |v| = k - 1$.

Now, to the general case, s such $\gcd(p, s) = 1$. Let k be the minimum positive integer which is the inverse of s modulo p. Then, $\sigma|_{\beta^k} = \sigma_\tau$ and β^k satisfies the hypotheses. Thus there exists $\alpha \in \mathcal{A}_p$ such that $\left(\beta^k\right)^\alpha = \tau$. Let k^{-1} be the inverse of k in $U(\mathbb{Z}_n)$; then $\beta^\alpha = \tau^{k^{-1}}$. There exists $\gamma \in N_{\mathcal{A}_p}\overline{<\tau>}$ which conjugates τ to $\tau^{k^{-1}}$ and so, $(\beta^\alpha)^{\gamma^{-1}} = \tau$. □

Lemma 9. *Let p be a prime number and $\beta \in \mathrm{Aut}(T_p)$ such that Suppose $[\beta, \beta^{\tau^x}] = e$ for all $x \in \mathbb{Z}$. Then, there exists a tree level m and a conjugate μ of τ such that $\beta \in \times_{p^m}\overline{\langle\mu\rangle}$ and there exists an index u of length m such that $\beta|_u = \mu$.*

Proof. Let m be the minimum tree level such that $\sigma_{\beta|_u} \neq e$ for some $|u| = m$. Therefore, $\sigma_{\beta|_u} = \sigma_\tau^s$ for some integer s such that $\gcd(p, s) = 1$ and so, $\mu = \beta|_u$ is conjugate to τ in $\mathrm{Aut}(T_p)$. Since $\beta \in \mathrm{Stab}(m)$, by Proposition 6 $[\mu, \beta|_v] = e$ for all indices v such that $|v| = m$. Therefore, $\beta|_v \in \overline{\langle\mu\rangle}$ for all v such that $|v| = m$. □

Theorem 4. *Let p be a prime number, $\sigma = (0, 1, \cdots, p-1) \in \Sigma_p$, $F = N_{\Sigma_p}(\langle\sigma\rangle)$, $\Gamma_0 = N_{\mathcal{A}}(\overline{<\tau>})$. Let G be a finitely generated solvable subgroup of $Aut\,(T_p)$ which contains the p-adic adding machine τ. Then, there exists an integer $t \geq 1$ such that G is conjugate to a subgroup of*

$$\times_p (\cdots (\times_p (\times_p \Gamma_0 \rtimes F) \rtimes F) \cdots) \rtimes F.$$

Proof. We may suppose G has derived length $d \geq 2$. Let B be the $(d-1)$-th term of the derived series of G. By Theorem 9, there exists a level t such that B is a subgroup of $V = \times_{p^t}\overline{\langle\mu\rangle}$ where $\mu = \tau^\alpha$ for some $\alpha \in Aut\,(T_n)$.

We will show that G is a subgroup of

$$\dot{J} = \times_p (\cdots (\times_p (\times_p (\Gamma_0)^\alpha \rtimes \Sigma_p) \rtimes \Sigma_p) \cdots) \rtimes \Sigma_p,$$

where $\times_p$ appears t times.

Let $\gamma \in G \backslash \dot{J}$. Then there exists an index w of length t such that $\gamma|_w \notin (\Gamma_0)^\alpha$. Since τ is transitive on all levels of the tree , by Theorem 9, there exists $\beta \in B$ such that $\beta|_w = \mu^\eta$ for some $\eta \in U(\mathbb{Z}_p)$.

Write $v = w^\gamma$. Then,

$$(\beta^\gamma)|_v \overset{(9)}{=} \left(\beta|_{v^{\gamma^{-1}}}\right)^{\gamma|_{v^{\gamma^{-1}}}} = (\beta|_w)^{\gamma|_w} \notin \overline{\langle\mu\rangle},$$

and this implies $\beta^\gamma \notin B \leq \overline{\langle\mu\rangle}$ and $\gamma \notin G$. Hence, G is a subgroup of $\dot{J}$.

Now, since G is a solvable group containing τ, there exist G_i $(0 \leq i \leq t)$ solvable subgroups of Σ_p containing $\sigma = (0, 1, \cdots, p-1)$ such that G is a subgroup of

$$R_t(\alpha) = \times_p (\cdots (\times_p (\times_p (\Gamma_0)^\alpha \rtimes G_1) \rtimes G_2) \cdots) \rtimes G_t.$$

Since for all i, we have $G_i \leq F$ we may substitute the $G_i's$ by F. Finally, $R_t(\alpha)$ is a conjugate of $R_t(1)$ by the diagonal automorphism $\alpha^{(t)}$. $\square$

7. Two cases for n even

7.1. The case $\sigma_\beta = (\sigma_\tau)^{\frac{n}{2}}$.

Theorem 5. *Let n be an even number, $\beta \in \mathcal{A}_n$ such that $\sigma_\beta = \sigma_\tau^{\frac{n}{2}}$ and $[\beta, \beta^{\tau^x}] = e$ for all $x \in \mathbb{Z}$. Then $H = \langle \beta|_i \ (0 \leq i \leq n-1), \tau\rangle$ is a metabelian subgroup of $\mathcal{A}_n$.*

Proof. Define the subgroup

$$R = \left\langle [\beta|_t, \tau^k],\ \beta|_i\beta|_{i+\frac{n}{2}},\ \beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})} \mid k \in \mathbb{Z} \text{ and } i,j,t \in Y \right\rangle.$$

Denote $\Delta_{\frac{n}{2}}(i,j)$ by $\Delta(i,j)$.

We will prove that N is an abelian normal subgroup of H.

(I) R is normal in H :

– $\left\langle [\beta|_i, \tau^k] \right\rangle^H \leq R$:

$$[\beta|_{i+\frac{n}{2}}, \tau^k]^{\beta|_j} \overset{(18)}{=} [\beta|_i, \tau^k]^{\tau^{\Delta(j,i)}};$$

– $\left\langle \beta|_i\beta_{i+\frac{n}{2}} \right\rangle^H \leq R$:

$$\begin{aligned}(\beta|_i\beta|_{i+\frac{n}{2}})^{\tau^k} &= \left(\beta|_i\beta|_{i+\frac{n}{2}}\right).[\beta|_i\beta|_{i+\frac{n}{2}}, \tau^k] \\ &= \left(\beta|_i\beta|_{i+\frac{n}{2}}\right)[\beta|_i, \tau^k]^{\beta|_{i+\frac{n}{2}}}[\beta|_{i+\frac{n}{2}}, \tau^k]\end{aligned}$$

$$\overset{(18)}{=} \left(\beta|_i\beta|_{i+\frac{n}{2}}\right)[\beta|_{i+\frac{n}{2}}, \tau^k]^{\tau^{\Delta(i+\frac{n}{2},i+\frac{n}{2})}}[\beta|_{i+\frac{n}{2}}, \tau^k] \overset{\text{Prop.8}}{=} \beta|_i\beta|_{i+\frac{n}{2}}[\beta|_{i+\frac{n}{2}}, \tau^k]^2$$

$$(\beta|_i\beta|_{i+\frac{n}{2}})^{\beta|_j} = \left(\beta|_j^{-1}\beta|_i\beta|_{i+\frac{n}{2}}\beta|_j\right)\tau^{\Delta(j+\frac{n}{2},i+\frac{n}{2})}\tau^{-\Delta(j+\frac{n}{2},i+\frac{n}{2})} \tag{27}$$

$$\stackrel{(17)}{=} \left(\beta|_j^{-1}\beta|_i\right)\tau^{\Delta(j,i)}\left(\beta|_{j+\frac{n}{2}}\beta|_i\right)\tau^{-\Delta(j+\frac{n}{2},i+\frac{n}{2})}$$

$$= \left(\beta|_j^{-1}\beta|_i\beta|_{j+\frac{n}{2}}\right)\tau^{\Delta(j,i)}.[\tau^{\Delta(j,i)},\beta|_{j+\frac{n}{2}}].\beta|_i\tau^{-\Delta(j+\frac{n}{2},i+\frac{n}{2})}$$

$$\stackrel{(17)}{=} \left(\beta|_j^{-1}\right)\tau^{\Delta(j+\frac{n}{2},i+\frac{n}{2})}\left(\beta|_j\beta|_{i+\frac{n}{2}}\right). [\tau^{\Delta(j,i)},\beta|_{j+\frac{n}{2}}].\beta|_i\tau^{-\Delta(j+\frac{n}{2},i+\frac{n}{2})}$$

$$= \tau^{\Delta(j+\frac{n}{2},i+\frac{n}{2})}.[\tau^{\Delta(j+\frac{n}{2},i+\frac{n}{2})},\beta|_j]. \beta|_{i+\frac{n}{2}}[\tau^{\Delta(j,i)},\beta|_{j+\frac{n}{2}}]\beta|_i\tau^{-\Delta(j+\frac{n}{2},i+\frac{n}{2})}$$

$$\stackrel{Prop.8}{=} \tau^{-\Delta(j,i)}[\tau^{-\Delta(j,i)},\beta|_j].\beta|_{i+\frac{n}{2}}. [\tau^{\Delta(j,i)},\beta|_{j+\frac{n}{2}}]\beta|_i\tau^{\Delta(j,i)}$$

$$\stackrel{(18)}{=} \tau^{-\Delta(j,i)}\beta|_{i+\frac{n}{2}}.[\tau^{-\Delta(j,i)},\beta|_{j+\frac{n}{2}}]^{\tau^{\Delta(j,i)}}.[\tau^{\Delta(j,i)},\beta|_{j+\frac{n}{2}}].\beta|_i\tau^{\Delta(j,i)}$$

$$\stackrel{(14)}{=} \left(\beta|_{i+\frac{n}{2}}\beta|_i\right)^{\tau^{\Delta(j,i)}}.$$

– $\left\langle\beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}\right\rangle^H \leq R$:

$$\begin{aligned}
&(\beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})})^{\tau^k} = \beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}.[\beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})},\tau^k]\\
= \;& \beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}.[\beta|_j^2,\tau^k]^{\tau^{-\Delta(j,j+\frac{n}{2})}}\\
= \;& \beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}\left([\beta|_j,\tau^k]^{\beta|_j}.[\beta|_j,\tau^k]\right)^{\tau^{-\Delta(j,j+\frac{n}{2})}}\\
\stackrel{(18)}{=} \;& \beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}\left([\beta|_{j+\frac{n}{2}},\tau^k]^{\tau^{\Delta(j,j+\frac{n}{2})}}.[\beta|_j,\tau^k]\right)^{\tau^{-\Delta(j,j+\frac{n}{2})}}\\
= \;& \beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}[\beta|_{j+\frac{n}{2}},\tau^k][\beta|_j,\tau^k]^{\tau^{-\Delta(j,j+\frac{n}{2})}}.
\end{aligned}$$

By Proposition 8 and 9, we can show

$$\left(\beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}\right)^{\beta|_i} = \left(\beta|_{j+\frac{n}{2}}^2\tau^{-\Delta(j+\frac{n}{2},j)}[\tau^{-\Delta(j+\frac{n}{2},j)},\beta|_{j+\frac{n}{2}}]\right)^{\tau^{\Delta(i,j)}}. \tag{28}$$

(II) R is an abelian subgroup:

$$[\beta|_i,\tau^k]^{\beta|_j\tau^t} \stackrel{Prop.9}{=} [\beta|_i,\tau^k]^{\tau^t\beta|_j}; \tag{29}$$

(30)

$$[\beta|_i,\tau^k]^{\beta|_j\beta|_{j+\frac{n}{2}}} \stackrel{(18)}{=} [\beta|_{i+\frac{n}{2}},\tau^k]^{\tau^{\Delta(j,i+\frac{n}{2})}\beta|_{j+\frac{n}{2}}} \stackrel{(29)}{=} [\beta|_{i+\frac{n}{2}},\tau^k]^{\beta|_{j+\frac{n}{2}}\tau^{\Delta(j,i+\frac{n}{2})}}$$

$$\overset{(18)}{=} [\beta|_i, \tau^k]^{\tau^{\Delta(j+\frac{n}{2},i)+\Delta(j,i+\frac{n}{2})}} \overset{\text{Prop.}8}{=} [\beta|_i, \tau^k]$$

(31)

$$\begin{aligned} & [\beta|_i, \tau^k]^{\beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}} \overset{(18)}{=} [\beta|_{i+\frac{n}{2}}, \tau^k]^{\tau^{\Delta(j,i+\frac{n}{2})}\beta|_j\tau^{-\Delta(j,j+\frac{n}{2})}} \\ \overset{(29)}{=} \; & [\beta|_{i+\frac{n}{2}}, \tau^k]^{\beta|_j\tau^{\Delta(j,i+\frac{n}{2})-\Delta(j,j+\frac{n}{2})}} \overset{(18)}{=} [\beta|_i, \tau^k]^{\tau^{\Delta(j,i)+\Delta(j,i+\frac{n}{2})-\Delta(j,j+\frac{n}{2})}} \\ \overset{\text{Prop.}8}{=} \; & [\beta|_i, \tau^k] \end{aligned}$$

$$\begin{aligned} \left(\beta|_i\beta|_{i+\frac{n}{2}}\right)^{\beta|_j\beta|_{j+\frac{n}{2}}} & \overset{(27)}{=} \left(\beta|_{i+\frac{n}{2}}\beta|_i\right)^{\tau^{\Delta(j,i)}\beta|_{j+\frac{n}{2}}} \\ & = \left(\beta|_{i+\frac{n}{2}}\beta|_i\right)^{\left(\beta|_{j+\frac{n}{2}}\tau^{\Delta(j,i)}[\tau^{\Delta(j,i)},\beta|_{j+\frac{n}{2}}]\right)} \\ & \overset{(27)}{=} \left(\beta|_i\beta|_{i+\frac{n}{2}}\right)^{\left(\tau^{\Delta(j+\frac{n}{2},i+\frac{n}{2})+\Delta(j,i)}.[\tau^{\Delta(j,i)},\beta|_{j+\frac{n}{2}}]\right)} \\ & \overset{\text{Prop.}8}{=} \left(\beta|_i\beta|_{i+\frac{n}{2}}\right)^{[\tau^{\Delta(j,i)},\beta|_{j+\frac{n}{2}}]} \\ & \overset{(30)}{=} \beta|_i\beta|_{i+\frac{n}{2}} \end{aligned}$$

$$\begin{aligned} (\beta|_i\beta|_{i+\frac{n}{2}})^{\beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}} & \overset{(27)}{=} (\beta|_{i+\frac{n}{2}}\beta|_i)^{\tau^{\Delta(j,i)}\beta|_j\tau^{-\Delta(j,j+\frac{n}{2})}} \\ & = (\beta|_{i+\frac{n}{2}}\beta|_i)^{\beta|_j\tau^{\Delta(j,i)}[\tau^{\Delta(j,i)},\beta|_j]\tau^{-\Delta(j,j+\frac{n}{2})}} \\ & = (\beta|_i\beta|_{i+\frac{n}{2}})^{\tau^{\Delta(j,i+\frac{n}{2})+\Delta(j,i)}[\tau^{\Delta(j,i)},\beta|_j]\tau^{-\Delta(j,j+\frac{n}{2})}} \\ & \overset{\text{Prop.}8}{=} (\beta|_i\beta|_{i+\frac{n}{2}})^{[\tau^{\Delta(j,i)},\beta|_j]^{\tau^{\Delta(j+\frac{n}{2},j)}}} \\ & \overset{\text{Prop.}9}{=} \beta|_i\beta|_{i+\frac{n}{2}} \end{aligned}$$

Let

$$\alpha = \beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}[\tau^{-\Delta(j,j+\frac{n}{2})}, \beta|_j]. \tag{32}$$

Then,

$$\begin{aligned} & \left(\beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}\right)^{\beta|_i^2\tau^{-\Delta(i,i+\frac{n}{2})}} \\ \overset{(28)}{=} \; & \left(\beta|_{j+\frac{n}{2}}^2\tau^{-\Delta(j+\frac{n}{2},j)}.[\tau^{-\Delta(j+\frac{n}{2},j)}, \beta|_{j+\frac{n}{2}}]\right)^{\tau^{\Delta(i,j)}\beta|_i\tau^{-\Delta(i,i+\frac{n}{2})}} \end{aligned}$$

$$= \left(\beta|_{j+\frac{n}{2}}^2\tau^{-\Delta(j+\frac{n}{2},j)}.[\tau^{-\Delta(j+\frac{n}{2},j)}, \beta|_{j+\frac{n}{2}}]\right)^{\left(\beta|_i\tau^{\Delta(i,j)}.[\tau^{\Delta(i,j)},\beta|_i].\tau^{-\Delta(i,i+\frac{n}{2})}\right)}$$

$$= \left(\left(\beta|_{j+\frac{n}{2}}^2\tau^{-\Delta(j+\frac{n}{2},j)}\right)^{\beta|_i}.[\tau^{-\Delta(j+\frac{n}{2},j)}, \beta|_{j+\frac{n}{2}}]^{\beta|_i}\right)^{\left(\tau^{\Delta(i,j)}.[\tau^{\Delta(i,j)},\beta|_i].\tau^{-\Delta(i,i+\frac{n}{2})}\right)}$$

$$\overset{(18)}{=} \left(\left(\beta|_{j+\frac{n}{2}}^2\tau^{-\Delta(j+\frac{n}{2},j)}\right)^{\beta|_i}.[\tau^{-\Delta(j+\frac{n}{2},j)}, \beta|_j]^{\tau^{\Delta(i,j)}}\right)^{\left(\tau^{\Delta(i,j)}.[\tau^{\Delta(i,j)},\beta|_i].\tau^{-\Delta(i,i+\frac{n}{2})}\right)}$$

$$\overset{(28)}{=} \left(\alpha^{\tau^{\Delta(i,j+\frac{n}{2})}}.[\tau^{-\Delta(j+\frac{n}{2},j)}, \beta|_j]^{\tau^{\Delta(i,j)}}\right)^{\left(\tau^{\Delta(i,j)}.[\tau^{\Delta(i,j)},\beta|_i].\tau^{-\Delta(i,i+\frac{n}{2})}\right)}$$

$$= \left(\alpha.[\tau^{-\Delta(j+\frac{n}{2},j)}, \beta|_j]^{\tau^{\Delta(i,j)-\Delta(i,j+\frac{n}{2})}}\right)^{\left(\tau^{\Delta(i,j+\frac{n}{2})+\Delta(i,j)}.[\tau^{\Delta(i,j)},\beta|_i].\tau^{-\Delta(i,i+\frac{n}{2})}\right)}$$

$$\overset{\text{Prop.8}}{=} \left(\alpha.[\tau^{-\Delta(j+\frac{n}{2},j)}, \beta|_j]^{\tau^{\Delta(j+\frac{n}{2},j)}}\right)^{\left(\tau^{\Delta(i,i+\frac{n}{2})}[\tau^{\Delta(i,j)},\beta|_i]\tau^{-\Delta(i,i+\frac{n}{2})}\right)}$$

$$\overset{(32)}{=} \left(\beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}[\tau^{-\Delta(j,j+\frac{n}{2})}, \beta|_j][\tau^{\Delta(j+\frac{n}{2},j)}, \beta|_j]^{-1}\right)^{[\tau^{\Delta(i,j)},\beta|_i]^{\tau^{-\Delta(i,i+\frac{n}{2})}}}$$

$$\overset{\text{Prop.8}}{=} \left(\beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}\right)^{[\tau^{\Delta(i,j)},\beta|_i]^{\tau^{-\Delta(i,i+\frac{n}{2})}}}$$

$$\overset{\text{Prop.9 e (31)}}{=} \beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}.$$

Moreover, since

$$\begin{aligned} R(\beta|_i)\, R(\beta|_j) &= R(\beta|_i)(\beta|_j) \overset{\text{Prop.5}}{=} R\tau^{\Delta(j,i+\frac{n}{2})}\beta|_{j+\frac{n}{2}}\beta|_{i+\frac{n}{2}}\tau^{\Delta(j,i+\frac{n}{2})} \\ &= R\beta|_{j+\frac{n}{2}}\beta|_{i+\frac{n}{2}}\tau^{2\Delta(j,i+\frac{n}{2})} = R\beta|_j^{-1}\beta|_i^{-1}\tau^{2\Delta(j,i+\frac{n}{2})} \\ &= R\beta|_j^{-1}\beta|_j^2\tau^{-\Delta(j,j+\frac{n}{2})}\beta|_i^{-1}\beta|_i^2\tau^{-\Delta(i,i+\frac{n}{2})}\tau^{2\Delta(j,i+\frac{n}{2})} \\ &= R\beta|_j\beta|_i\tau^{-\Delta(j,j+\frac{n}{2})-\Delta(i,i+\frac{n}{2})+2\Delta(j,i+\frac{n}{2})} \\ &\overset{\text{Prop.8}}{=} R\beta|_j\beta|_i = R\beta|_jN\beta|_i \end{aligned}$$

and

$$R\beta|_i = R\beta|_{i+\frac{n}{2}}^{-1}, \quad R\beta|_i^2 = R\tau^{\Delta(i,i+\frac{n}{2})}, \forall i,j \in Y,$$

we conclude $\dfrac{H}{R}$ is a homomorphic image of

$$\mathbb{Z} \times \underbrace{C_2 \times \cdots \times C_2}_{\frac{n}{2}\ \text{terms}}.$$

□

7.2. **The case σ_β transposition.** We prove in this section part *(II) (ii)* of Theorem B.

Theorem 6. *Let n be an even number, B an abelian subgroup of $\mathcal{A}_n$ normalized by τ. Suppose $\beta = (\beta|_0, \beta|_1, \cdots, \beta|_{n-1})\sigma_\beta \in B$ where σ_β is a transposition. Then $H = \langle \beta|_i \ (0 \le i \le n-1), \tau\rangle$ is a metabelian group.*

We prove progressively that

$$\begin{aligned} N &= \left\langle [\beta|_i, \tau^k] \mid k \in \mathbb{Z}, i \in Y \right\rangle, \\ U &= \left\langle N, \quad \beta|_j \mid j \neq 0, \frac{n}{2} \right\rangle, \\ V &= \left\langle U, \quad \beta|_{\frac{n}{2}}\beta|_0, \quad \tau\,(\beta|_0)^2 \right\rangle \end{aligned}$$

are normal abelian subgroups of H, from which it follows that $\frac{H}{V}$ is cyclic and therefore H metabelian.

Lemma 10. *The degree of the tree n is even and σ_β is $\langle\sigma_\tau\rangle$-conjugate to the transposition $\left(0, \frac{n}{2}\right)$.*

Proof. On conjugating by an appropriate power of σ_τ, we may assume $\sigma_\beta = (0, j)$. The conjugates of σ_β by σ_τ^i produce $(i, j+i)$. In particular, $(j, 2j)$ is a conjugate which is supposed to commute with $(0, j)$. Therefore, $\{0, j\} = \{j, 2j\}$, $2j = 0$ modulo(n), $n = 2n'$ and $j = n'$. $\square$

We go back to part (I) of the Proposition 7,

$$\begin{aligned} &\left(\tau^v|_{(i)\sigma_\tau^{-v}}\right)^{-1}\left(\beta|_{(i)\sigma_\tau^{-v}}\right)\left(\tau^v|_{(i)\sigma_\tau^{-v}\sigma_\beta}\right)\left(\beta|_{(i)\sigma_\tau^{-v}\sigma_\beta\sigma_\tau^v}\right) \\ = \quad &(\beta|_i)\left(\tau^v|_{(i)\sigma_\beta\sigma_\tau^{-v}}\right)^{-1}\left(\beta|_{(i)\sigma_\beta\sigma_\tau^{-v}}\right)\left(\tau^v|_{(i)\sigma_\beta\sigma_\tau^{-v}\sigma_\beta}\right) \end{aligned}$$

and set in it $j = (i)\,\sigma_\tau^{-v}$, $v = kn + r$, $r = \overline{v}$ to obtain

$$(\tau^v)|_j^{-1}\beta|_j(\tau^v)|_{(j)\sigma_\beta}\beta|_{(j)\sigma_\beta\sigma_\tau^v} \tag{33}$$

$$= \beta|_{(j)\sigma_\tau^v}(\tau^v)|^{-1}_{(j)\sigma_\tau^v\sigma_\beta\sigma_\tau^{-v}}\beta|_{(j)\sigma_\tau^v\sigma_\beta\sigma_\tau^{-v}}(\tau^v)_{(j)\sigma_\tau^v\sigma_\beta\sigma_\tau^{-v}\sigma_\beta}. \tag{34}$$

Proposition 11. *The following cases hold for different pairs (j, r).*

- *For $j = 0$ there are 3 subcases*
  - *If $r = 0$, then*

$$[\beta|_0, \tau^k]^{\beta|_{\frac{n}{2}}} = [\beta|_{\frac{n}{2}}, \tau^k], \ \forall k \in \mathbb{Z}; \tag{35}$$

  - *If $r = \frac{n}{2}$, then*

$$\beta|_0\tau\beta|_0 = \beta|_{\frac{n}{2}}\tau^{-1}\beta|_{\frac{n}{2}}, \tag{36}$$

and

$$[\beta|_0, \tau^k]^{\tau\beta|_0} = [\beta|_{\frac{n}{2}}, \tau^k], \forall k \in \mathbb{Z}. \tag{37}$$

  - *If $r \neq 0$ and $r \neq \frac{n}{2}$, then*

$$\tau^{\delta(\frac{n}{2}, r)}\beta|_0\beta|_{\frac{n}{2}+r} = \beta|_r\tau^{\delta(\frac{n}{2}, r)}\beta|_0, \forall r \in Y - \{0, \frac{n}{2}\} \tag{38}$$

and

$$[\beta|_0, \tau^k]^{\beta|_r} = [\beta|_0, \tau^k], \forall k \in \mathbb{Z}. \tag{39}$$

- *For* $j = \frac{n}{2}$ *there are* 3 *subcases*
  - *If* $r = 0$, *then*

$$[\beta|_{\frac{n}{2}}, \tau^k]^{\beta|_0} = [\beta|_0, \tau^k],\ \forall k \in \mathbb{Z}; \tag{40}$$

  - *If* $r = \frac{n}{2}$, *then*

$$\tau^{-1}\beta|_{\frac{n}{2}}^2 = \beta|_0^2\tau, \tag{41}$$

and

$$[\beta|_{\frac{n}{2}}, \tau^k]^{\beta|_{\frac{n}{2}}\tau^{-1}} = [\beta|_0, \tau^k], \forall k \in \mathbb{Z}; \tag{42}$$

  - *If* $r \neq 0$ *and* $r \neq \frac{n}{2}$, *then*

$$\tau^{-\delta(\frac{n}{2},r)}\beta|_{\frac{n}{2}}\beta|_r = \beta|_{\frac{n}{2}+r}\tau^{-\delta(\frac{n}{2},r)}\beta|_{\frac{n}{2}}, \forall r \in Y - \{0, \frac{n}{2}\} \tag{43}$$

and

$$[\beta|_{\frac{n}{2}}, \tau^k]^{\beta|_r} = [\beta|_{\frac{n}{2}}, \tau^k], \forall k \in \mathbb{Z}, \forall r \in Y - \{0, \frac{n}{2}\}. \tag{44}$$

- *For* $j \neq 0$ *and* $j \neq \frac{n}{2}$, *there are* 5 *subcases:*
  - *If* $j \neq n - r$ *and* $j \neq \frac{n}{2} - r$, *then*

$$\beta|_j\beta_t = \beta|_t\beta|_j, \forall j, t \in Y - \{0, \frac{n}{2}\} \tag{45}$$

and

$$[\beta|_j, \tau^k]^{\beta|_t} = [\beta|_j, \tau^k], \forall j, t \in Y - \{0, \frac{n}{2}\} \tag{46}$$

  - *If* $j = n - r$ *and* $0 < r < \frac{n}{2}$, *then*

$$\tau^{-1}\beta|_{j+\frac{n}{2}}\tau\beta|_0 = \beta|_0\beta|_j, \forall j \in \{1, 2, \cdots, \frac{n}{2} - 1\} \tag{47}$$

and

$$[\beta|_{j+\frac{n}{2}}, \tau^k]^{\tau\beta|_0} = [\beta|_j, \tau^k], \forall j \in \{1, 2, \cdots, \frac{n}{2} - 1\} \tag{48}$$

  - *If* $j = n - r$ *and* $\frac{n}{2} < r \leq n - 1$, *then*

$$\beta|_j\beta|_0 = \beta|_0\beta|_{\frac{n}{2}+j}, \forall j \in \{1, \cdots, \frac{n}{2} - 1\} \tag{49}$$

and

$$[\beta|_j, \tau^k]^{\beta|_0} = [\beta|_{\frac{n}{2}+j}, \tau^k], \forall k \in \mathbb{Z}, \forall j \in \{1, \cdots, \frac{n}{2} - 1\} \tag{50}$$

– *If* $j = \frac{n}{2} - r$ *and* $0 < r < \frac{n}{2}$, *then*

$$\beta|_j\beta|_{\frac{n}{2}} = \beta|_{\frac{n}{2}}\tau^{-1}\beta|_{j+\frac{n}{2}}\tau, \forall j \in \{1, \cdots, \frac{n}{2} - 1\} \tag{51}$$

and

$$[\beta|_j, \tau^k]^{\beta|_{\frac{n}{2}}\tau^{-1}} = [\beta|_{\frac{n}{2}+j}, \tau^k], \forall k \in \mathbb{Z}, \forall j \in \{1, \cdots, \frac{n}{2} - 1\} \tag{52}$$

– *If* $j = \frac{n}{2} - r$ *and* $\frac{n}{2} < r \leq n - 1$, *then*

$$\beta|_{\frac{n}{2}}\beta|_j = \beta|_{\frac{n}{2}+j}\beta|_{\frac{n}{2}}, \forall j \in \{1, \cdots, \frac{n}{2} - 1\} \tag{53}$$

and

$$[\beta|_j, \tau^k] = [\beta|_{\frac{n}{2}+j}, \tau^k]^{\beta|_{\frac{n}{2}}}, \forall k \in \mathbb{Z}, \forall j \in \{1, \cdots, \frac{n}{2} - 1\}. \tag{54}$$

Proof. We will prove just the last case. As $j \notin \{0, \frac{n}{2}, n - r, \frac{n}{2} - r\}$, we have

$(j)\,\sigma_\tau^v = (j)\,\sigma_\beta\sigma_\tau^v = j+r$ and $(j)\,\sigma_\beta = (j)\,\sigma_\tau^v\sigma_\beta\sigma_\tau^{-v} = (j)\,\sigma_\tau^v\sigma_\beta\sigma_\tau^{-v}\sigma_\beta = j$.

Therefore,

$$\big((\tau^v)|_j^{-1}\beta|_j(\tau^v)|_j\beta|_{j+r} = \beta|_{j+r}(\tau^v)|_j^{-1}\beta|_j(\tau^v)_j, \forall v \in \mathbb{Z}\big)$$

$$\Leftrightarrow \big(\tau^{-k-\delta(j,r)}\beta|_j\tau^{k+\delta(j,r)}\beta|_{j+r} = \beta|_{j+r}\tau^{-k-\delta(j,r)}\beta|_j\tau^{k+\delta(j,r)}, \forall k \in \mathbb{Z}\big)$$

$$\Leftrightarrow \big(\beta|_j[\beta|_j, \tau^{k+\delta(j,r)}]\beta|_{j+r} = \beta|_{j+r}\beta|_j[\beta|_j, \tau^{k+\delta(j,r)}], \forall k \in \mathbb{Z}\big),$$

$$\beta|_j\beta_t = \beta|_t\beta|_j, \forall j, t \in Y - \{0, \frac{n}{2}\} \tag{55}$$

and

$$[\beta|_j, \tau^k]^{\beta|_t} = [\beta|_j, \tau^k], \forall j, t \in Y - \{0, \frac{n}{2}\}. \tag{56}$$

□

Lemma 11. $N = \big\langle[\beta|_i, \quad \tau^k] \mid k \in \mathbb{Z}, i \in Y\big\rangle$ *is an abelian normal subgroup of* H.

Proof. Define

$$N_i = \big\langle[\beta|_i, \tau^k] \mid k \in \mathbb{Z}\big\rangle$$

for each $i \in Y$. Then, $N = \langle N_i \mid i \in Y\rangle$, each N_i is an abelian subgroup normalized by τ and

$$[\beta|_i, \tau^k]^{\beta|_j^{-1}} = [\beta|_i, \tau^k], \forall k \in \mathbb{Z}, \forall i, j \in Y, j \neq 0, \frac{n}{2} \tag{57}$$

We have $[N_i, N_j] = 1, \forall i, j \in Y, j \neq 0, \frac{n}{2}$, because

$$\begin{aligned} [\beta|_i, \tau^k]^{[\beta|_j, \tau^t]} &= [\beta|_i, \tau^k]^{\beta|_j^{-1}\tau^{-t}\beta|_j\tau^t} \overset{(57)}{=} [\beta|_i, \tau^k]^{\tau^{-t}\beta|_j\tau^t} \\ &\overset{(14)}{=} \left([\beta|_i, \tau^{-t}]^{-1}[\beta|_i, \tau^{k-t}]\right)^{\beta|_j\tau^t} \\ &\overset{(57)}{=} \left([\beta|_i, \tau^{-t}]^{-1}[\beta|_i, \tau^{k-t}]\right)^{\tau^t} \\ \overset{(14)}{=} [\beta|_i, \tau^k]^{\tau^{-t}\tau^t} &= [\beta|_i, \tau^k], \forall k, t \in \mathbb{Z}, \end{aligned}$$

$\forall i, j \in Y, j \neq 0, \frac{n}{2}$.

Furthermore, $[N_0, N_{\frac{n}{2}}] = 1$, because

$$\begin{aligned} [\beta|_{\frac{n}{2}}, \tau^k]^{[\beta|_0, \tau^t]} &= [\beta|_{\frac{n}{2}}, \tau^k]^{\beta|_0^{-1}\tau^{-t}\beta|_0\tau^t} \overset{(37)}{=} [\beta|_0, \tau^k]^{\tau\tau^{-t}\beta|_0\tau^t} \\ &\overset{(14)}{=} \left([\beta|_0, \tau^{-t}]^{-1}[\beta|_0, \tau^{k-t}]\right)^{\tau\beta|_0\tau^t} \\ &\overset{(37)}{=} \left([\beta|_{\frac{n}{2}}, \tau^{-t}]^{-1}[\beta|_{\frac{n}{2}}, \tau^{k-t}]\right)^{\tau^t} \\ &\overset{(14)}{=} [\beta|_{\frac{n}{2}}, \tau^k]^{\tau^{-t}\tau^t} = [\beta|_{\frac{n}{2}}, \tau^k], \forall k, t \in \mathbb{Z}. \end{aligned}$$

Therefore N is abelian.

Now, equation (57) implies

$$N_i = N_i^{\beta|_j} = N_i^{\beta|_j^{-1}}, \forall i, j \in Y, j \neq 0, \frac{n}{2}; \tag{58}$$

equations (14), (35) imply

$$\left\{ N_{\frac{n}{2}} = N_0^{\beta|_0},\ N_0 = N_{\frac{n}{2}}^{\beta|_0^{-1}} \right. ; \tag{59}$$

equation (40) implies

$$\left\{ N_0 = N_{\frac{n}{2}}^{\beta|_0},\ N_{\frac{n}{2}} = N_0^{\beta|_0^{-1}} \right. ; \tag{60}$$

equations (14), (42) imply

$$\left\{ N_0 = N_{\frac{n}{2}}^{\beta|_{\frac{n}{2}}},\ N_{\frac{n}{2}} = N_0^{\beta|_{\frac{n}{2}}^{-1}} \right. ; \tag{61}$$

equations (14), (48) imply

$$\left\{ N_j = N_{j+\frac{n}{2}}^{\beta|_0},\ N_{j+\frac{n}{2}} = N_j^{\beta|_0^{-1}} \right. , \forall j \in \{1, \cdots, \frac{n}{2} - 1\}; \tag{62}$$

equations (14) and (50) imply

$$\left\{ N_{j+\frac{n}{2}} = N_j^{\beta|_0},\ N_j = N_{j+\frac{n}{2}}^{\beta|_0^{-1}} \right. , \forall j \in \{1, \cdots, \frac{n}{2} - 1\}; \tag{63}$$

equations (14) (52) imply

$$\text{(64)} \qquad \Big\{ N_{j+\frac{n}{2}} = N_j^{\beta|_{\frac{n}{2}}},\ N_j = N_{j+\frac{n}{2}}^{\beta|_{\frac{n}{2}}^{-1}}, \forall j \in \{1, \cdots, \frac{n}{2} - 1\};$$

equations (14), (54) imply

$$\text{(65)} \qquad \Big\{ N_j = N_{j+\frac{n}{2}}^{\beta|_{\frac{n}{2}}},\ N_{j+\frac{n}{2}} = N_j^{\beta|_{\frac{n}{2}}^{-1}}, \forall j \in \{1, \cdots, \frac{n}{2} - 1\}.$$

Thus (57)-(65) prove

$$\begin{aligned} N &= \langle N_i \mid i \in Y \rangle \\ &= \big\langle [\beta|_i, \tau^k] \mid \forall i, k \in \mathbb{Z} \big\rangle \end{aligned}$$

is an abelian normal subgroup of H. □

Lemma 12. $U = \big\langle N, \ \ \beta|_j \mid j \neq 0, \frac{n}{2} \big\rangle$ *is a normal abelian subgroup of* H.

Proof. Lemma 11 and equations (39), (44), (45) and (46) show that U is abelian.

The fact that N is normal in H, together with the following assertions prove that U is normal in H.

Let $J = \big\langle \beta_0, \beta_{\frac{n}{2}}, \tau \big\rangle$. Then, for $j \in Y - \{0, \frac{n}{2}\}$, we have

(I) $\langle \beta|_j \rangle^J \leq U$:

$$\beta|_j^{\tau^t} = \beta|_j[\beta|_j, \tau^t];$$

$$\beta|_j^{\beta|_0} \overset{(49)}{=} \beta|_{j+\frac{n}{2}};$$

$$\beta|_j^{\beta|_0^{-1}} \overset{(47)}{=} \tau^{-1}\beta|_{j+\frac{n}{2}}\tau = \beta|_{j+\frac{n}{2}}[\beta|_{j+\frac{n}{2}}, \tau];$$

$$\beta|_j^{\beta|_{\frac{n}{2}}} \overset{(51)}{=} \tau^{-1}\beta|_{j+\frac{n}{2}}\tau = \beta|_{j+\frac{n}{2}}[\beta|_{j+\frac{n}{2}}, \tau];$$

$$\beta|_j^{\beta|_{\frac{n}{2}}^{-1}} \overset{(53)}{=} \beta|_{j+\frac{n}{2}};$$

(II) $\big\langle \beta|_{j+\frac{n}{2}} \big\rangle^J \leq U$:

$$\beta|_{j+\frac{n}{2}}^{\tau^t} = \beta|_{j+\frac{n}{2}}[\beta|_{j+\frac{n}{2}}, \tau^t];$$

$$\beta|_{j+\frac{n}{2}}^{\beta|_0} \overset{(47)}{=} \beta|_0^{-1}\tau\beta|_0\beta|_j\beta|_0^{-1}\tau^{-1}\beta|_0$$

$$= \big([\beta|_0, \tau]^{-1}\big)^{\tau^{-1}} \beta|_j^{\tau^{-1}} [\beta|_0, \tau]^{\tau^{-1}} \in U;$$

$$\beta|_{j+\frac{n}{2}}^{\beta|_0^{-1}} \overset{(49)}{=} \beta|_j \in U;$$

$$\beta|_{j+\frac{n}{2}}^{\beta|_{\frac{n}{2}}} \overset{(53)}{=} \beta|_j \in U;$$

$$\begin{aligned}\beta|_{j+\frac{n}{2}}^{\beta|_{\frac{n}{2}}^{-1}} &\overset{(51)}{=} \beta|_{\frac{n}{2}}\tau\beta|_{\frac{n}{2}}^{-1}\beta|_j\beta|_{\frac{n}{2}}\tau^{-1}\beta|_{\frac{n}{2}}^{-1} \\ &= [\beta|_{\frac{n}{2}},\tau]^{\beta|_{\frac{n}{2}}^{-1}\tau^{-1}}\beta|_j^{\tau^{-1}}\left([\beta|_{\frac{n}{2}},\tau]^{-1}\right)^{\beta|_{\frac{n}{2}}^{-1}\tau^{-1}}.\end{aligned}$$

Hence, U is a normal abelian subgroup of H. □

Lemma 13. *$V = \left\langle U, \quad \beta|_{\frac{n}{2}}\beta|_0, \quad \tau\beta|_0^2\right\rangle$ is a normal abelian subgroup of H.*

Proof. Lemma 12 together with the following assertions prove that V is a normal abelian subgroup of H.

For $j \in Y - \{0, \frac{n}{2}\}, k \in \mathbb{Z}$, and $J = \left\langle \beta|_0, \beta_{\frac{n}{2}}, \tau, \right\rangle$, we prove

(I) $\beta|_{\frac{n}{2}}\beta|_0 \in C_H(U)$:

$$(\beta|_j)^{\beta|_{\frac{n}{2}}\beta|_0} \overset{(51)}{=} (\beta|_{j+\frac{n}{2}})^{\tau\beta|_0} \overset{(47)}{=} \beta|_j;$$

$$(\beta|_{j+\frac{n}{2}})^{\beta|_{\frac{n}{2}}\beta|_0} \overset{(53)}{=} (\beta|_j)^{\beta|_0} \overset{(49)}{=} \beta|_{j+\frac{n}{2}};$$

$$\begin{aligned}[\beta|_j,\tau^k]^{\beta|_{\frac{n}{2}}\beta|_0} = [\beta|_j,\tau^k]^{\beta|_{\frac{n}{2}}\tau^{-1}\tau\beta|_0} &\overset{(52)}{=} [\beta|_{j+\frac{n}{2}},\tau^k]^{\tau\beta|_0} \\ &\overset{(48)}{=} [\beta|_j,\tau^k];\end{aligned}$$

$$[\beta|_{j+\frac{n}{2}},\tau^k]^{\beta|_{\frac{n}{2}}\beta|_0} \overset{(54)}{=} [\beta|_j,\tau^k]^{\beta|_0} \overset{(50)}{=} [\beta|_{j+\frac{n}{2}},\tau^k];$$

$$[\beta|_0,\tau^k]^{\beta|_{\frac{n}{2}}\beta|_0} \overset{(35)}{=} [\beta|_{\frac{n}{2}},\tau^k]^{\beta|_0} \overset{(40)}{=} [\beta|_0,\tau^k];$$

$$\begin{aligned}[\beta|_{\frac{n}{2}},\tau^k]^{\beta|_{\frac{n}{2}}\beta|_0} &= [\beta|_{\frac{n}{2}},\tau^k]^{\beta|_{\frac{n}{2}}\tau^{-1}\tau\beta|_0} \\ &\overset{(42)}{=} [\beta|_0,\tau^k]^{\tau\beta|_0} \overset{(37)}{=} [\beta|_{\frac{n}{2}},\tau^k];\end{aligned}$$

(II) $\tau\beta|_0^2 \in C_H(U)$:

$$\begin{aligned}\beta|_j^{\tau\beta|_0^2} &= (\beta|_j[\beta|_j,\tau])^{\beta|_0^2} = (\beta|_j^{\beta|_0}[\beta|_j,\tau]^{\beta|_0})^{\beta|_0} \\ &\overset{(49),(50)}{=} (\beta|_{j+\frac{n}{2}}[\beta|_{j+\frac{n}{2}},\tau])^{\beta|_0} = \beta|_{j+\frac{n}{2}}^{\tau\beta|_0} \overset{(47)}{=} \beta|_j;\end{aligned}$$

$$(\beta|_{j+\frac{n}{2}})^{\tau\beta|_0^2} \overset{(47)}{=} \beta|_j^{\beta|_0} \overset{(49)}{=} \beta|_{j+\frac{n}{2}};$$

$$[\beta|_0,\tau^k]^{\tau\beta|_0^2} \overset{(37)}{=} [\beta|_{\frac{n}{2}},\tau^k]^{\beta|_0} \overset{(40)}{=} [\beta|_0,\tau^k];$$

$$\begin{aligned}[\beta|_{\frac{n}{2}},\tau^k]^{\tau\beta|_0^2} &\overset{(14)}{=} ([\beta|_{\frac{n}{2}},\tau]^{-1}[\beta|_{\frac{n}{2}},\tau^{k+1}])^{\beta|_0^2} \\ &\overset{(40)}{=} ([\beta|_0,\tau]^{-1}[\beta|_0,\tau^{k+1}])^{\beta|_0}\end{aligned}$$

$$\overset{(14)}{=} [\beta|_0,\tau^k]^{\tau\beta|_0} \overset{(37)}{=} [\beta|_{\frac{n}{2}},\tau^k];$$

$$[\beta|_j,\tau^k]^{\tau\beta|_0^2} \overset{(14)}{=} ([\beta|_j,\tau]^{-1}[\beta|_j,\tau^{k+1}])^{\beta|_0^2}$$

$$\overset{(50)}{=} ([\beta|_{j+\frac{n}{2}},\tau]^{-1}[\beta|_{j+\frac{n}{2}},\tau^{k+1}])^{\beta|_0}$$

$$\overset{(14)}{=} [\beta|_{j+\frac{n}{2}},\tau^k]^{\tau\beta|_0} \overset{(48)}{=} [\beta|_j,\tau^k];$$

$$[\beta|_{j+\frac{n}{2}},\tau^k]^{\tau\beta|_0^2} \overset{(48)}{=} [\beta|_j,\tau^k]^{\beta|_0} \overset{(50)}{=} [\beta|_{j+\frac{n}{2}},\tau^k];$$

(III) $\tau\beta|_0^2 \in C_H(\beta|_{\frac{n}{2}}\beta|_0)$:

$$\begin{aligned}(\beta|_{\frac{n}{2}}\beta|_0)^{\tau\beta|_0^2} &= \beta|_0^{-2}\tau^{-1}\beta|_{\frac{n}{2}}\beta|_0\tau\beta|_0^2\\ &\overset{(36)}{=} \beta|_0^{-2}\tau^{-1}\beta|_{\frac{n}{2}}\beta|_{\frac{n}{2}}\tau^{-1}\beta|_{\frac{n}{2}}\beta|_0\end{aligned}$$

$$\begin{aligned}&= \beta|_0^{-2}\tau^{-1}\beta|_{\frac{n}{2}}^2\tau^{-1}\beta|_{\frac{n}{2}}\beta|_0 = (\tau\beta|_0^2)^{-1}\beta|_{\frac{n}{2}}^2\tau^{-1}\beta|_{\frac{n}{2}}\beta|_0\\ &\overset{(41)}{=} \beta|_{\frac{n}{2}}\beta|_0;\end{aligned}$$

(IV) $\left\langle\beta|_{\frac{n}{2}},\beta|_0\right\rangle^J \leq V$:

$$(\beta|_{\frac{n}{2}}\beta|_0)^{\tau^k} = \beta|_{\frac{n}{2}}\beta|_0[\beta|_{\frac{n}{2}}\beta|_0,\tau^k] = \beta|_{\frac{n}{2}}\beta|_0[\beta|_{\frac{n}{2}},\tau^k]^{\beta|_0}[\beta|_0,\tau^k];$$

$$(\beta|_{\frac{n}{2}}\beta|_0)^{\beta|_0} = \beta|_0^{-1}\beta|_{\frac{n}{2}}\beta|_0^2 = \beta|_0^{-1}\beta|_{\frac{n}{2}}\tau^{-1}\tau\beta|_0^2 = \beta|_0^{-1}\beta|_{\frac{n}{2}}^{-1}\beta|_{\frac{n}{2}}^2\tau^{-1}\tau\beta|_0^2$$

$$= (\beta|_{\frac{n}{2}}\beta|_0)^{-1}(\tau\beta|_0^2)^2;$$

$$\beta|_{\frac{n}{2}}\beta|_0 \overset{(t)}{=} (\tau\beta|_0^2)^2((\beta|_{\frac{n}{2}}\beta|_0)^{-1})^{\beta|_0};$$

$$(\beta|_{\frac{n}{2}}\beta|_0)^{\beta|_0^{-1}} \overset{(u)}{=} ((\tau\beta|_0^2)^2)^{\beta|_0^{-1}}(\beta|_{\frac{n}{2}}\beta|_0)^{-1};$$

$$(\beta|_{\frac{n}{2}}\beta|_0)^{\beta|_{\frac{n}{2}}^{-1}} = \beta|_{\frac{n}{2}}^2\beta|_0\beta|_{\frac{n}{2}}^{-1} = \beta|_{\frac{n}{2}}^2\tau^{-1}\tau\beta|_0\beta|_0\beta|_0^{-1}\beta|_{\frac{n}{2}}^{-1}$$

$$\overset{(41)}{=} (\tau\beta|_0^2)^2\beta|_0^{-1}\beta|_{\frac{n}{2}}^{-1} = (\tau\beta|_0^2)^2(\beta|_{\frac{n}{2}}\beta|_0)^{-1};$$

$$(\beta|_{\frac{n}{2}}\beta|_0)^{\beta|_{\frac{n}{2}}} \overset{(x)}{=} (\beta|_{\frac{n}{2}}\beta|_0)^{-1}((\tau\beta|_0^2)^2)^{\beta|_{\frac{n}{2}}}$$

.

(V) $\langle \tau\beta|_0^2\rangle^J \leq V$:

$$(\tau\beta|_0^2)^{\tau^k} = \tau(\beta|_0^2)^{\tau^k} = \tau\beta|_0^2[\beta|_0^2, \tau^k] = \tau\beta|_0^2[\beta|_0, \tau^k]^{\beta|_0}[\beta|_0, \tau^k];$$

$$(\tau\beta|_0^2)^{\beta|_0} = \beta|_0^{-1}\tau\beta|_0^2\beta|_0 = \tau\tau^{-1}\beta|_0^{-1}\tau\beta|_0\beta|_0^2 = \tau[\tau, \beta|_0]\beta|_0^2$$

$$= \tau[\tau, \beta|_0]\tau^{-1}\tau\beta|_0^2 = ([\beta|_0, \tau]^{-1})^{\tau^{-1}}\tau\beta|_0^2;$$

$$(\tau\beta|_0^2)^{\beta|_0^{-1}} = \beta|_0\tau\beta|_0 = \tau\beta|_0[\beta|_0, \tau]\beta|_0 = \tau\beta|_0^2[\beta|_0, \tau]^{\beta|_0};$$

$$(\tau\beta|_0^2)^{\beta|_{\frac{n}{2}}^{-1}} \overset{(p)}{=} \left((\tau\beta|_0^2)^{\beta|_0^{-1}}([\beta|_0, \tau]^{-1})^{\beta|_0}\right)^{\beta|_{\frac{n}{2}}^{-1}}$$

$$= (\tau\beta|_0^2)^{\beta|_0^{-1}\beta|_{\frac{n}{2}}^{-1}}([\beta|_0, \tau]^{-1})^{\beta|_0\beta|_{\frac{n}{2}}^{-1}}$$

$$= (\tau\beta|_0^2)^{(\beta|_{\frac{n}{2}}\beta|_0)^{-1}}([\beta|_0, \tau]^{-1})^{\beta|_0\beta|_{\frac{n}{2}}^{-1}} \overset{(g)}{=} \tau\beta|_0^2([\beta|_0, \tau]^{-1})^{\beta|_0\beta|_{\frac{n}{2}}^{-1}};$$

$$(\tau\beta|_0^2)^{\beta|_{\frac{n}{2}}} \overset{(q)}{=} \tau\beta|_0^2[\beta|_0, \tau]^{\beta|_0}.$$

□

8. Solvable groups for $n = 4$.

Let B be an abelian subgroup of $\mathcal{A}_4 = Aut(T_4)$ normalized by τ and let $\beta \in B$. Then, by Proposition 5 , $\sigma_\beta \in D = \langle (0,1,2,3), (0,2)\rangle$ the unique Sylow 2-subgroup of Σ_4 which contains $\sigma = \sigma_\tau = (0,1,2,3)$.

The normalizer of $\overline{\langle\tau\rangle}$ here is $\Gamma_0 = N_{\mathcal{A}_4}\left(\overline{\langle\tau\rangle}\right) = \langle \Lambda, \iota\rangle$ where Λ is the monic normalizer and $\iota = \iota^{(1)}\,(0,3)\,(1,2)$ inverts τ.

Given a group W, the subgroup generated by the square of its elements is denoted by W^2 which contains the derived subgroup W'.

Lemma 14. *Let $L = L\,(D)$ be the layer closure of D above. If $\gamma \in L^2$ then $\gamma\tau$ is a conjugate of τ.*

Proof. If $\alpha \in L$ then $\sigma_{\alpha^2} \in \langle \sigma^2\rangle$ and the product in any order of the states $(\alpha^2)\,|_i$ $(0 \leq i \leq 3)$ belongs to $S = L^2$.

Let $\gamma \in S$. Then $\gamma\tau$ is transitive on the 1*st* level of the tree and $(\gamma\tau)^4$ is inactive with conjugate 1st level states, the first state being

$$(\gamma|_0)\,(\gamma|_1)\,(\gamma|_2)\,(\gamma|_3)\,\tau \text{ if } \sigma_\gamma = e,$$

and

$$(\gamma|_0)\,(\gamma|_3)\,(\gamma|_2)\,(\gamma|_1)\,\tau \text{ if } \sigma_\gamma = \sigma^2;$$

in both cases it is an element of $S^2\tau$. Therefore, $\gamma\tau$ is transitive on the 2nd level of the tree. Now we use induction to prove that $\gamma\tau$ is transitive on all levels of the tree. □

8.1. **Cases** $\sigma_\beta \in \{(0,3)(1,2),(0,1)(2,3)\}$**.** We will show that these cases cannot occur. We note that $(0,1)(2,3)$ and $(0,3)(1,2)$ are conjugate by σ_τ. Since the argument for β applies as well for β^τ it is sufficient to consider the first case.

Suppose $\sigma_\beta = (0,1)(2,3)$. Then,

$$\beta^\tau = \left(\tau^{-1}\left(\beta|_3\right), \beta|_0, \beta|_1, \beta|_2\tau\right)\left(\sigma_\beta\right)^{\sigma_\tau}.$$

On substituting $\alpha = \beta^\tau$ in $\theta = [\beta,\alpha]$ and in (7)

$$\theta|_{(i)\sigma_{\alpha\beta}} = \left(\beta|_{(i)\sigma_\alpha}\right)^{-1}\left(\alpha|_i\right)^{-1}\left(\beta|_i\right)\left(\alpha|_{(i)\sigma_\beta}\right), \forall i \in Y. \tag{66}$$

we get $\theta = e$ and

$$e = \left(\beta|_{(i)\sigma_{\beta^\tau}}\right)^{-1}\left(\beta^\tau|_i\right)^{-1}\left(\beta|_i\right)\left(\beta^\tau|_{(i)\sigma_\beta}\right), \forall i \in Y \tag{67}$$

and so for the index $i = 0$, we get

$$\begin{aligned} e &= \left(\beta|_3\right)^{-1}\left(\tau^{-1}\left(\beta|_3\right)\right)^{-1}\left(\beta|_0\right)\left(\beta|_0\right), \\ e &= \left(\beta|_3\right)^{-2}\tau\left(\beta|_0\right)^2 \end{aligned}$$

which is impossible.

8.2. **Cases** $\sigma_\beta \in \{(0,2),(1,3)\}$**.**

Lemma 15. *Let $\alpha, \gamma \in Aut(T_4)$ be such that*

$$\begin{aligned} \sigma_\alpha, \sigma_\gamma &\in \langle (0,1,2,3),(0,2)\rangle, \\ \tau^{-1}\alpha^2 &= \gamma^2\tau, \\ [\alpha,\tau^k]^\gamma &= [\gamma,\tau^k] \end{aligned}$$

for all $k \in \mathbb{Z}$. Then,

$$\begin{aligned} \sigma_\alpha, \sigma_\gamma &\in \langle\sigma\rangle; \\ \sigma_\alpha\sigma_\gamma &= \sigma^{\pm 1}. \end{aligned}$$

Proof. From the second and third equations above, we have $\sigma^{-1}\sigma_\alpha^2 = \sigma_\gamma^2\sigma$ and $[\sigma_\alpha,\sigma^k]^{\sigma_\gamma} = [\sigma_\gamma,\sigma^k]$.

(i) Suppose $\sigma_\gamma^2 = e$. Then $\sigma_\alpha^2 = \sigma^2$ and therefore, $\sigma_\alpha = \sigma^{\pm1}$, $[\sigma_\alpha,\sigma^k]^{\sigma_\gamma} = [\sigma_\gamma,\sigma^k] = e$ for all k; thus, $\sigma_\gamma \in \langle\sigma\rangle$ and from the supposition, $\sigma_\gamma \in \langle\sigma^2\rangle$, $\sigma_\alpha\sigma_\gamma = \sigma^{\pm1}$ follow.

(ii) Suppose $o(\sigma_\gamma) = 4$. Then, $\sigma_\gamma = \sigma^{\pm1}$ and $\sigma_\alpha^2 = e$. Since $[\sigma_\alpha,\sigma^k]^{\sigma_\gamma} = e$ for all k, we obtain $\sigma_\alpha \in \langle\sigma\rangle$, $\sigma_\alpha^2 = e$ and $\sigma_\alpha \in \langle\sigma^2\rangle$ .Therefore, $\sigma_\alpha\sigma_\gamma = \sigma^{\pm1}$. ☐

(1) Suppose $\sigma_\beta = (0,2)$. Then by the analysis in Section 7.2,

$$V = \left\langle [\beta|_i,\tau^k], \beta|_1, \beta|_3, \beta|_2\beta|_0, \tau\beta|_0^2 \mid i \in Y\right\rangle$$

is an abelian normal subgroup of H.

By Lemma 14 , $\tau\beta|_0^2 = \mu$ is a conjugate of τ. As V is abelian, there exist $\xi, t_1, t_2 \in \mathbb{Z}_4$ such that

$$\mu = \tau\beta|_0^2, \beta|_2\beta|_0 = \mu^{\xi}, \beta|_1 = \mu^{t_1}, \beta|_3 = \mu^{t_2}.$$

Therefore,

$$\beta|_2 = \mu^{\xi}\beta|_0^{-1}, \tau = \mu\beta|_0^{-2}.$$

On substituting $\gamma = \beta_0$ and $\alpha = \beta_2$ in Lemma 15, we obtain $\sigma_{\alpha\gamma} = \sigma_{\beta|_2\beta|_0} = \sigma^{\pm 1}$. Thus, from $\beta|_2\beta|_0 = \mu^{\xi}$, we reach $\xi \in U(Z_4)$.

By (41), we have

$$\beta|_2^2\tau^{-1} = \tau\beta|_0^2.$$

It follows then that

$$\begin{aligned} \mu^{\xi}\beta|_0^{-1}\mu^{\xi}\beta|_0^{-1}\beta|_0^2\mu^{-1} &= \mu, \\ \left(\mu^{\xi}\right)^{\beta|_0} &= \mu^{2-\xi}. \end{aligned}$$

Therefore,

$$\mu^{\beta|_0} = \mu^{\frac{2-\xi}{\xi}} \tag{68}$$

where $\frac{2-\xi}{\xi} \in \mathbb{Z}_4^1$.

By Equation (49) we have

$$\beta|_1^{\beta|_0} = \beta|_3.$$

From this it follows that

$$\left(\mu^{t_1}\right)^{\beta|_0} = \mu^{t_2},\ \mu^{t_1\frac{2-\xi}{\xi}} = \mu^{t_2},\ t_2 = t_1\frac{2-\xi}{\xi}.$$

We have reached the form of β,

$$\beta = (\beta|_0, \mu^{t_1}, \mu^{\xi}\beta|_0^{-1}, \mu^{t_1\frac{2-\xi}{\xi}})(0,2)$$

where $\mu = \tau^{\alpha}$ for some $\alpha \in Aut(T_4)$.

Now, since

$$\beta|_0 = \left(\lambda_{\frac{2-\xi}{\xi}}\tau^m\right)^{\alpha}$$

for some $m \in \mathbb{Z}_4$, we have

$$\begin{aligned} \mu^{t_1} &= (\tau^{t_1})^{\alpha}, \\ \mu^{\xi}\beta|_0^{-1} &= \left(\tau^{\xi}\left(\lambda_{\frac{2-\xi}{\xi}}\tau^m\right)^{-1}\right)^{\alpha} \\ &= \left(\lambda_{\frac{\xi}{2-\xi}}\tau^{(\xi-m)\frac{\xi}{2-\xi}}\right)^{\alpha}. \end{aligned}$$

Thus

$$\beta = (\lambda_{\frac{2-\xi}{\xi}}\tau^m, \tau^{t_1}, \lambda_{\frac{\xi}{2-\xi}}\tau^{(\xi-m)\frac{\xi}{2-\xi}}, \tau^{t_1\frac{2-\xi}{\xi}})^{\alpha^{(1)}}(0,2)$$

and

$$\begin{aligned}\tau &= \mu\beta|_0^{-2} \\ &= \left(\tau\left(\lambda_{\frac{2-\xi}{\xi}}\tau^m\right)^{-2}\right)^{\alpha} \\ &= \left(\lambda_{(\frac{\xi}{2-\xi})^2}\tau^{\left(1-\frac{2m}{\xi}\right)\left(\frac{\xi}{2-\xi}\right)^2}\right)^{\alpha}\end{aligned}$$

We note that in case $\xi = 1$, β has the form

$$\beta = (\tau^m, \tau^{t_1}, \tau^{1-m}, \tau^{t_1})^{\alpha^{(1)}}(0,2)$$

where $\tau = (\tau^{1-2m})^{\alpha}$ and therefore,

$$\beta = (\tau^{\frac{m}{1-2m}}, \tau^{\frac{t_1}{1-2m}}, \tau^{\frac{1-m}{1-2m}}, \tau^{\frac{t_1}{1-2m}})(0,2).$$

(2) Suppose $\sigma_\beta = (1,3)$. Then, $\gamma = \beta^\tau$ satisfies $[\gamma, \gamma^{\tau^k}] = e$. Therefore the previous case applies and we have

$$\gamma = (\lambda_{\frac{2-\xi}{\xi}}\tau^m, \tau^{t_1}, \lambda_{\frac{\xi}{2-\xi}}\tau^{(\xi-m)\frac{\xi}{2-\xi}}, \tau^{t_1\frac{2-\xi}{\xi}})^{\alpha^{(1)}}(0,2),$$

where

$$\tau = \left(\lambda_{(\frac{\xi}{2-\xi})^2}\tau^{\left(1-\frac{2m}{\xi}\right)\left(\frac{\xi}{2-\xi}\right)^2}\right)^{\alpha} = (e, e, e, \left(\lambda_{(\frac{\xi}{2-\xi})^2}\tau^{\left(1-\frac{2m}{\xi}\right)\left(\frac{\xi}{2-\xi}\right)^2}\right)^{\alpha})\sigma_\tau.$$

Hence, β has the form

$$\beta = \gamma^{\tau^{-1}} = (\tau^{t_1}, \lambda_{\frac{2-\xi}{\xi}}\tau^{1+m-\xi}, \tau^{t_1\frac{2-\xi}{\xi}}, \lambda_{\frac{\xi}{2-\xi}}\tau^{(1-m)\frac{\xi}{2-\xi}})^{\alpha^{(1)}}(1,3).$$

8.3. **The case $\sigma_\beta = (\sigma_\tau)^2 = (0,2)\,(1,3)$.** We know that

$$V = \left\langle N, \beta|_i\beta|_{i+2}, \beta|_j^2\tau^{-\Delta(j,j+2)} \mid i,j,t \in Y \text{ and } k \in \mathbb{Z}\right\rangle$$

is an abelian normal subgroup of H and

$$\tau^{\Delta(i,j)}\beta|_{i+2}\beta|_j\tau^{\Delta(i,j)} = \beta|_{j+2}\beta|_i, \tag{69}$$

by analysis of the case 7.1.

From Lemmas 12 and 13, we have

$$\tau\beta|_0^2 = \mu,\ \beta|_2\beta|_0 = \mu^{\xi_0},\ \beta|_3\beta|_1 = \mu^{\xi_1},\ \tau\beta|_1^2 = \mu^{\xi_2}$$

where $\mu = \tau^\alpha$ and $\xi_0, \xi_1, \xi_2 \in U(\mathbb{Z}_4)$. Therefore,

$$\tau = \mu\beta|_0^{-2} \tag{70}$$

$$\beta|_2 = \mu^{\xi_0}\beta|_0^{-1} \tag{71}$$

$$\beta|_3 = \mu^{\xi_1}\beta|_1^{-1} \tag{72}$$

$$\tau = \mu^{\xi_2}\beta|_1^{-2}. \tag{73}$$

Now, we let i, j take their values from Y in (69). Note that (i, j) and (j, i) produce equivalent equations and the case where $i = j$ is a tautology. Thus we have to treat $(i, j) = (0, 1), (0, 2), (1, 3), (2, 3), (0, 3), (1, 2)$. Indeed, the last two cases turn out to be superfluous.

(i) Substitute $i = 0, j = 2$ in (69), to obtain

$$\beta|_2^2\tau^{-1} = \tau\beta|_0^2 \tag{74}$$

Use (70) and (71) in (74) to get

$$\mu^{\xi_0}\beta|_0^{-1}\mu^{\xi_0}\beta|_0^{-1}\beta|_0^2\mu^{-1} = \mu$$

and so,

$$(\mu^{\xi_0})^{\beta|_0} = \mu^{2-\xi_0}.$$

Therefore,

$$\mu^{\beta|_0} = \mu^{\frac{2-\xi_0}{\xi_0}} \tag{75}$$

Since $\frac{2-\xi_0}{\xi_0} \in \mathbb{Z}_4^1$, we find

$$\beta|_0 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}\right)^\alpha. \tag{76}$$

From (71), we have

$$\beta|_2 = \mu^{\xi_0}\beta|_0^{-1} = \left(\tau^{\xi_0}\tau^{-m_0}\lambda_{\frac{\xi_0}{2-\xi_0}}\right)^\alpha = \left(\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}\right)^\alpha. \tag{77}$$

(ii) Substitute $i = 1, j = 3$ in (69) to get

$$\beta|_3^2\tau^{-1} = \tau\beta|_1^2. \tag{78}$$

On using (72) and (73) in (78), we obtain

$$\mu^{\xi_1}\beta|_1^{-1}\mu^{\xi_1}\beta|_1^{-1}\beta|_1^2\mu^{-\xi_2} = \mu^{\xi_2}$$

and so,

$$(\mu^{\xi_1})^{\beta|_1} = \mu^{2\xi_2-\xi_1}.$$

Therefore,

$$\mu^{\beta|_1} = \mu^{\frac{2\xi_2-\xi_1}{\xi_1}}. \tag{79}$$

Since $\frac{2\xi_2-\xi_1}{\xi_1} \in \mathbb{Z}_4^1$, we have

$$\beta|_1 = \left(\lambda_{\frac{2\xi_2-\xi_1}{\xi_1}}\tau^{m_1}\right)^\alpha. \tag{80}$$

By (72), we find

(81)
$$\beta|_3 = \mu^{\xi_1}\beta|_1^{-1} = \left(\tau^{\xi_1}\tau^{-m_1}\lambda_{\frac{\xi_1}{2\xi_2-\xi_1}}\right)^{\alpha} = \left(\lambda_{\frac{\xi_1}{2\xi_2-\xi_1}}\tau^{(\xi_1-m_1)\frac{\xi_1}{2\xi_2-\xi_1}}\right)^{\alpha}.$$

(iii) Substitute $i=0, j=1$ in (69) to get

$$\beta|_2\beta|_1 = \beta|_3\beta|_0. \tag{82}$$

Use (76), (77), (80) and (81) in (82), to obtain

$$\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}\lambda_{\frac{2\xi_2-\xi_1}{\xi_1}}\tau^{m_1} = \lambda_{\frac{\xi_1}{2\xi_2-\xi_1}}\tau^{(\xi_1-m_1)\frac{\xi_1}{2\xi_2-\xi_1}}\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}$$

and so,

$$\lambda_{\frac{\xi_0}{2-\xi_0}\frac{2\xi_2-\xi_1}{\xi_1}}\tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}\frac{2\xi_2-\xi_1}{\xi_1}+m_1} = \lambda_{\frac{\xi_1}{2\xi_2-\xi_1}\frac{2-\xi_0}{\xi_0}}\tau^{(\xi_1-m_1)\frac{\xi_1}{2\xi_2-\xi_1}\frac{2-\xi_0}{\xi_0}+m_0}.$$

Therefore,

$$\left(\frac{\xi_1}{2\xi_2-\xi_1}\right)^2 = \left(\frac{\xi_0}{2-\xi_0}\right)^2 \tag{83}$$

and

$$(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}\frac{2\xi_2-\xi_1}{\xi_1}+m_1 = (\xi_1-m_1)\frac{\xi_1}{2\xi_2-\xi_1}\frac{2-\xi_0}{\xi_0}+m_0. \tag{84}$$

(iv) Substitute $i=2, j=3$ in (69) to get

$$\beta|_0\beta|_3 = \beta|_1\beta|_2. \tag{85}$$

Use (76), (77), (80) and (81) in (85), to obtain

$$\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}\lambda_{\frac{\xi_1}{2\xi_2-\xi_1}}\tau^{(\xi_1-m_1)\frac{\xi_1}{2\xi_2-\xi_1}} = \lambda_{\frac{2\xi_2-\xi_1}{\xi_1}}\tau^{m_1}\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}$$

and so,

$$\lambda_{\frac{\xi_0}{2-\xi_0}\frac{\xi_1}{2\xi_2-\xi_1}}\tau^{m_0\frac{\xi_1}{2\xi_2-\xi_1}+(\xi_1-m_1)\frac{\xi_1}{2\xi_2-\xi_1}} = \lambda_{\frac{2\xi_2-\xi_1}{\xi_1}\frac{\xi_0}{2-\xi_0}}\tau^{m_1\frac{\xi_0}{2-\xi_0}+(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}.$$

Therefore,

$$\left(\frac{\xi_1}{2\xi_2-\xi_1}\right)^2 = \left(\frac{\xi_0}{2-\xi_0}\right)^2$$

and

$$m_0\frac{\xi_1}{2\xi_2-\xi_1}+(\xi_1-m_1)\frac{\xi_1}{2\xi_2-\xi_1} = m_1\frac{\xi_0}{2-\xi_0}+(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}. \tag{86}$$

We have from (83)

$$\frac{\xi_0}{2-\xi_0} = \pm\frac{\xi_1}{2\xi_2-\xi_1}. \tag{87}$$

(a) If

$$\frac{\xi_0}{2-\xi_0} = \frac{\xi_1}{2\xi_2-\xi_1},$$

then

$$2\xi_2\xi_0 - \xi_1\xi_0 = 2\xi_1 - \xi_1\xi_0,$$

and so,

$$\xi_2 = \frac{\xi_1}{\xi_0}. \tag{88}$$

From (84), we get

$$m_1 = \frac{\xi_1-\xi_0}{2} + m_0. \tag{89}$$

(b) If

$$\frac{\xi_0}{2-\xi_0} = -\frac{\xi_1}{2\xi_2-\xi_1}$$

then by (84) and (86),

$$m_0 - \xi_0 + m_1 = m_1 - \xi_1 + m_0$$

$$m_0 + \xi_1 - m_1 = -m_1 - \xi_0 + m_0,$$

which implies $\xi_1 = \xi_0 = 0$,which is impossible.

Now by (88) and (89), we have

$$\beta|_1 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{\frac{\xi_1-\xi_0}{2}+m_0}\right)^{\alpha} \tag{90}$$

and

$$\beta|_3 = \left(\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{\left(\frac{\xi_1+\xi_0}{2}-m_0\right)\frac{\xi_0}{2-\xi_0}}\right)^{\alpha}. \tag{91}$$

Therefore,

$$\beta = (\beta|_0, \beta|_1, \beta|_2, \beta|_3)(0,2)(1,3)$$

where $\beta|_0, \beta|_1, \beta|_2$ and $\beta|_3$ are described in (76),(90), (77) and (91), respectively and

$$\begin{aligned}\tau &= \mu\beta|_0^{-2} \\ &= \left(\tau\left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}\right)^{-2}\right)^{\alpha} \\ &= \left(\lambda_{(\frac{\xi_0}{2-\xi_0})^2}\tau^{\left(1-\frac{2m_0}{\xi_0}\right)\left(\frac{\xi_0}{2-\xi_0}\right)^2}\right)^{\alpha}.\end{aligned}$$

(v) The cases $(i,j) = (1,2),(0,3)$ in (69) do not add any more information about β.

Summarizing, we have found

$$\beta|_0 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}\right)^{\alpha},\ \beta|_1 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{\frac{\xi_1-\xi_0}{2}+m_0}\right)^{\alpha}, \tag{92}$$

$$\beta|_2 = \left(\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}\right)^{\alpha}, \beta|_3 = \left(\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{\left(\frac{\xi_1+\xi_0}{2}-m_0\right)\frac{\xi_0}{2-\xi_0}}\right)^{\alpha}, \tag{93}$$

$$\tau = \left(\lambda_{(\frac{\xi_0}{2-\xi_0})^2}\tau^{\left(1-\frac{2m_0}{\xi_0}\right)\left(\frac{\xi_0}{2-\xi_0}\right)^2}\right)^{\alpha} \tag{94}$$

In the particular case where $\xi_0 = 1$, β has the form

$$\beta = (\tau^{\frac{m_0}{1-2m_0}}, \tau^{\frac{\frac{\xi_1-1}{2}+m_0}{1-2m_0}}, \tau^{\frac{1-m_0}{1-2m_0}}, \tau^{\frac{\frac{\xi_1+1}{2}-m_0}{1-2m_0}})(0,2)(1,3)$$

where $\tau = (\tau^{1-2m_0})^{\alpha}$.

8.4. **Cases** $\sigma_\beta \in \{e, \sigma_\tau, \sigma_\tau^{-1}\}$**.** (1) Suppose $\sigma_\beta = e$ and let β stabilize the kth level of the tree. Then by Proposition 6, we have

$$[\beta|_u, \beta|_v^{\tau^{\xi}}] = e, \text{ for all } u,v \in \mathcal{M} \text{ with } |u| = |v| = k.$$

Therefore, $\dot{N} = \langle \beta|_w \mid |w| = k, w \in \mathcal{M}\rangle$ is abelian and so is its its normal closure $\dot{M}$ under $\left\langle \dot{N}, \tau \right\rangle$. Also, active elements in $\dot{M}$ are characterized in 8.1, 8.2, 8.3 and 8.4. In particular, there exists $\kappa \in \dot{M}$ such that $\sigma_\kappa = (0,2)(1,3)$ and $\beta \in \times_{p^k} C(\kappa)$.

(2) Suppose $\sigma_\beta = \sigma_\tau = (0,1,2,3)$. Then, clearly

$$\beta^2 = (\beta|_0\beta|_1,\ \beta|_1\beta|_2,\ \beta|_2\beta|_3,\ \beta|_3\beta|_0)(0,2)(1,3)$$

satisfies $[\beta^2, (\beta^2)^{\tau^k}] = e$ for all $k \in \mathbb{Z}_4$. Therefore, by the previous analysis, we have

$$\beta|_0\beta|_1 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}\right)^{\alpha}, \tag{95}$$

$$\beta|_1\beta|_2 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{\frac{\xi_1-\xi_0}{2}+m_0}\right)^{\alpha}, \tag{96}$$

$$\beta|_2\beta|_3 = \left(\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}\right)^{\alpha}, \tag{97}$$

$$\beta|_3\beta|_0 = \left(\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{\left(\frac{\xi_1+\xi_0}{2}-m_0\right)\frac{\xi_0}{2-\xi_0}}\right)^{\alpha}, \tag{98}$$

$$\tau = \left(\lambda_{(\frac{\xi_0}{2-\xi_0})^2}\tau^{\left(1-\frac{2m_0}{\xi_0}\right)\left(\frac{\xi_0}{2-\xi_0}\right)^2}\right)^{\alpha}. \tag{99}$$

Therefore,

$$\beta|_0\beta|_1\beta|_2\beta|_3 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}\right)^{\alpha} = \left(\tau^{\frac{\xi_0^2}{2-\xi_0}}\right)^{\alpha},$$

$$\beta|_1\beta|_2\beta|_3\beta|_0 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{\frac{\xi_1-\xi_0}{2}+m_0}\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{\left(\frac{\xi_1+\xi_0}{2}-m_0\right)\frac{\xi_0}{2-\xi_0}}\right)^{\alpha} = \left(\tau^{\frac{\xi_1\xi_0}{2-\xi_0}}\right)^{\alpha}.$$

Thus,

$$\left(\tau^{\frac{\xi_0^2}{2-\xi_0}}\right)^{\alpha\beta|_0} = \left(\tau^{\frac{\xi_1\xi_0}{2-\xi_0}}\right)^{\alpha}$$

and

$$(\tau^{\alpha})^{\beta|_0} = \left(\tau^{\frac{\xi_1}{\xi_0}}\right)^{\alpha} \tag{100}$$

Substitute $\eta = \frac{\xi_1}{\xi_0}$ in (100) to get

$$\beta|_0 = (\psi_\eta\tau^{m_1})^{\alpha}, \tag{101}$$

where

$$\psi_\eta = \begin{cases} \lambda_\eta, & \text{if } \eta \in \mathbb{Z}_4^1 \\ \theta\lambda_{-\eta}, & \text{if } -\eta \in \mathbb{Z}_4^1 \end{cases}, \tag{102}$$

$$\theta = \theta^{(1)}(e, \tau^{-1}, \tau^{-1}, \tau^{-1})(1,3)$$

(an invertor of τ). Note that

$$\psi_\eta\lambda_\xi = \psi_\eta\psi_\xi = \psi_{\eta\xi} = \psi_{\xi\eta} = \psi_\xi\psi_\eta = \lambda_\xi\psi_\eta$$

for all $\xi \in \mathbb{Z}_4^1$.

By (95) and (101),

$$\beta|_1 = \left(\tau^{-m_1}\psi_{\eta^{-1}}\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{m_0}\right)^{\alpha} = \left(\psi_{\frac{2-\xi_0}{\eta\xi_0}}\tau^{-m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right)+m_0}\right)^{\alpha}. \tag{103}$$

Also, by (96) and (101),

$$\begin{aligned}\beta|_2 &= \left(\tau^{m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right)-m_0}\psi_{\frac{\eta\xi_0}{2-\xi_0}}\lambda_{\frac{2-\xi_0}{\xi_0}}\tau^{\frac{\eta\xi_0-\xi_0}{2}+m_0}\right)^{\alpha}\\ &= \left(\psi_{\eta}\tau^{\left[m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right)-m_0\right]\eta+\frac{\eta\xi_0-\xi_0}{2}+m_0}\right)^{\alpha}.\end{aligned} \tag{104}$$

Furthermore, by (98) and (101),

$$\begin{aligned}\beta|_3 &= \left(\lambda_{\frac{\xi_0}{2-\xi_0}}\tau^{\left(\frac{\eta\xi_0+\xi_0}{2}-m_0\right)\frac{\xi_0}{2-\xi_0}}\tau^{-m_1}\psi_{\eta^{-1}}\right)^{\alpha}\\ &= \left(\psi_{\frac{\xi_0}{\eta(2-\xi_0)}}\tau^{\left[\left(\frac{\eta\xi_0+\xi_0}{2}-m_0\right)\frac{\xi_0}{2-\xi_0}-m_1\right]\eta^{-1}}\right)^{\alpha}.\end{aligned} \tag{105}$$

Setting $i=1$ and $t=2$ em (17), we obtain

$$\beta|_0\beta|_2 = \beta|_1^2. \tag{106}$$

Use (101), (103), (104) and (105) in (106), to get

$$\begin{aligned}&\psi_{\eta}\tau^{m_1}\psi_{\eta}\tau^{\left[m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right)-m_0\right]\eta+\frac{\eta\xi_0-\xi_0}{2}+m_0}\\ = \;&\psi_{\frac{2-\xi_0}{\eta\xi_0}}\tau^{-m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right)+m_0}\psi_{\frac{2-\xi_0}{\eta\xi_0}}\tau^{-m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right)+m_0}\end{aligned} \tag{107}$$

which is the same as

$$\begin{aligned}&\psi_{\eta^2}\tau^{m_1\eta+\left[m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right)-m_0\right]\eta+\frac{\eta\xi_0-\xi_0}{2}+m_0}\\ = \;&\psi_{\left(\frac{2-\xi_0}{\eta\xi_0}\right)^2}\tau^{\left[-m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right)+m_0\right]\left(\frac{2-\xi_0}{\eta\xi_0}\right)-m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right)+m_0}.\end{aligned} \tag{108}$$

Therefore,

$$\eta^2 = \left(\frac{2-\xi_0}{\eta\xi_0}\right)^2 \tag{109}$$

and

$$\begin{aligned}&m_1\eta+\left[m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right)-m_0\right]\eta+\frac{\eta\xi_0-\xi_0}{2}+m_0\\ = &\left[-m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right)+m_0\right]\left(\frac{2-\xi_0}{\eta\xi_0}\right)-m_1\left(\frac{2-\xi_0}{\eta\xi_0}\right)+m_0\end{aligned}$$

(a) Suppose

$$\eta = -\frac{2-\xi_0}{\eta\xi_0} \tag{110}$$

(or what is the same

$$\left(\eta^2-1\right)\xi_0 = -2). \tag{111}$$

Then on substituting this in the above equation, we get

$$(\eta-1)\,\xi_0 = 0$$

contradicting the previous equation.

(b) Suppose

$$\eta = \frac{2-\xi_0}{\eta\xi_0}. \tag{112}$$

Then,

$$\xi_0 = \frac{2}{\eta^2+1} \tag{113}$$

and this leads to

$$m_0 = 2m_1 + \frac{\eta-1}{2\eta(\eta^2+1)}. \tag{114}$$

On substituting (113) and (114) in(103), (104), (105) and (99), we find

$$\beta|_1 = \left(\psi_\eta \tau^{m_1(2-\eta)+\frac{\eta-1}{2\eta(\eta^2+1)}}\right)^\alpha \tag{115}$$

$$\beta|_2 = \left(\psi_\eta \tau^{m_1(\eta^2-2\eta+2)+\frac{\eta^2-1}{2\eta(\eta^2+1)}}\right)^\alpha, \tag{116}$$

$$\beta|_3 = \left(\psi_{\eta^{-3}} \tau^{\frac{2\eta^2+\eta+1}{2\eta^4(\eta^2+1)}-m_1\left(\frac{\eta^2+2}{\eta^3}\right)}\right)^\alpha, \tag{117}$$

$$\tau = \left(\psi_{\eta^{-4}} \tau^{\frac{\eta+1}{2\eta^5}-2m_1\left(\frac{\eta^2+1}{\eta^4}\right)}\right)^\alpha. \tag{118}$$

Substitute $i=0$, $t=1$ in (17), to get

$$\beta|_3\beta|_1 = \tau\beta|_0^2. \tag{119}$$

Using (101), (115), (116), (117) and (118) in (119), we obtain

$$\begin{aligned} & \psi_{\eta^{-3}}\tau^{\frac{2\eta^2+\eta+1}{2\eta^4(\eta^2+1)}-m_1\left(\frac{\eta^2+2}{\eta^3}\right)}\psi_\eta\tau^{m_1(2-\eta)+\frac{\eta-1}{2\eta(\eta^2+1)}} \\ = \; & \psi_{\eta^{-4}}\tau^{\frac{\eta+1}{2\eta^5}-2m_1\left(\frac{\eta^2+1}{\eta^4}\right)}\psi_\eta\tau^{m_1}\psi_\eta\tau^{m_1}. \end{aligned}$$

Thus,

$$\begin{aligned} & \psi_{\eta^{-2}}\tau^{\frac{2\eta^2+\eta+1}{2\eta^3(\eta^2+1)}-m_1\left(\frac{\eta^2+2}{\eta^2}\right)+m_1(2-\eta)+\frac{\eta-1}{2\eta(\eta^2+1)}} \\ = \; & \psi_{\eta^{-2}}\tau^{\frac{\eta+1}{2\eta^3}-2m_1\left(\frac{\eta^2+1}{\eta^2}\right)+m_1\eta+m_1}, \end{aligned}$$

which implies

$$(\eta-1)m_1 = 0 \tag{120}$$

and thus,

$$m_1 = 0 \text{ or } \eta = 1.$$

- If $m_1 = 0$ we get

$$\begin{aligned} \beta &= (\psi_\eta, \psi_\eta\tau^{\frac{\eta-1}{2\eta(\eta^2+1)}}, \psi_\eta\tau^{\frac{\eta^2-1}{2\eta(\eta^2+1)}}, \psi_{\eta^{-3}}\tau^{\frac{2\eta^2+\eta+1}{2\eta^4(\eta^2+1)}})^{\alpha^{(1)}}\sigma_\tau \\ &= \tau^\gamma, \end{aligned} \tag{121}$$

where

$$\gamma = \left(\lambda_{\frac{2}{\eta^2(\eta^2+1)}}\right)^{(1)}(e, \psi_\eta, \psi_{\eta^2}\tau^{\frac{\eta-1}{2\eta(\eta^2+1)}}, \psi_{\eta^3}\tau^{\frac{2\eta^2-n-1}{2\eta(\eta^2+1)}})\alpha^{(1)} \tag{122}$$

and

$$\tau = \left(\psi_{\eta^{-4}}\tau^{\frac{\eta+1}{2\eta^5}}\right)^\alpha. \tag{123}$$

- If $\eta = 1$ we get

$$\beta = (\tau^{m_1}, \tau^{m_1}, \tau^{m_1}, \tau^{1-3m_1})^{\alpha^{(1)}}(0,1,2,3) \tag{124}$$

and

$$\tau = \left(\tau^{1-4m_1}\right)^\alpha, \tag{125}$$

which produce

$$\begin{aligned} \beta &= (\tau^{\frac{m_1}{1-4m_1}}, \tau^{\frac{m_1}{1-4m_1}}, \tau^{\frac{m_1}{1-4m_1}}, \tau^{\frac{1-3m_1}{1-4m_1}})(0,1,2,3) \\ &= (\tau^{\frac{m_1}{1-4m_1}}, \tau^{\frac{m_1}{1-4m_1}}, \tau^{\frac{m_1}{1-4m_1}}, \tau^{\frac{m_1}{1-4m_1}})\tau \\ &= \tau^{\frac{4m_1}{1-4m_1}}\tau = \tau^{\frac{1}{1-4m_1}} = \tau^{\lambda_{\frac{1}{1-4m_1}}} \end{aligned} \tag{126}$$

(3) Suppose $\sigma_\beta = \sigma_\tau^{-1} = (0,3,2,1)$. Then, β^{-1} satisfies the previous case. Therefore, as θ inverts τ, we have

$$\beta = \left(\beta^{-1}\right)^{-1} = \left(\tau^{\gamma}\right)^{-1} = (\tau)^{\theta\gamma} \tag{127}$$

or

$$\beta = \tau^{\theta\lambda_{\frac{1}{1-4m_1}}}, \tag{128}$$

where $m_1 \in \mathbb{Z}_4$,

$$\gamma = \left(\lambda_{\frac{2}{\eta^2(\eta^2+1)}}\right)^{(1)} (e, \psi_\eta, \psi_{\eta^2}\tau^{\frac{\eta-1}{2\eta(\eta^2+1)}}, \psi_{\eta^3}\tau^{\frac{2\eta^2-n-1}{2\eta(\eta^2+1)}})\alpha^{(1)}, \tag{129}$$

$\eta \in U(\mathbb{Z}_4)$ and

$$\tau = \left(\psi_{\eta^{-4}}\tau^{\frac{\eta+1}{2\eta^5}}\right)^{\alpha}. \tag{130}$$

8.5. **Final Step.** We finish the proof of the second part of Theorem A. In order to treat the remaining case where the activity of β is a 4-cycle, we use the fact that $\beta^2 \in B$, which we have already described. Next, from the description of the centralizer of β^2, we are able to pin down the form of β.

Proposition 12. *Let $\beta = (\beta|_0,\ \beta|_1,\ \beta|_2,\ \beta|_3)(0,2)(1,3)$ be such that $(\beta|_0)(\beta|_2) = \tau^{\theta_1}$ and $(\beta|_1)(\beta|_3) = \tau^{\theta_2}$, for some $\theta_1, \theta_2 \in Aut(T_4)$. Then, β is conjugate to τ^2.*

Proof. Let $\alpha = (e, e, \beta|_0^{-1}, \beta|_3^{-1})$. Then,

$$\beta^{\alpha} = (e, e,\ \beta|_0\beta|_2\ , \beta|_1\beta|_3)(0,2)(1,3). \tag{131}$$

Therefore, substituting $\beta|_0\beta|_2 = \tau^{\theta_1}$ and $\beta|_1\beta|_3 = \tau^{\theta_2}$ in the above equation, we have

$$\beta^{\alpha} = (e, e, \tau^{\theta_1}, \tau^{\theta_2})(0,2)(1,3).$$

Conjugating β^{α} by $\gamma = (\theta_1^{-1}, \theta_2^{-1}, \theta_1^{-1}, \theta_2^{-1})$ produces

$$\beta^{\alpha\gamma} = \tau^2.$$

□

We show below that active elements of B produce elements within B conjugate to τ^2.

Proposition 13. *Let $\beta \in B$ with nontrivial σ_β. Then*

(i) *If $\sigma_\beta = \sigma_\tau^2$, then β is a conjugate of τ^2.*
(ii) *If $\sigma_\beta \in \{(0,2),(1,3)\}$, then $\beta\beta^{\tau}$ is a conjugate τ^2.*
(iii) *If $\sigma_\beta \in \{\sigma_\tau, \sigma_\tau^{-1}\}$, then β^2 is a conjugate of τ^2.*

Proof. It is enough to prove (i), since (ii), (iii) are just special cases.

If $\sigma_\beta = \sigma_\tau^2$, then

$$\beta|_0 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}} \tau^{m_0}\right)^\alpha, \ \beta|_1 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}} \tau^{\frac{\xi_1-\xi_0}{2}+m_0}\right)^\alpha, \tag{132}$$

$$\beta|_2 = \left(\lambda_{\frac{\xi_0}{2-\xi_0}} \tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}\right)^\alpha, \beta|_3 = \left(\lambda_{\frac{\xi_0}{2-\xi_0}} \tau^{\left(\frac{\xi_1+\xi_0}{2}-m_0\right)\frac{\xi_0}{2-\xi_0}}\right)^\alpha, \tag{133}$$

$$\tau = \left(\lambda_{(\frac{\xi_0}{2-\xi_0})^2} \tau^{\left(1-\frac{2m_0}{\xi_0}\right)\left(\frac{\xi_0}{2-\xi_0}\right)^2}\right)^\alpha, \tag{134}$$

where $\xi_0, \xi_1 \in U(\mathbb{Z}_4), \ m_0 \in \mathbb{Z}_4$.

Therefore,

$$\beta|_0\beta|_2 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}} \tau^{m_0} \lambda_{\frac{\xi_0}{2-\xi_0}} \tau^{(\xi_0-m_0)\frac{\xi_0}{2-\xi_0}}\right)^\alpha = \left(\tau^{\frac{\xi_0^2}{2-\xi_0}}\right)^\alpha = (\tau)^{\psi\frac{\xi_0^2}{2-\xi_0}\alpha}$$

$$\beta|_1\beta|_3 = \left(\lambda_{\frac{2-\xi_0}{\xi_0}} \tau^{\frac{\xi_1-\xi_0}{2}+m_0} \lambda_{\frac{\xi_0}{2-\xi_0}} \tau^{\left(\frac{\xi_1+\xi_0}{2}-m_0\right)\frac{\xi_0}{2-\xi_0}}\right)^\alpha = \left(\tau^{\frac{\xi_1\xi_0}{2-\xi_0}}\right)^\alpha = \tau^{\psi\frac{\xi_1\xi_0}{2-\xi_0}\alpha}$$

It follows from Proposition 12, that β is a conjugate of τ^2. □

Corollary 4. *Suppose $\beta \in B$ is an active element. Then, B is conjugate to a subgroup of $C(\tau^2)$.*

Proposition 14. *Let $\gamma \in C(\tau^2)$. Then,*

$$\gamma = (\tau^{m_0}, \tau^{m_1}, \tau^{m_0+\delta((0)\sigma_\gamma,\ 2)}, \tau^{m_1+\delta((1)\sigma_\gamma,\ 2)})\sigma_\gamma, \tag{135}$$

where $m_0, m_1 \in \mathbb{Z}_4, \sigma_\gamma \in C_{\Sigma_4}(\sigma^2)$.

Proof. Write $\gamma = (\gamma|_0, \gamma|_1, \gamma|_2, \gamma|_3)\sigma_\gamma$. Then $\tau^2\gamma = \gamma\tau^2$ translates to

$$\begin{aligned} & (e, e, \tau, \tau)(0,2)(1,3)(\gamma|_0, \gamma|_1, \gamma|_2, \gamma|_3)\sigma_\gamma \\ = \ & (\gamma|_0, \gamma|_1, \gamma|_2, \gamma|_3)\sigma_\gamma(e, e, \tau, \tau)(0,2)(1,3), \end{aligned}$$

and this in turn,

$$\begin{aligned} & (\gamma|_2, \gamma|_3, \tau\gamma|_0, \tau\gamma|_1)(0,2)(1,3)\sigma_\gamma \\ = \ & (\gamma|_0, \gamma|_1, \gamma|_2, \gamma|_3). \\ & \sigma_\gamma(\tau^{\delta(0,2)}, \tau^{\delta(1,2)}, \tau^{\delta(2,2)}, \tau^{\delta(3,2)})(0,2)(1,3) \\ = \ & (\gamma|_0, \gamma|_1, \gamma|_2, \gamma|_3) \\ & (\tau^{\delta((0)\sigma_\gamma,2)}, \tau^{\delta((1)\sigma_\gamma,2)}, \tau^{\delta((2)\sigma_\gamma,2)}, \tau^{\delta((3)\sigma_\gamma,2)})\sigma_\gamma(0,2)(1,3) \\ = \ & (\gamma|_0\tau^{\delta((0)\sigma_\gamma,2)}, \gamma|_1\tau^{\delta((1)\sigma_\gamma,2)}, \gamma|_2\tau^{\delta((2)\sigma_\gamma,2)}, \gamma|_3\tau^{\delta((3)\sigma_\gamma,2)})\sigma_\gamma(0,2)(1,3) \end{aligned}$$

Thus,

$$\begin{cases} \gamma|_2 = \gamma|_0 \tau^{\delta((0)\sigma_\gamma,2)}, \\ \gamma|_3 = \gamma|_1 \tau^{\delta((1)\sigma_\gamma,2)}, \\ \tau\gamma|_0 = \gamma|_2 \tau^{\delta((2)\sigma_\gamma,2)}, \\ \tau\gamma|_1 = \gamma|_3 \tau^{\delta((3)\sigma_\gamma,2)}. \end{cases}$$

Hence,

$$\begin{cases} \gamma|_2 = \gamma|_0 \tau^{\delta((0)\sigma_\gamma,2)},\ \gamma|_3 = \gamma|_1 \tau^{\delta((1)\sigma_\gamma,2)}, \\ \tau^{\gamma|_0} = \tau^{\delta((0)\sigma_\gamma,2)+\delta((2)\sigma_\gamma,2)} = \tau,\ \tau^{\gamma|_1} = \tau^{\delta((1)\sigma_\gamma,2)+\delta((3)\sigma_\gamma,2)} = \tau \end{cases}.$$

Therefore, there exist $m_0, m_1 \in \mathbb{Z}_4$ such that

$$\begin{cases} \gamma|_0 = \tau^{m_0},\ \gamma|_1 = \tau^{m_1}, \\ \gamma|_2 = \tau^{m_0+\delta((0)\sigma_\gamma,2)},\ \gamma|_3 = \tau^{m_1+\delta((1)\sigma_\gamma,2)} \end{cases}.$$

Hence, γ has the form

$$\gamma = (\tau^{m_0}, \tau^{m_1}, \tau^{m_0+\delta((0)\sigma_\gamma,2)}, \tau^{m_1+\delta((1)\sigma_\gamma,2)})\sigma_\gamma, \tag{136}$$

where $\sigma_\gamma \in C_{\Sigma_4}(\sigma^2)$. □

Corollary 5. *The centralizer of τ^2 in $\mathcal{A}_4$ is*

$$C(\tau^2) = \langle (e, e, \tau, e)(0, 2),\ \tau,\ (\tau^{m_0}, \tau^{m_1}, \tau^{m_0}, \tau^{m_1}) \mid m_0, m_1 \in \mathbb{Z}_4 \rangle.$$

Corollary 6. *Let $\gamma \in C(\tau^2)$ be such that $\sigma_\gamma \in \langle (0, 2)(1, 3) \rangle$. Then*

$$\gamma \in \left\langle (\tau^{m_0}, \tau^{m_1}, \tau^{m_0}, \tau^{m_1}),\ \tau^2 \mid m_0, m_1 \in \mathbb{Z}_4 \right\rangle.$$

Proposition 15. *Let $\dot{H} = \langle (\tau^{m_0}, \tau^{m_1}, \tau^{m_0}, \tau^{m_1}), \tau^2 \mid m_0, m_1 \in \mathbb{Z}_4 \rangle$. Then the normalizer $N_{\mathcal{A}_4}(\dot{H})$ is the group*

$$\left\langle C(\tau^2), (\psi_{2m_0+1}, \psi_{2m_1+1}, \psi_{2m_0+1}\tau^{m_0}, \psi_{2m_1+1}\tau^{m_1}) \mid m_0, m_1 \in \mathbb{Z}_4 \right\rangle,$$

where, for each $\eta \in U(\mathbb{Z}_4)$, ψ_η is defined by (102) and

$$\tau^{\psi_\eta} = \tau^\eta.$$

Proof. Note that $\dot{H}$ is an abelian group. Let $\alpha \in N_{\mathcal{A}_4}(\dot{H})$. Then,

$$(\tau^2)^\alpha = (\tau^{m_0}, \tau^{m_1}, \tau^{m_0+1}, \tau^{m_1+1})(0, 2)(1, 3),$$

where $m_0, m_1 \in \mathbb{Z}_4$.

Suppose α is inactive. Then,

$$\begin{aligned} & (\tau^{m_0}, \tau^{m_1}, \tau^{m_0+1}, \tau^{m_1+1})(0, 2)(1, 3) \\ = \ & (\alpha|_0^{-1}, \alpha|_1^{-1}, \alpha|_2^{-1}, \alpha|_3^{-1})(e, e, \tau, \tau)(0, 2)(1, 3)(\alpha|_0, \alpha|_1, \alpha|_2, \alpha|_3) \\ = \ & (\alpha|_0^{-1}, \alpha|_1^{-1}, \alpha|_2^{-1}, \alpha|_3^{-1})(e, e, \tau, \tau)(\alpha|_2, \alpha|_3, \alpha|_0, \alpha|_1)(0, 2)(1, 3) \\ = \ & (\alpha|_0^{-1}\alpha|_2, \alpha|_1^{-1}\alpha|_3, \alpha|_2^{-1}\tau\alpha|_0, \alpha|_3^{-1}\tau\alpha|_1)(0, 2)(1, 3) \end{aligned}$$

which produces

$$\begin{cases} \alpha|_0^{-1}\alpha|_2 = \tau^{m_0},\ \alpha|_1^{-1}\alpha|_3 = \tau^{m_1}, \\ \alpha|_2^{-1}\tau\alpha|_0 = \tau^{m_0+1},\ \alpha|_3^{-1}\tau\alpha|_1 = \tau^{m_1+1} \end{cases}.$$

Therefore,

$$\begin{cases} \alpha|_2 = \alpha|_0\tau^{m_0},\ \alpha|_3 = \alpha|_1\tau^{m_1}, \\ \alpha|_0^{-1}\tau\alpha|_0 = \tau^{2m_0+1},\ \alpha|_1^{-1}\tau\alpha|_1 = \tau^{2m_1+1}. \end{cases}$$

Thus,

$$\alpha = (\alpha|_0, \alpha|_1, \alpha|_2, \alpha|_3) = (\psi_{2m_0+1}, \psi_{2m_1+1}, \psi_{2m_0+1}\tau^{m_0}, \psi_{2m_1+1}\tau^{m_1})$$

satisfies

$$(\tau^2)^\alpha = (\tau^{m_0}, \tau^{m_1}, \tau^{m_0+1}, \tau^{m_1+1})(0,2)(1,3).$$

□

Theorem 7. *Let G be a finitely generated solvable subgroup of $Aut(T_4)$ which contains τ. Then, G is a subgroup of*

$$\times_4 \left(\cdots\left(\times_4 \left(\times_4 N_{\mathcal{A}_4}(H)^\alpha \rtimes S_4\right) \rtimes S_4\right)\cdots\right) \rtimes S_4 \tag{137}$$

for some $\alpha \in \mathcal{A}_4$.

Proof. As in the case $n = p$, we assume G has derived length $d \geq 2$ and let B be the $(d-1)$th term of the derived series of G. Then, B is an abelian group normalized by τ. On analyzing the case 8.4 and the final step, there exists a level t such that B is a subgroup of $\dot{V} = \times_{4^k} C(\mu^2)$,where $\mu = \tau^\alpha$ for some $\alpha \in \mathcal{A}_4$ and where $\sigma_{\mu^2} = (0,2)(1,3)$. There also exists $\beta \in B$ such that $\beta|_u = \mu^2$ for some index $u \in \mathcal{M}$.

Moreover, if T is the normalizer of $C(\tau^2)$, then clearly, T^α is the normalizer of $C(\mu^2)$.

We will show now that G is a subgroup of

$$\dot{J} = \times_4 \left(\cdots\left(\times_4 \left(\times_4 N_{\mathcal{A}_4}(H)^\alpha \rtimes S_4\right) \rtimes S_4\right)\cdots\right) \rtimes S_4$$

where the cartesian product $\times_4$appears t times..

Let $\gamma \notin \dot{J}$. Since $\gamma \notin \dot{J}$, there exists $w \in \mathcal{M}$ having $|w| = t$ and $\gamma|_w \notin T^\alpha$. Since τ is transitive on all levels of the tree, by Corollary 6 we can conjugate β by an appropriate power of τ to get $\theta \in B$ such that

$$\theta|_w = \mu^2 \text{ or } \theta|_w = \left(\mu^2\right)^\tau = \left((\tau^{m_0}, \tau^{m_1}, \tau^{m_0+1}, \tau^{m_1+1})(0,2)(1,3)\right)^\alpha,$$

where $m_0, m_1 \in \mathbb{Z}_4$. Thus, for $v = w^\gamma$ we have

$$\left(\theta^\gamma\right)|_v \overset{(9)}{=} \theta|_{v^{\gamma^{-1}}}^{\gamma_{v^{\gamma^{-1}}}} = \theta|_w^{\gamma|_w} \notin C(\mu^2)$$

which implies $\theta^\gamma \notin B \leq \dot{V}$ and $\gamma \notin G$. Hence, G is a subgroup of $\dot{J}$. □

References

[1] Bass H., Espinar O., Rockmore D., Tresser C., Cyclic Renormalization and the Automorphism Groups of Rooted Trees, Lecture Notes in Mathematics 1621, Springer, Berlin, 1995.

[2] Sidki, S., *Regular Trees and their Automorphisms,* Monografias de Matemática, No **56**, Impa, Rio de Janeiro, 1998.

[3] Brunner, A., Sidki S., Vieira A. C., *A just-nonsolvable torsion-free group defined on the binary tree,* J. Algebra **211** (1999), 99-114.

[4] Grigorchuk R. I.,. Nekrachevych, V., I. Suschanskii, I., *Automata, dynamical systems, and groups,* Proc. Steklov Institute **231** (2000), 128-203.

[5] Sidki, S., Automorphisms of one-rooted trees: growth, circuit structure and acyclicity, Journal of Mathematical Sciences, Vol. **100**, no 1 (2000),

[6] Sidki, S., Silva, E.F., A family of just-nonsolvable torsion-free groups defined on n-ary trees, In Atas da XVI Escola de Álgebra, Brasília, Matematica Contemporânea **21** (2001).

[7] Sidki, S., The Binary Adding Machine and Solvable Groups, International Journal of Algebra and Computation, Vol. **13**, no 1 (2003), 95-110.

[8] Jones, G. A., Cyclic regular subgroups of primitive permutation groups. J. Group Theory 5 (2002), no. 4, 403–407.

[9] Sidki, S., Just-Non-(abelian by P-type) Groups, Progress in Math., 248, (2005) 389-402.

[10] Nekrashevych, V., *Self-similar groups,* volume 117 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2005.

[11] Vorobets, M., Vorobets,Y., On a free group of transformations defined by an automaton, Geom. Dedicata 124 (2007) 237–249.

E-mail address: jsrocha74@gmail.com

E-mail address: sidki@mat.unb.br

Instituto Federal de Educação, Ciência e Tecnologia de Goiás, Campus Formosa, 73800-000, Formosa - GO, Brazil

Departamento de Matemática, Universidade de Brasília, 70910-900, Brasília-DF, Brazil