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Structure Constants of Diagonal Reduction Algebras of $\mathfrak{gl}$ Type

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Abstract. We describe, in terms of generators and relations, the reduction algebra, related to the diagonal embedding of the Lie algebra $\mathfrak{gl}_n$ into $\mathfrak{gl}_n \oplus \mathfrak{gl}_n$. Its representation theory is related to the theory of decompositions of tensor products of $\mathfrak{gl}_n$-modules.

Key words: reduction algebra; extremal projector; Zhelobenko operators

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Contents

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1 Introduction

This paper completes the work [7]: it contains a derivation of basic relations for the diagonal reduction algebras of $\mathfrak{gl}$ type, their low dimensional examples and properties.

Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{k} \subset \mathfrak{g}$ its reductive Lie subalgebra and V an irreducible finite-dimensional $\mathfrak{g}$-module, which decomposes, as an $\mathfrak{k}$-module, into a direct sum of irreducible $\mathfrak{k}$-modules V_i with certain multiplicities m_i,

$$V \approx \sum_i V_i \otimes W_i. \tag{1.1}$$

Here $W_i = \mathrm{Hom}_{\mathfrak{k}}(V_i, V)$ are the spaces of multiplicities, $m_i = \dim W_i$. While the multiplicities m_i present certain combinatorial data, the spaces W_i of multiplicities itself may exhibit a 'hidden structure' of modules over certain special algebras [4]. The well-known example is the Olshanski *centralizer construction* [9], where $\mathfrak{g} = \mathfrak{gl}_{n+m}$, $\mathfrak{k} = \mathfrak{gl}_m$ and the spaces W_i carry the structure of irreducible Yangian $Y(\mathfrak{gl}_n)$-modules.

In general, the multiplicity spaces W_i are irreducible modules over the centralizer $\mathrm{U}(\mathfrak{g})^{\mathfrak{k}}$ of $\mathfrak{k}$ in the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ [8]. However, these centralizers have a rather complicated algebraic structure and are hardly convenient for applications. Besides, under the above assumptions, the direct sum $W = \oplus_i W_i$ becomes a module over the *reduction* (or Mickelsson) algebra. The reduction algebra is defined as follows. Suppose $\mathfrak{k}$ is given with a triangular decomposition

$$\mathfrak{k} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}. \tag{1.2}$$

Denote by I_+ the left ideal of $\mathrm{A} := \mathrm{U}(\mathfrak{g})$, generated by elements of $\mathfrak{n}$, $\mathrm{I}_+ := \mathrm{A}\mathfrak{n}$. Then the reduction algebra $\mathrm{S}^{\mathfrak{n}}(\mathrm{A})$, related to the pair $(\mathfrak{g}, \mathfrak{k})$, is defined as the quotient $\mathrm{Norm}(\mathrm{I}_+)/\mathrm{I}_+$ of the normalizer of the ideal I_+ over I_+. It is equipped with a natural structure of the associative algebra. By definition, for any $\mathfrak{g}$-module V the space $V^{\mathfrak{n}}$ of vectors, annihilated by $\mathfrak{n}$, is a module over $\mathrm{S}^{\mathfrak{n}}(\mathrm{A})$. Since V is finite-dimensional, $V^{\mathfrak{n}}$ is isomorphic to $\oplus_i W_i$, so the latter space can be viewed as an $\mathrm{S}^{\mathfrak{n}}(\mathrm{A})$-module as well; the zero-weight component of $\mathrm{S}^{\mathfrak{n}}(\mathrm{A})$, which contains a quotient of the centralizer $\mathrm{U}(\mathfrak{g})^{\mathfrak{k}}$, preserves each multiplicity space W_i. The representation theory of the reduction algebra $\mathrm{S}^{\mathfrak{n}}(\mathrm{A})$ is closely related to the theory of branching rules $\mathfrak{g} \downarrow \mathfrak{k}$ for the restrictions of representations of $\mathfrak{g}$ to $\mathfrak{k}$.

The reduction algebra simplifies after the localization over the multiplicative set generated by elements $h_\gamma + k$, where γ ranges through the set of roots of $\mathfrak{k}$, $k \in \mathbb{Z}$, and h_γ is the coroot corresponding to γ. Let $\overline{\mathrm{U}}(\mathfrak{h})$ be the localization of the universal enveloping algebra $\mathrm{U}(\mathfrak{h})$ of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$ over the above multiplicative set. The localized reduction algebra $\mathrm{Z}^{\mathfrak{n}}(\mathrm{A})$ is an algebra over the commutative ring $\overline{\mathrm{U}}(\mathfrak{h})$; the principal part of the defining relations is quadratic but the relations may contain linear or degree 0 terms, see [10, 6].

Besides, the reduction algebra admits another description as a (localized) double coset space $\mathfrak{n}_-\mathrm{A}\backslash\mathrm{A}/\mathrm{A}\mathfrak{n}$ endowed with the multiplication map defined by means of the insertion of the extremal projector [6] of Asherova–Smirnov–Tolstoy [3]. The centralizer $\mathrm{A}^{\mathfrak{k}}$ is a subalgebra of $\mathrm{Z}^{\mathfrak{n}}(\mathrm{A})$.

It was shown in [7] that the general reduction algebra $\mathrm{Z}^{\mathfrak{n}}(\mathrm{A})$ admits a presentation over $\overline{\mathrm{U}}(\mathfrak{h})$ such that the defining relations are ordering relations for the generators, in an arbitrary order, compatible with the natural partial order on $\mathfrak{h}^*$. The set of ordering relations for the general reduction algebra $\mathrm{Z}^{\mathfrak{n}}(\mathrm{A})$ was shown in [7] to be "algorithmically efficient" in the sense that any expression in the algebra can be ordered with the help of this set.

The structure constants of the reduction algebra are in principle determined with the help of the extremal projector P or the tensor J studied by Arnaudon, Buffenoir, Ragoucy and Roche [1]. However the explicit description of the algebra is hardly achievable directly.

In the present paper, we are interested in the special restriction problem, when $\mathfrak{g}$ is the direct sum of two copies of a reductive Lie algebra $\mathfrak{a}$ and $\mathfrak{k}$ is the diagonally embedded $\mathfrak{a}$. The resulting reduction algebra for the symmetric pair $(\mathfrak{a}\oplus\mathfrak{a},\mathfrak{a})$ we call *diagonal reduction* algebra $\mathrm{DR}(\mathfrak{a})$ of $\mathfrak{a}$. The theory of branching rules for $\mathfrak{a}\oplus\mathfrak{a}\downarrow\mathfrak{a}$ is the theory of decompositions of the tensor products of $\mathfrak{a}$-modules into a direct sum of irreducible $\mathfrak{a}$-modules.

We restrict ourselves here to the Lie algebra $\mathfrak{a}=\mathfrak{gl}_n$ of the general linear group. In this situation finite-dimensional irreducible modules over $\mathfrak{g}$ are tensor products of two irreducible $\mathfrak{gl}_n$-modules, the decomposition (1.1) is the decomposition of the tensor product into the direct sum of irreducible $\mathfrak{gl}_n$-modules, and the multiplicities m_i are the Littlewood–Richardson coefficients.

The reduction algebra $\mathrm{DR}(\mathfrak{gl}_n)$ for brevity will be denoted further by Z_n.

In [7] we suggested a set $\mathfrak{R}$ of relations for the algebra Z_n. We demonstrated that the set $\mathfrak{R}$ is equivalent, over $\overline{\mathrm{U}}(\mathfrak{h})$, to the set of the defining ordering relations provided that all relations from the set $\mathfrak{R}$ are valid.

The main goal of the present paper is the verification of all relations from the system $\mathfrak{R}$. There are two principal tools in our derivation. First, we use the braid group action by the Zhelobenko automorphisms of reduction algebras [10, 6]. Second, we employ the stabilization phenomenon, discovered in [7], for the multiplication rule and for the defining relations with respect to the standard embeddings $\mathfrak{gl}_n\hookrightarrow\mathfrak{gl}_{n+1}$; stabilization provides a natural way of extending relations for Z_n to relations for Z_{n+1} (Z_n is not a subalgebra of Z_{n+1}). We perform necessary calculations for low n (at most $n=4$); the braid group action and the stabilization law allow to extend the results for general n.

As an illustration, we write down the complete lists of defining relations in the form of ordering relations for the reduction algebras $\mathrm{DR}(\mathfrak{sl}_3)$ and $\mathrm{DR}(\mathfrak{sl}_2)$. Although for a finite n the task of deriving the set of defining (ordering) relations for $\mathrm{DR}(\mathfrak{sl}_n)$ is achievable in a finite time, it is useful to have the list of relations for small n in front of the eyes.

We return to the stabilization and cut phenomena and make more precise statements concerning now the embedding of the Lie algebra $\mathfrak{gl}_n\oplus\mathfrak{gl}_1$ into the Lie algebra $\mathfrak{gl}_{n+1}$ (more generally, of $\mathfrak{gl}_n\oplus\mathfrak{gl}_m$ into $\mathfrak{gl}_{n+m}$). As a consequence we find that cutting preserves the centrality: the cut of a central element of the algebra Z_{n+m} is central in the algebra $\mathrm{Z}_n\otimes\mathrm{Z}_m$. We also show that, similarly to the Harish-Chandra map, the restriction of the cutting to the center is a homomorphism. As an example, we derive the Casimir operators for the algebra $\mathrm{DR}(\mathfrak{sl}_2)$ by cutting the Casimir operators for the algebra $\mathrm{DR}(\mathfrak{sl}_3)$.

The relations in the diagonal reduction algebra have a quadratic and a degree zero part. The algebra, defined by the homogeneous quadratic part of the relations, tends, in a quite simple regime, to a commutative algebra (the homogeneous algebra can be thus considered as a “dynamical” deformation of a commutative algebra; “dynamical” here means that the left and right multiplications by elements of the ring $\mathrm{U}(\mathfrak{h})$ differ). This observation about the limit is used in the proof in [7] of the completeness of the set of derived relations over the field of fractions of $\mathrm{U}(\mathfrak{h})$. We prove the completeness by establishing the equivalence between the set of derived relations and the set of ordering relations.

The stabilization law enables one to give a definition of the reduction “algebra” Z_∞ related to the diagonal embedding of the inductive limit $\mathfrak{gl}_\infty$ of $\mathfrak{gl}_n$ into $\mathfrak{gl}_\infty\oplus\mathfrak{gl}_\infty$ (strictly speaking, Z_∞ is not an algebra, some relations have an infinite number of terms).

We also discuss the diagonal reduction algebra for the special linear Lie algebra $\mathfrak{sl}_n$; it is a direct tensor factor in Z_n.

Such a precise description, as the one we give for Z_n, is known for a few examples of the reduction algebras: the most known is related to the embedding of $\mathfrak{gl}_n$ to $\mathfrak{gl}_{n+1}$ [10]. Its representation theory was used for the derivation of precise formulas for the action of the generators of $\mathfrak{gl}_n$ on the Gelfand–Zetlin basic vectors [2]. The reduction algebra for the pair $(\mathfrak{gl}_n,\mathfrak{gl}_{n+1})$ is based on the root embedding $\mathfrak{gl}_n\subset\mathfrak{gl}_{n+1}$ of Lie algebras. In contrast to this example, the diagonal reduction algebra $\mathrm{DR}(\mathfrak{a})$ is based on the diagonal embedding of $\mathfrak{a}$ into $\mathfrak{a}\oplus\mathfrak{a}$, which is not a root embedding of reductive Lie algebras.

2 Notation

Let $\mathcal{E}_{ij}$, $i,j=1,\dots,n$, be the standard generators of the Lie algebra $\mathfrak{gl}_n$, with the commutation relations

$$[\mathcal{E}_{ij},\mathcal{E}_{kl}]=\delta_{jk}\mathcal{E}_{il}-\delta_{il}\mathcal{E}_{kj},$$

where δ_{jk} is the Kronecker symbol. We shall also use the root notation $\mathcal{H}_\alpha$, $\mathcal{E}_\alpha$, $\mathcal{E}_{-\alpha}$, … for elements of $\mathfrak{gl}_n$.

Let $\mathcal{E}^{(1)}_{ij}$ and $\mathcal{E}^{(2)}_{ij}$, $i,j=1,\dots,n$, be the standard generators of the two copies of the Lie algebra $\mathfrak{gl}_n$ in $\mathfrak{g}:=\mathfrak{gl}_n\oplus\mathfrak{gl}_n$,

$$[\mathcal{E}^{(a)}_{ij},\mathcal{E}^{(b)}_{kl}]=\delta_{ab}\big(\delta_{jk}\mathcal{E}^{(a)}_{il}-\delta_{il}\mathcal{E}^{(a)}_{kj}\big).$$

Set

$$e_{ij}:=\mathcal{E}^{(1)}_{ij}+\mathcal{E}^{(2)}_{ij},\qquad E_{ij}:=\mathcal{E}^{(1)}_{ij}-\mathcal{E}^{(2)}_{ij}.$$

The elements e_{ij} span the diagonally embedded Lie algebra $\mathfrak{k}\simeq\mathfrak{gl}_n$, while E_{ij} form an adjoint $\mathfrak{k}$-module $\mathfrak{p}$. The Lie algebra $\mathfrak{k}$ and the space $\mathfrak{p}$ constitute a symmetric pair, that is, $[\mathfrak{k},\mathfrak{k}]\subset\mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}]\subset\mathfrak{p}$, and $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{k}$:

$$[e_{ij},e_{kl}]=\delta_{jk}e_{il}-\delta_{il}e_{kj},\qquad [e_{ij},E_{kl}]=\delta_{jk}E_{il}-\delta_{il}E_{kj},\qquad [E_{ij},E_{kl}]=\delta_{jk}e_{il}-\delta_{il}e_{kj}.$$

In the sequel, h_a means the element e_{aa} of the Cartan subalgebra $\mathfrak{h}$ of the subalgebra $\mathfrak{k}\in\mathfrak{gl}_n\oplus\mathfrak{gl}_n$ and h_{ab} the element $e_{aa}-e_{bb}$.

Let $\{\varepsilon_a\}$ be the basis of $\mathfrak{h}^*$ dual to the basis $\{h_a\}$ of $\mathfrak{h}$, $\varepsilon_a(h_b)=\delta_{ab}$. We shall use as well the root notation h_α, e_α, $e_{-\alpha}$ for elements of $\mathfrak{k}$, and H_α, E_α, $E_{-\alpha}$ for elements of $\mathfrak{p}$.

The Lie subalgebra $\mathfrak{n}$ in the triangular decomposition (1.2) is spanned by the root vectors e_{ij} with $i<j$ and the Lie subalgebra $\mathfrak{n}_-$ by the root vectors e_{ij} with $i>j$. Let $\mathfrak{b}_+$ and $\mathfrak{b}_-$ be the corresponding Borel subalgebras, $\mathfrak{b}_+=\mathfrak{h}\oplus\mathfrak{n}$ and $\mathfrak{b}_-=\mathfrak{h}\oplus\mathfrak{n}_-$. Denote by Δ_+ and Δ_- the sets of positive and negative roots in the root system $\Delta=\Delta_+\cup\Delta_-$ of $\mathfrak{k}$: Δ_+ consists of roots $\varepsilon_i-\varepsilon_j$ with $i<j$ and Δ_- consists of roots $\varepsilon_i-\varepsilon_j$ with $i>j$. Let Q be the root lattice, $\mathrm{Q}:=\{\gamma\in\mathfrak{h}^*\,|\,\gamma=\sum_{\alpha\in\Delta_+,n_\alpha\in\mathbb{Z}}n_\alpha\alpha\}$. It contains the positive cone Q_+,

$$\mathrm{Q}_+:=\Big\{\gamma\in\mathfrak{h}^*\,|\,\gamma=\sum_{\alpha\in\Delta_+,n_\alpha\in\mathbb{Z},\ n_\alpha\geq 0}n_\alpha\alpha\Big\}.$$

For $\lambda,\mu\in\mathfrak{h}^*$, the notation

$$\lambda>\mu \tag{2.1}$$

means that the difference $\lambda-\mu$ belongs to Q_+, $\lambda-\mu\in\mathrm{Q}_+$. This is a partial order in $\mathfrak{h}^*$.

We fix the following action of the cover of the symmetric group S_n (the Weyl group of the diagonal $\mathfrak{k}$) on the Lie algebra $\mathfrak{gl}_n\oplus\mathfrak{gl}_n$ by automorphisms

$$\acute{\sigma}_i(x):=\mathrm{Ad}_{\exp(e_{i,i+1})}\mathrm{Ad}_{\exp(-e_{i+1,i})}\mathrm{Ad}_{\exp(e_{i,i+1})}(x),$$

so that

$$\acute{\sigma}_i(e_{kl})=(-1)^{\delta_{ik}+\delta_{il}}e_{\sigma_i(k)\sigma_i(l)},\qquad \acute{\sigma}_i(E_{kl})=(-1)^{\delta_{ik}+\delta_{il}}E_{\sigma_i(k)\sigma_i(l)}.$$

Here $\sigma_i = (i, i+1)$ is an elementary transposition in the symmetric group. We extend naturally the above action of the cover of S_n to the action by automorphisms on the associative algebra $\mathrm{A} \equiv \mathrm{A}_n := \mathrm{U}(\mathfrak{gl}_n) \otimes \mathrm{U}(\mathfrak{gl}_n)$. The restriction of this action to $\mathfrak{h}$ coincides with the natural action $\sigma(h_k) = h_{\sigma(k)}$, $\sigma \in \mathrm{S}_n$, of the Weyl group on the Cartan subalgebra.

Besides, we use the shifted action of S_n on the polynomial algebra $\mathrm{U}(\mathfrak{h})$ (and its localizations) by automorphisms; the shifted action is defined by

$$\sigma \circ h_k := h_{\sigma(k)} + k - \sigma(k), \qquad k = 1, \dots, n; \qquad \sigma \in \mathrm{S}_n. \tag{2.2}$$

It becomes the usual action for the variables

$$\mathring{h}_k := h_k - k, \qquad \mathring{h}_{ij} := \mathring{h}_i - \mathring{h}_j; \tag{2.3}$$

by (2.2) for any $\sigma \in \mathrm{S}_n$ we have

$$\sigma \circ \mathring{h}_k = \mathring{h}_{\sigma(k)}, \qquad \sigma \circ \mathring{h}_{ij} = \mathring{h}_{\sigma(i)\sigma(j)}.$$

It will be sometimes convenient to denote the commutator $[a, b]$ of two elements a and b of an associative algebra by

$$\hat{a}(b) := [a, b]. \tag{2.4}$$

3 Reduction algebra Z_n

In this section we recall the definition of the reduction algebras, in particular the diagonal reduction algebras of the $\mathfrak{gl}$ type. We introduce the order for which the ordering relations for the algebra Z_n will be discussed. The formulas for the Zhelobenko automorphisms for the algebra Z_n are given; some basic facts about the standard involution, anti-involution and central elements for the algebra Z_n are presented at the end of the section.

1. Let $\overline{\mathrm{U}}(\mathfrak{h})$ and $\bar{\mathrm{A}}$ be the rings of fractions of the algebras $\mathrm{U}(\mathfrak{h})$ and A with respect to the multiplicative set, generated by elements

$$h_{ij} + l, \qquad l \in \mathbb{Z}, \quad 1 \le i < j \le n.$$

Define Z_n to be the double coset space of $\bar{\mathrm{A}}$ by its left ideal $\bar{\mathrm{I}}_+ := \bar{\mathrm{A}}\mathfrak{n}$, generated by elements of $\mathfrak{n}$, and the right ideal $\bar{\mathrm{I}}_- := \mathfrak{n}_-\bar{\mathrm{A}}$, generated by elements of $\mathfrak{n}_-$, $\mathrm{Z}_n := \bar{\mathrm{A}}/(\bar{\mathrm{I}}_+ + \bar{\mathrm{I}}_-)$.

The space Z_n is an associative algebra with respect to the multiplication map

$$a \diamond b := aPb. \tag{3.1}$$

Here P is the extremal projector [3] for the diagonal $\mathfrak{gl}_n$. It is an element of a certain extension of the algebra $\mathrm{U}(\mathfrak{gl}_n)$ satisfying the relations $e_{ij}P = Pe_{ji} = 0$ for all i and j such that $1 \le i < j \le n$.

The algebra Z_n is a particular example of a reduction algebra; in our context, Z_n is defined by the coproduct (the diagonal inclusion) $\mathrm{U}(\mathfrak{gl}_n) \to \mathrm{A}$.

2. The main structure theorems for the reduction algebras are given in [7, Section 2].

In the sequel we choose a weight linear basis $\{p_K\}$ of $\mathfrak{p}$ ($\mathfrak{p}$ is the $\mathfrak{k}$-invariant complement to $\mathfrak{k}$ in $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$) and equip it with a total order $\prec$. The total order $\prec$ will be compatible with the partial order $<$ on $\mathfrak{h}^*$, see (2.1), in the sense that $\mu_K < \mu_L \;\Rightarrow\; p_K \prec p_L$. We shall sometimes write $I \prec J$ instead of $p_I \prec p_J$. For an arbitrary element $a \in \bar{\mathrm{A}}$ let $\widetilde{a}$ be its image in the reduction algebra; in particular, $\widetilde{p_K}$ is the image in the reduction algebra of the basic vector $p_K \in \mathfrak{p}$.

3. In our situation we choose the set of vectors E_{ij}, $i,j=1,\dots,n$, as a basis of the space $\mathfrak{p}$. The weight of E_{ij} is $\varepsilon_i-\varepsilon_j$. The compatibility of a total order $\prec$ with the partial order $<$ on $\mathfrak{h}^*$ means the condition

$$E_{ij}\prec E_{kl}\qquad \text{if}\quad i-j>k-l.$$

The order in each subset $\{E_{ij}|i-j=a\}$ with a fixed a can be chosen arbitrarily. For instance, we can set

$$E_{ij}\prec E_{kl}\qquad \text{if}\quad i-j>k-l\quad \text{or}\quad i-j=k-l\quad \text{and}\quad i>k. \tag{3.2}$$

Denote the images of the elements E_{ij} in Z_n by z_{ij}. We use also the notation t_i for the elements z_{ii} and $t_{ij}:=t_i-t_j$ for the elements $z_{ii}-z_{jj}$. The order (3.2) induces as well the order on the generators z_{ij} of the algebra Z_n:

$$z_{ij}\prec z_{kl}\ \Leftrightarrow\ E_{ij}\prec E_{kl}.$$

The statement (d) in the paper [7, Section 2] implies an existence of structure constants $\mathrm{B}_{(ab),(cd),(ij),(kl)}\in\overline{\mathrm{U}}(\mathfrak{h})$ and $\mathrm{D}_{(ab),(cd)}\in\overline{\mathrm{U}}(\mathfrak{h})$ such that for any $a,b,c,d=1,\dots,n$ we have

$$z_{ab}\diamond z_{cd}=\sum_{i,j,k,l:z_{ij}\preceq z_{kl}}\mathrm{B}_{(ab),(cd),(ij),(kl)}z_{ij}\diamond z_{kl}+\mathrm{D}_{(ab),(cd)}. \tag{3.3}$$

In particular, the algebra Z_n (in general, the reduction algebra related to a symmetric pair $(\mathfrak{k},\mathfrak{p})$, $\mathfrak{g}:=\mathfrak{k}+\mathfrak{p}$) is $\mathbb{Z}_2$-graded; the degree of z_{ab} is 1 and the degree of any element from $\overline{\mathrm{U}}(\mathfrak{h})$ is 0.

The relations (3.3) together with the weight conditions

$$[h,z_{ab}]=(\varepsilon_a-\varepsilon_b)(h)z_{ab}$$

are the defining relations for the algebra Z_n.

Note that the denominators of the structure constants $\mathrm{B}_{(ab),(cd),(ij),(kl)}$ and $\mathrm{D}_{(ab),(cd)}$ are products of linear factors of the form $\mathring{h}_{ij}+\ell$, $i<j$, where $\ell\geq -1$ is an integer, see [7].

4. The algebra Z_n can be equipped with the action of Zhelobenko automorphisms [6]. Denote by $\check{\mathrm{q}}_i$ the Zhelobenko automorphism $\check{\mathrm{q}}_i:\mathrm{Z}_n\to\mathrm{Z}_n$ corresponding to the transposition $\sigma_i\in\mathrm{S}_n$. It is defined as follows [6]. First we define a map $\check{\mathrm{q}}_i:\mathrm{A}\to\bar{\mathrm{A}}/\bar{\mathrm{I}}_+$ by

$$\check{\mathrm{q}}_i(x):=\sum_{k\geq 0}\frac{(-1)^k}{k!}\hat{e}^k_{i,i+1}(\acute{\sigma}_i(x))e^k_{i+1,i}\prod_{a=1}^{k}(h_{i,i+1}-a+1)^{-1}\qquad \mathrm{mod}\ \bar{\mathrm{I}}_+. \tag{3.4}$$

Here $\hat{e}_{i,i+1}$ stands for the adjoint action of the element $e_{i,i+1}$, see (2.4). The operator $\check{\mathrm{q}}_i$ has the property

$$\check{\mathrm{q}}_i(hx)=(\sigma_i\circ h)\check{\mathrm{q}}_i(x) \tag{3.5}$$

for any $x\in\mathrm{A}$ and $h\in\mathfrak{h}$; $\sigma\circ h$ is defined in (2.2). With the help of (3.5), the map $\check{\mathrm{q}}_i$ can be extended to the map (denoted by the same symbol) $\check{\mathrm{q}}_i:\bar{\mathrm{A}}\to\bar{\mathrm{A}}/\bar{\mathrm{I}}_-$ by setting $\check{\mathrm{q}}_i(a(h)x)=(\sigma_i\circ a(h))\check{\mathrm{q}}_i(x)$ for any $x\in\mathrm{A}$ and $a(h)\in\overline{\mathrm{U}}(\mathfrak{h})$. One can further prove that $\check{\mathrm{q}}_i(\bar{\mathrm{I}}_+)=0$ and $\check{\mathrm{q}}_i(\bar{\mathrm{I}}_-)\subset(\bar{\mathrm{I}}_-+\bar{\mathrm{I}}_+)/\bar{\mathrm{I}}_+$, so that $\check{\mathrm{q}}_i$ can be viewed as a linear operator $\check{\mathrm{q}}_i:\mathrm{Z}_n\to\mathrm{Z}_n$. Due to [6], this is an algebra automorphism, satisfying (3.5).

The operators $\check{\mathrm{q}}_i$ satisfy the braid group relations [10]:

$$\check{\mathrm{q}}_i\check{\mathrm{q}}_{i+1}\check{\mathrm{q}}_i=\check{\mathrm{q}}_{i+1}\check{\mathrm{q}}_i\check{\mathrm{q}}_{i+1},$$
$$\check{\mathrm{q}}_i\check{\mathrm{q}}_j=\check{\mathrm{q}}_j\check{\mathrm{q}}_i,\quad |i-j|>1,$$

and the inversion relation [6]:

$$\check{\mathrm{q}}_i^2(x) = \frac{1}{h_{i,i+1}+1}\acute{\sigma}_i^2(x)(h_{i,i+1}+1), \qquad x \in \mathrm{Z}_n. \tag{3.6}$$

In particular, $\check{\mathrm{q}}_i^2(x) = x$ if x is of zero weight.

5. The Chevalley anti-involution ϵ in $\mathrm{U}(\mathfrak{gl}_n \oplus \mathfrak{gl}_n)$, $\epsilon(e_{ij}) := e_{ji}$, $\epsilon(E_{ij}) := E_{ji}$, induces the anti-involution ϵ in the algebra Z_n:

$$\epsilon(z_{ij}) = z_{ji}, \qquad \epsilon(h_k) = h_k. \tag{3.7}$$

Besides, the outer automorphism of the Dynkin diagram of $\mathfrak{gl}_n$ induces the involutive automorphism ω of Z_n,

$$\omega(z_{ij}) = (-1)^{i+j+1} z_{j'i'}, \qquad \omega(h_k) = -h_{k'}, \tag{3.8}$$

where $i' = n+1-i$. The operations ϵ and ω commute, $\epsilon\omega = \omega\epsilon$.

Central elements of the subalgebra $\mathrm{U}(\mathfrak{gl}_n) \otimes 1 \subset \mathrm{A}$, generated by n Casimir operators of degrees $1, \dots, n$, as well as central elements of the subalgebra $1 \otimes \mathrm{U}(\mathfrak{gl}_n) \subset \mathrm{A}$ project to central elements of the algebra Z_n. In particular, central elements of degree 1 project to central elements

$$I^{(n,h)} := h_1 + \cdots + h_n \tag{3.9}$$

and

$$I^{(n,t)} := t_1 + \cdots + t_n \tag{3.10}$$

of the algebra Z_n. The difference of central elements of degree two projects to the central element

$$\sum_{i=1}^{n} (h_i - 2i) t_i \tag{3.11}$$

of the algebra Z_n. The images of other Casimir operators are more complicated.

4 Main results

This section contains the principal results of the paper. We first give preliminary information on the new basis in which the defining relations for the algebra Z_n can be written down in an economical fashion. The braid group action on the new generators is then explicitly given in Subsection 4.2. The complete set of the defining relations for the algebra Z_n is written down in Subsection 4.3. The regime for which both the set of the derived defining relations and the set of the defining ordering relation have a controllable "limiting behavior" is introduced in Subsection 4.4. Subsection 4.5 deals with the diagonal reduction algebra for $\mathfrak{sl}_n$; the quadratic Casimir operator for $\mathrm{DR}(\mathfrak{sl}_n)$ as well as for the diagonal reduction algebra for an arbitrary semi-simple Lie algebra $\mathfrak{k}$ is given there. Subsection 4.6 is devoted to the stabilization and cut phenomena with respect to the embedding of the Lie algebra $\mathfrak{gl}_n \oplus \mathfrak{gl}_m$ into the Lie algebra $\mathfrak{gl}_{n+m}$; the theorem about the behavior of the centers of the diagonal reduction algebra under the cutting is proved there.

4.1 New variables

We shall use the following elements of $\overline{\mathrm{U}}(\mathfrak{h})$:

$$A_{ij} := \frac{\mathring{h}_{ij}}{\mathring{h}_{ij}-1}, \quad A'_{ij} := \frac{\mathring{h}_{ij}-1}{\mathring{h}_{ij}}, \quad B_{ij} := \frac{\mathring{h}_{ij}-1}{\mathring{h}_{ij}-2}, \quad B'_{ij} := \frac{\mathring{h}_{ij}-2}{\mathring{h}_{ij}-1}, \quad C'_{ij} := \frac{\mathring{h}_{ij}-3}{\mathring{h}_{ij}-2},$$

the variables $\mathring{h}_{ij}$ are defined in (2.3). Note that $A_{ij}A'_{ij} = B_{ij}B'_{ij} = 1$.

Define elements $\mathring{t}_1, \dots, \mathring{t}_n \in \mathrm{Z}_n$ by

$$\mathring{t}_1 := t_1, \quad \mathring{t}_2 := \check{\mathrm{q}}_1(t_1), \quad \mathring{t}_3 := \check{\mathrm{q}}_2\check{\mathrm{q}}_1(t_1), \quad \dots, \quad \mathring{t}_n := \check{\mathrm{q}}_{n-1}\cdots\check{\mathrm{q}}_2\check{\mathrm{q}}_1(t_1).$$

Using (3.4) we find the relations

$$\begin{aligned} &\check{\mathrm{q}}_i(t_i) = -\frac{1}{\mathring{h}_{i,i+1}-1}t_i + \frac{\mathring{h}_{i,i+1}}{\mathring{h}_{i,i+1}-1}t_{i+1}, \qquad \check{\mathrm{q}}_i(t_{i+1}) = \frac{\mathring{h}_{i,i+1}}{\mathring{h}_{i,i+1}-1}t_i - \frac{1}{\mathring{h}_{i,i+1}-1}t_{i+1}, \\ &\check{\mathrm{q}}_i(t_k) = t_k, \qquad k \neq i, i+1, \end{aligned} \tag{4.1}$$

which can be used to convert the definition (4.1) into a linear over the ring $\overline{\mathrm{U}}(\mathfrak{h})$ change of variables:

$$\begin{aligned} \mathring{t}_l &= t_l \prod_{j=1}^{l-1} A_{jl} - \sum_{k=1}^{l-1} t_k \frac{1}{\mathring{h}_{kl}-1} \prod_{j=1}^{k-1} A_{jl}, \\ t_l &= \mathring{t}_l \prod_{j=1}^{l-1} A'_{jl} + \sum_{k=1}^{l-1} \mathring{t}_k \frac{1}{\mathring{h}_{kl}} \prod_{\substack{j=1 \\ j\neq k}}^{l-1} A'_{jk}. \end{aligned} \tag{4.2}$$

For example,

$$\begin{aligned} \mathring{t}_2 &= -\frac{1}{\mathring{h}_{12}-1}t_1 + \frac{\mathring{h}_{12}}{\mathring{h}_{12}-1}t_2, \qquad t_2 = \frac{1}{\mathring{h}_{12}}\mathring{t}_1 + \frac{\mathring{h}_{12}-1}{\mathring{h}_{12}}\mathring{t}_2, \\ \mathring{t}_3 &= -\frac{1}{\mathring{h}_{13}-1}t_1 - \frac{\mathring{h}_{13}}{(\mathring{h}_{13}-1)(\mathring{h}_{23}-1)}t_2 + \frac{\mathring{h}_{13}\mathring{h}_{23}}{(\mathring{h}_{13}-1)(\mathring{h}_{23}-1)}t_3, \\ t_3 &= \frac{\mathring{h}_{12}+1}{\mathring{h}_{12}\mathring{h}_{13}}\mathring{t}_1 + \frac{\mathring{h}_{12}-1}{\mathring{h}_{12}\mathring{h}_{23}}\mathring{t}_2 + \frac{(\mathring{h}_{13}-1)(\mathring{h}_{23}-1)}{\mathring{h}_{13}\mathring{h}_{23}}\mathring{t}_3. \end{aligned}$$

In terms of the new variables $\mathring{t}$'s, the linear in t central element (3.10) reads

$$\sum t_i = \sum \mathring{t}_i \prod_{a:a\neq i} \frac{\mathring{h}_{ia}+1}{\mathring{h}_{ia}}.$$

4.2 Braid group action

Since $\check{\mathrm{q}}_i^2(x) = x$ for any element x of zero weight, the braid group acts as its symmetric group quotient on the space of weight 0 elements. It follows from (4.1) and $\check{\mathrm{q}}_i(t_1) = t_1$ for all $i > 1$ that

$$\check{\mathrm{q}}_\sigma(\mathring{t}_i) = \mathring{t}_{\sigma(i)} \tag{4.3}$$

for any $\sigma \in \mathrm{S}_n$.

The action of the Zhelobenko automorphisms, see Section 3, on the generators z_{kl} looks as follows:

$$\begin{aligned} &\check{\mathrm{q}}_i(z_{ik}) = -z_{i+1,k}A_{i,i+1}, && \check{\mathrm{q}}_i(z_{ki}) = -z_{k,i+1}, \quad k \neq i, i+1, \\ &\check{\mathrm{q}}_i(z_{i+1,k}) = z_{i,k}, && \check{\mathrm{q}}_i(z_{k,i+1}) = z_{k,i}A_{i,i+1}, \quad k \neq i, i+1, \\ &\check{\mathrm{q}}_i(z_{i,i+1}) = -z_{i+1,i}A_{i,i+1}B_{i,i+1}, && \check{\mathrm{q}}_i(z_{i+1,i}) = -z_{i,i+1}, \\ &\check{\mathrm{q}}_i(z_{j,k}) = z_{j,k}, \quad j, k \neq i, i+1. \end{aligned} \tag{4.4}$$

Denote $i' = n + 1 - i$, as before. The braid group action (4.4) is compatible with the anti-involution ϵ and the involution ω (note that $\omega(\mathring{h}_{ij}) = \mathring{h}_{j'i'}$), see (3.7) and (3.8), in the following sense:

$$\epsilon\check{\mathrm{q}}_i = \check{\mathrm{q}}_i^{-1}\epsilon, \tag{4.5}$$

$$\omega\check{\mathrm{q}}_i = \check{\mathrm{q}}_{i'-1}\omega. \tag{4.6}$$

Let w_0 be the longest element of the Weyl group of $\mathfrak{gl}_n$, the symmetric group S_n. Similarly to the squares of the transformations corresponding to the simple roots, see (3.6), the action of $\check{\mathrm{q}}_{w_0}^2$ is the conjugation by a certain element of $\overline{\mathrm{U}}(\mathfrak{h})$.

Lemma 1. *We have*

$$\check{\mathrm{q}}_{w_0}^2(x) = S^{-1}xS, \tag{4.7}$$

where

$$S = \prod_{i,j:i<j} \mathring{h}_{ij}. \tag{4.8}$$

The proof shows that the formula (4.7) works for an arbitrary reductive Lie algebra, with $S = \prod_{\alpha\in\Delta_+} \mathring{h}_\alpha$.

Proposition 2. *The action of $\check{\mathrm{q}}_{w_0}$ on generators reads*

$$\check{\mathrm{q}}_{w_0}(z_{ij}) = (-1)^{i+j} z_{i'j'} \prod_{a:a<i'} A_{ai'} \prod_{b:b>j'} A_{j'b}, \tag{4.9}$$

$$\check{\mathrm{q}}_{w_0}(\mathring{t}_i) = \mathring{t}_{i'}. \tag{4.10}$$

The proofs of Lemma 1 and Proposition 2 are in Section 5.

4.3 Defining relations

To save space we omit in this section the symbol $\diamond$ for the multiplication in the algebra Z_n. It should not lead to any confusion since no other multiplication is used in this section.

Each relation which we will derive will be of a certain weight, equal to a sum of two roots. From general considerations the upper estimate for the number of terms in a quadratic relation of weight $\lambda = \alpha + \beta$ is the number $|\lambda|$ of quadratic combinations $z_{\alpha'}z_{\beta'}$ with $\alpha' + \beta' = \lambda$. There are several types, excluding the trivial one, $\lambda = 2(\varepsilon_i - \varepsilon_j)$, $|\lambda| = 1$:

1. $\lambda = \pm(2\varepsilon_i - \varepsilon_j - \varepsilon_k)$, where i, j and k are pairwise distinct. Then $|\lambda| = 2$.
2. $\lambda = \varepsilon_i - \varepsilon_j + \varepsilon_k - \varepsilon_l$ with pairwise distinct i, j, k and l. Then $|\lambda| = 4$.
3. $\lambda = \varepsilon_i - \varepsilon_j$, $i \neq j$. For $z_{\alpha'}z_{\beta'}$, there are $2(n-2)$ possibilities (subtype 3a) with $\alpha' = \varepsilon_i - \varepsilon_k$, $\beta' = \varepsilon_k - \varepsilon_j$ or $\alpha' = \varepsilon_k - \varepsilon_j$, $\beta' = \varepsilon_i - \varepsilon_k$ with $k \neq i, j$ and $2n$ possibilities (subtype 3b) with $\alpha' = 0$, $\beta' = \varepsilon_i - \varepsilon_j$ or $\alpha' = \varepsilon_i - \varepsilon_j$, $\beta' = 0$. Thus $|\lambda| = 4(n-1)$.

4. $\lambda=0$. There are n^2 possibilities (subtype 4a) with $\alpha'=0$, $\beta'=0$ and $n(n-1)$ possibilities (subtype 4b) with $\alpha'=\varepsilon_i-\varepsilon_j$, $\beta'=\varepsilon_j-\varepsilon_i$, $i\neq j$. Here $|\lambda|=n(2n-1)$.

Below we write down relations for each type (and subtype) separately. The relations of the types 1 and 2 have a simple form in terms of the original generators z_{ij}. To write the relations of the types 3 and 4, it is convenient to renormalize the generators z_{ij} with $i\neq j$. Namely, we set

$$\mathring{z}_{ij}=z_{ij}\prod_{k=1}^{i-1}A_{ki}. \tag{4.11}$$

In terms of the generators $\mathring{z}_{ij}$, the formulas (4.4) for the action of the automorphisms $\check{\mathrm{q}}_i$ translate as follows:

$$\begin{aligned}
&\check{\mathrm{q}}_i(\mathring{z}_{ik})=-\mathring{z}_{i+1,k}, && \check{\mathrm{q}}_i(\mathring{z}_{i+1,k})=\mathring{z}_{i,k}A_{i+1,i}, \quad k\neq i,i+1,\\
&\check{\mathrm{q}}_i(\mathring{z}_{ki})=-\mathring{z}_{k,i+1}, && \check{\mathrm{q}}_i(\mathring{z}_{k,i+1})=\mathring{z}_{k,i}A_{i,i+1}=A'_{i+1,i}\mathring{z}_{k,i}, \quad k\neq i,i+1,\\
&\check{\mathrm{q}}_i(\mathring{z}_{i,i+1})=-A'_{i+1,i}\mathring{z}_{i+1,i}, && \check{\mathrm{q}}_i(\mathring{z}_{i+1,i})=-\mathring{z}_{i,i+1}A_{i+1,i},\\
&\check{\mathrm{q}}_i(\mathring{z}_{j,k})=\mathring{z}_{j,k}, \quad j,k\neq i,i+1.
\end{aligned}$$

1. The relations of the type 1 are:

$$z_{ij}z_{ik}=z_{ik}z_{ij}A_{kj}, \qquad z_{ji}z_{ki}=z_{ki}z_{ji}A'_{kj}, \qquad \text{for} \quad j<k,\ i\neq j,k. \tag{4.12}$$

2. Denote

$$D_{ijkl}:=\left(\frac{1}{\mathring{h}_{ik}}-\frac{1}{\mathring{h}_{jl}}\right).$$

Then, for any four pairwise different indices i, j, k and l, we have the following relations of the type 2:

$$\begin{aligned}
&[z_{ij},z_{kl}]=z_{kj}z_{il}D_{ijkl}, \qquad i<k,\ j<l,\\
&z_{ij}z_{kl}-z_{kl}z_{ij}A'_{jl}A'_{lj}=z_{kj}z_{il}D_{ijkl}, \qquad i<k,\ j>l.
\end{aligned} \tag{4.13}$$

3a. Let $i\neq k\neq l\neq i$. Denote

$$\mathring{E}_{ikl}:=-\left((\mathring{t}_i-\mathring{t}_k)\frac{\mathring{h}_{il}+1}{\mathring{h}_{ik}\mathring{h}_{il}}+(\mathring{t}_k-\mathring{t}_l)\frac{\mathring{h}_{il}-1}{\mathring{h}_{kl}\mathring{h}_{il}}\right)\mathring{z}_{il}+\sum_{a:a\neq i,k,l}\mathring{z}_{al}\mathring{z}_{ia}\frac{B_{ai}}{\mathring{h}_{ka}+1}.$$

With this notation the first group of the relations of the type 3 is:

$$\begin{aligned}
&\mathring{z}_{ik}\mathring{z}_{kl}A'_{ik}-\mathring{z}_{kl}\mathring{z}_{ik}B_{ki}=\mathring{E}_{ikl}, \qquad i<k<l,\\
&\mathring{z}_{ik}\mathring{z}_{kl}A'_{ik}A'_{lk}B_{lk}-\mathring{z}_{kl}\mathring{z}_{ik}B_{ki}=\mathring{E}_{ikl}, \qquad i<l<k,\\
&\mathring{z}_{ik}\mathring{z}_{kl}A_{ki}-\mathring{z}_{kl}\mathring{z}_{ik}B_{ki}=\mathring{E}_{ikl}, \qquad k<i<l,\\
&\mathring{z}_{ik}\mathring{z}_{kl}A_{ki}A_{li}B'_{li}-\mathring{z}_{kl}\mathring{z}_{ik}B_{ki}=\mathring{E}_{ikl}, \qquad k<l<i,\\
&\mathring{z}_{ik}\mathring{z}_{kl}A'_{ik}A'_{lk}B_{lk}A_{li}B'_{li}-\mathring{z}_{kl}\mathring{z}_{ik}B_{ki}=\mathring{E}_{ikl}, \qquad l<i<k,\\
&\mathring{z}_{ik}\mathring{z}_{kl}A_{ki}A'_{lk}B_{lk}A_{li}B'_{li}-\mathring{z}_{kl}\mathring{z}_{ik}B_{ki}=\mathring{E}_{ikl}, \qquad l<k<i.
\end{aligned} \tag{4.14}$$

The relations (4.14) can be written in a more compact way with the help of both systems, z_{ij} and $\mathring{z}_{ij}$, of generators. Let now

$$E_{ikl}:=-\left((\mathring{t}_i-\mathring{t}_k)\frac{\mathring{h}_{il}+1}{\mathring{h}_{ik}\mathring{h}_{il}}+(\mathring{t}_k-\mathring{t}_l)\frac{\mathring{h}_{il}-1}{\mathring{h}_{kl}\mathring{h}_{il}}\right)z_{il}+\sum_{a:a\neq i,k,l}\mathring{z}_{al}z_{ia}\frac{B_{ai}}{\mathring{h}_{ka}+1}.$$

Then

$$z_{ik}\mathring{z}_{kl}A'_{ik} - \mathring{z}_{kl}z_{ik}B_{ki} = E_{ikl}, \qquad k<l,$$
$$z_{ik}\mathring{z}_{kl}A'_{ik}A'_{lk}B_{lk} - \mathring{z}_{kl}z_{ik}B_{ki} = E_{ikl}, \qquad l<k. \tag{4.15}$$

Moreover, after an extra redefinition: $\mathring{\mathring{z}}_{kl} = \mathring{z}_{kl}B_{lk}$ for $k>l$, the left hand side of the second line in (4.15) becomes, up to a common factor, the same as the left hand side of the first line, namely, it reads $(z_{ik}\,\mathring{\mathring{z}}_{kl}\,A'_{ik} - \,\mathring{\mathring{z}}_{kl}\,z_{ik}B_{ki})A'_{lk}$.

3b. Let $i\neq j\neq k\neq i$. The second group of relations of the type 3 reads:

$$\mathring{z}_{ij}\mathring{t}_i = \mathring{t}_i\mathring{z}_{ij}C'_{ji} - \mathring{t}_j\mathring{z}_{ij}\frac{1}{\mathring{h}_{ij}+2} - \sum_{a:a\neq i,j}\mathring{z}_{aj}\mathring{z}_{ia}\frac{1}{\mathring{h}_{ia}+2},$$
$$\mathring{z}_{ij}\mathring{t}_j = -\mathring{t}_i\mathring{z}_{ij}\frac{C'_{ji}}{\mathring{h}_{ij}-1} + \mathring{t}_j\mathring{z}_{ij}A_{ij}A'_{ji}B_{ji} + \sum_{a:a\neq i,j}\mathring{z}_{aj}\mathring{z}_{ia}A_{ij}A'_{ji}B_{ai}\mathring{h}_{ja}+1, \tag{4.16}$$
$$\mathring{z}_{ij}\mathring{t}_k = \mathring{t}_i\mathring{z}_{ij}\frac{(\mathring{h}_{ij}+3)B_{ji}}{(\mathring{h}_{ik}^2-1)(\mathring{h}_{jk}-1)} + \mathring{t}_j\mathring{z}_{ij}\frac{(\mathring{h}_{ij}+1)B_{ji}}{(\mathring{h}_{ik}-1)(\mathring{h}_{jk}-1)^2} + \mathring{t}_k\mathring{z}_{ij}A_{ik}A_{ki}A_{jk}B'_{jk}$$
$$- \mathring{z}_{kj}\mathring{z}_{ik}\frac{(\mathring{h}_{ij}+1)B_{ki}}{(\mathring{h}_{ik}-1)(\mathring{h}_{jk}-1)} - \sum_{a:a\neq i,j,k}\mathring{z}_{aj}\mathring{z}_{ia}\frac{\mathring{h}_{ij}+1}{(\mathring{h}_{ik}-1)(\mathring{h}_{jk}-1)}\frac{B_{ai}}{\mathring{h}_{ka}+1}.$$

4a. The relations of the weight zero (the type 4) are also divided into 2 groups. This is the first group of the relations:

$$[\mathring{t}_i, \mathring{t}_j] = 0. \tag{4.17}$$

As follows from the proof, the relations (4.17) hold for the diagonal reduction algebra for an arbitrary reductive Lie algebra: the images of the generators, corresponding to the Cartan subalgebra, commute.

4b. Finally, the second group of the relations of the type 4 is

$$[\mathring{z}_{ij}, \mathring{z}_{ji}] = \mathring{h}_{ij} - \frac{1}{\mathring{h}_{ij}}(\mathring{t}_i - \mathring{t}_j)^2 + \sum_{a:a\neq i,j}\left(\frac{1}{\mathring{h}_{ja}+1}\mathring{z}_{ai}\mathring{z}_{ia} - \frac{1}{\mathring{h}_{ia}+1}\mathring{z}_{aj}\mathring{z}_{ja}\right), \tag{4.18}$$

where $i\neq j$.

Main statement. Denote by $\mathfrak{R}$ the system (4.12), (4.13), (4.14), (4.16), (4.17) and (4.18) of the relations.

Theorem 3. *The relations $\mathfrak{R}$ are the defining relations for the weight generators z_{ij} and t_i of the algebra Z_n. In particular, the set* (3.3) *of ordering relations follows over $\overline{\mathrm{U}}(\mathfrak{h})$ from* (*and is equivalent to*) $\mathfrak{R}$.

The derivation of the system $\mathfrak{R}$ of the relations is given in Section 5. The validity in Z_n of relations from the set $\mathfrak{R}$, together with the results from [7], completes the proof of Theorem 3 (Section 5.4).

4.4 Limit

Let $\mathfrak{R}^{\prec}$ be the set of ordering relations (3.3). Denote by $\mathfrak{R}_0$ the homogeneous (quadratic) part of the system $\mathfrak{R}$ and by $\mathfrak{R}_0^{\prec}$ the homogeneous part of the system $\mathfrak{R}^{\prec}$.

1. Placing coefficients from $\overline{\mathrm{U}}(\mathfrak{h})$ in all relations from $\mathfrak{R}_0$ to the same side (to the right, for example) from the monomials $\widetilde{p_L}\diamond\widetilde{p_M}$, one can give arbitrary numerical values to the variables h_α (α's are roots of $\mathfrak{k}$).

The structure of the extremal projector P or the recurrence relation (5.4) implies that the system $\mathfrak{R}_0$ admits, for an arbitrary reductive Lie algebra, the limit at $h_{\alpha_i} = c_i h$, $h \to \infty$ (α_i ranges through the set of simple positive roots of $\mathfrak{k}$ and c_i are generic positive constants). Moreover, this homogeneous algebra becomes the usual commutative (polynomial) algebra in this limit; so this limiting behavior of the system $\mathfrak{R}_0$, used in the proof, generalizes to a wider class of reduction algebras, related to a pair $(\mathfrak{g}, \mathfrak{k})$ as in the introduction.

2. The limiting procedure from paragraph **1** establishes the bijection between the set of relations and the set of unordered pairs (L, M), where L, M are indices of basic vectors of $\mathfrak{p}$. The proof in [7] shows that over $\mathcal{D}(\mathfrak{h})$ the system $\mathfrak{R}$ can be rewritten in the form of ordering relations for an arbitrary order on the set $\{\widetilde{p_L}\}$ of generators. Here $\mathcal{D}(\mathfrak{h})$ is the field of fractions of the ring $\mathrm{U}(\mathfrak{h})$.

By definition, the relations from $\mathfrak{R}^{\prec}$ are labeled by pairs (L, M) with $L > M$. The above bijection induces therefore a bijection between the sets $\mathfrak{R}$ and $\mathfrak{R}^{\prec}$.

4.5 $\mathfrak{sl}_n$

1. Denote the subalgebra of Z_n, generated by two central elements (3.9) and (3.10), by Y_n; the algebra Y_n is isomorphic to Z_1.

Since the extremal projector for $\mathfrak{sl}_n$ is the same as for $\mathfrak{gl}_n$, the diagonal reduction algebra $\mathrm{DR}(\mathfrak{sl}_n)$ for $\mathfrak{sl}_n$ is naturally a subalgebra of Z_n. The subalgebra $\mathrm{DR}(\mathfrak{sl}_n)$ is complementary to Y_n in the sense that $\mathrm{Z}_n = Y_n \otimes \mathrm{DR}(\mathfrak{sl}_n)$.

The algebra $\mathrm{DR}(\mathfrak{sl}_n)$ is generated by z_{ij}, $i, j = 1, \dots, n$, $i \neq j$, and $t_{i,i+1} := t_i - t_{i+1}$, $i = 1, \dots, n-1$ (and the Cartan subalgebra $\mathfrak{h}$, generated by $h_{i,i+1}$, of the diagonally embedded $\mathfrak{sl}_n$). The elements $t_{i,i+1}$ form a basis in the space of "traceless" combinations $\sum c_m t_m$ (traceless means that $\sum c_m = 0$), $c_m \in \overline{\mathrm{U}}(\mathfrak{h})$.

2. The action of the braid group restricts onto the traceless subspace:

$$\check{\mathrm{q}}_i(t_{i-1,i}) = t_{i-1,i} + \frac{\mathring{h}_{i,i+1}}{\mathring{h}_{i,i+1} - 1} t_{i,i+1}, \qquad \check{\mathrm{q}}_i(t_{i+1,i+2}) = \frac{\mathring{h}_{i,i+1}}{\mathring{h}_{i,i+1} - 1} t_{i,i+1} + t_{i+1,i+2},$$

$$\check{\mathrm{q}}_i(t_{i,i+1}) = -\frac{\mathring{h}_{i,i+1} + 1}{\mathring{h}_{i,i+1} - 1} t_{i,i+1}, \qquad \check{\mathrm{q}}_i(t_{k,k+1}) = t_{k,k+1}, \quad k \neq i-1, i, i+1.$$

The traceless subspace with respect to the generators t_i and the traceless subspace with respect to the generators $\mathring{t}_i$ (that is, the space of linear combinations $\sum c_m \mathring{t}_m$, $c_m \in \overline{\mathrm{U}}(\mathfrak{h})$, with $\sum c_m = 0$) coincide. Indeed, in the expression of t_l as a linear combination of $\mathring{t}_k$'s (the second line in (4.2)), we find, calculating residues and the value at infinity, that the sum of the coefficients is 1,

$$\prod_{j=1}^{l-1} A'_{jl} + \sum_{k=1}^{l-1} \frac{1}{\mathring{h}_{kl}} \prod_{\substack{j=1 \\ j \neq k}}^{l-1} A'_{jk} = 1.$$

Therefore, in the decomposition of the difference $t_i - t_j$ as a linear combination of $\mathring{t}_k$'s, the sum of the coefficients vanishes, so it is traceless with respect to $\mathring{t}_k$'s; $t_{l,l+1}$ is a linear combination of $\mathring{t}_{12}, \mathring{t}_{23}, \dots, \mathring{t}_{l,l+1}$ (and vice versa). It should be however noted that in contrast to (4.2), the coefficients in these combinations do not factorize into a product of linear monomials, the lowest example is $\mathring{t}_{34}$:

$$\mathring{t}_{12} = \frac{\mathring{h}_{12}}{\mathring{h}_{12} - 1} t_{12}, \qquad \mathring{t}_{23} = \frac{\mathring{h}_{23}}{\mathring{h}_{13} - 1} \left(-\frac{1}{\mathring{h}_{12} - 1} t_{12} + \frac{\mathring{h}_{13}}{\mathring{h}_{23} - 1} t_{23} \right),$$

$$\mathring{t}_{34} = \frac{\mathring{h}_{34}}{\mathring{h}_{14}-1}\left(-\frac{1}{\mathring{h}_{13}-1}t_{12} - \frac{\mathring{h}_{14}(\mathring{h}_{13}-1)+\mathring{h}_{23}(\mathring{h}_{24}-1)}{(\mathring{h}_{13}-1)(\mathring{h}_{23}-1)(\mathring{h}_{24}-1)}t_{23} + \frac{\mathring{h}_{14}\mathring{h}_{24}}{(\mathring{h}_{24}-1)(\mathring{h}_{34}-1)}t_{34}\right).$$

3. One can directly see that the commutations between z_{ij} and the differences $t_k - t_l$ close. The renormalization (4.11) is compatible with the $\mathfrak{sl}$-condition and, as we have seen, the set $\{t_{i,i+1}\}$ of generators can be replaced by the set $\{\mathring{t}_{i,i+1}\}$. Therefore, one can work with the generators $\mathring{z}_{ij}$, $i, j = 1, \dots, n$, $i \neq j$, and $\mathring{t}_{i,i+1} := t_i - t_{i+1}$, $i = 1, \dots, n-1$. A direct look at the relations (4.12), (4.13), (4.14), (4.16), (4.17) and (4.18) shows that the only non-trivial verification concerns the relations (4.16); one has to check here the following assertion: when $\mathring{z}$ moves through $\mathring{t}_{i,i+1}$, only traceless combinations of $\mathring{t}_l$'s appear in the right hand side. Write a relation from the list (4.16) in the form $\mathring{z}_{ij}\mathring{t}_l = \sum_m \chi_m^{(i,j,l,m)}\mathring{t}_m\mathring{z}_{ij} + \cdots$, $\chi_m^{(i,j,l,m)} \in \overline{\mathrm{U}}(\mathfrak{h})$, where dots stand for terms with $\mathring{z}\mathring{z}$. The assertion follows from the direct observation that for all i, j and l the sum of the coefficients $\chi_m^{(i,j,l,m)}$ is 1, $\sum_m \chi_m^{(i,j,l,m)} = 1$.

4. With the help of the central elements (3.9), (3.10) and (3.11) one can build a unique linear in t's traceless combination:

$$\sum_{i=1}^{n}(h_i - 2i)t_i - \left(\frac{1}{n}\sum_{i=1}^{n} h_i - n - 1\right)\sum_{j=1}^{n} t_j.$$

It clearly depends only on the differences $h_i - h_j$ and belongs therefore to the center of the subalgebra $\mathrm{DR}(\mathfrak{sl}_n)$.

One can write this central element in the form

$$\sum_{u,v=1}^{n-1} \mathsf{C}^{uv} h_{u,u+1}t_{v,v+1} + \sum_{v=1}^{n-1}(n-v)vt_{v,v+1} = \sum_{u,v=1}^{n-1} \mathsf{C}^{uv}(\mathring{h}_{u,u+1}+1)t_{v,v+1}, \tag{4.19}$$

where C^{uv} is the inverse Cartan matrix of $\mathfrak{sl}_n$.

In general, let $\mathfrak{k}$ be a semi-simple Lie algebra of rank r with the Cartan matrix a_{ij}. Let b_{ij} be the symmetrized Cartan matrix and $(\, , \,)$ the scalar product on $\mathfrak{h}^*$ induced by the invariant non-degenerate bilinear form on $\mathfrak{k}$, so that

$$a_{ij} = d_i b_{ij}, \qquad b_{ij} = (\alpha_i, \alpha_j), \qquad d_i = 2/(\alpha_i, \alpha_i).$$

For each $i = 1, \dots, r$ let $\alpha_i^\vee$ be the coroot vector corresponding to the simple root α_i, so that $\alpha_j(\alpha_i^\vee) = a_{ij}$. Let d_{ij} be the matrix, inverse to $c_{ij} = d_i b_{ij} d_j$. Let $\rho \in \mathfrak{h}^*$ be the half-sum of all positive roots. Write

$$\rho = \frac{1}{2}\sum_{i=1}^{r} n_i \alpha_i,$$

where n_i are nonnegative integers. Let t_{α_i} be the images of $H_{\alpha_i} = \alpha_i^{\vee(1)} - \alpha_i^{\vee(2)}$ in the diagonal reduction algebra $\mathrm{DR}(\mathfrak{k})$ and $h_{\alpha_i} = \alpha_i^{\vee(1)} + \alpha_i^{\vee(2)}$ be the coroot vectors of the diagonally embedded Lie algebra $\mathfrak{k}$. The generalization of the central element (4.19) to the reduction algebra $\mathrm{DR}(\mathfrak{k})$ reads

$$\sum_{i,i=1}^{r} d_{ij} h_{\alpha_i} t_{\alpha_j} + \sum_{i=1}^{r} n_i(\alpha_i, \alpha_i)t_{\alpha_i}.$$

4.6 Stabilization and cutting

In [7] we discovered the stabilization and cut phenomena which are heavily used in our derivation of the set of defining relations for the diagonal reduction algebras of $\mathfrak{gl}$-type. The consideration in [7] uses the standard (by the first coordinates) embedding of $\mathfrak{gl}_n$ into $\mathfrak{gl}_{n+1}$. In this subsection we shall make several more precise statements about the stabilization and cut considering now the embedding of $\mathfrak{gl}_n \oplus \mathfrak{gl}_1$ into $\mathfrak{gl}_{n+1}$ (more generally, $\mathfrak{gl}_n \oplus \mathfrak{gl}_m$ into $\mathfrak{gl}_{n+m}$). These precisions are needed to establish the behavior of the center of the diagonal reduction algebra: namely we shall see that cutting preserves the centrality.

Notation: $\mathfrak{h}$ in this subsection denotes the Cartan subalgebra of $\mathfrak{gl}_{n+m}$.

Consider an embedding of $\mathfrak{gl}_n \oplus \mathfrak{gl}_m$ into $\mathfrak{gl}_{n+m}$, given by an assignment $e_{ij} \mapsto e_{ij}$, $i,j = 1,\dots,n$, and $e_{ab} \mapsto e_{n+a,n+b}$, $a,b = 1,\dots,m$, where e_{kl} in the source are the generators of $\mathfrak{gl}_n \oplus \mathfrak{gl}_m$ and target e_{kl} are in $\mathfrak{gl}_{n+m}$. This rule together with the similar rule $E_{ij} \mapsto E_{ij}$ and $E_{ab} \mapsto E_{n+a,n+b}$ defines an embedding of the Lie algebra $(\mathfrak{gl}_n \oplus \mathfrak{gl}_m) \oplus (\mathfrak{gl}_n \oplus \mathfrak{gl}_m)$ into the Lie algebra $\mathfrak{gl}_{n+m} \oplus \mathfrak{gl}_{n+m}$ and of the enveloping algebras $\mathrm{A}_n \otimes \mathrm{A}_m = \mathrm{U}(\mathfrak{gl}_n \oplus \mathfrak{gl}_n) \otimes \mathrm{U}(\mathfrak{gl}_m \oplus \mathfrak{gl}_m)$ into $\mathrm{A}_{n+m} = \mathrm{U}(\mathfrak{gl}_{n+m} \oplus \mathfrak{gl}_{n+m})$. This embedding clearly maps nilpotent subalgebras of $\mathfrak{gl}_n \oplus \mathfrak{gl}_m$ to the corresponding nilpotent subalgebras of $\mathfrak{gl}_{n+m}$ and thus defines an embedding $\iota_{n,m}$: $\mathrm{Z}_n \otimes \mathrm{Z}_m \to \mathrm{Z}_{n+m}$ of the corresponding double coset spaces. However, the map $\iota_{n,m}$ is not a homomorphism of algebras. This is because the multiplication maps are defined with the help of projectors, which are different for $\mathfrak{gl}_n \oplus \mathfrak{gl}_m$ and $\mathfrak{gl}_{n+m}$.

However, as we will explain now we can control certain differences between the two multiplication maps. Let $\mathrm{V}_{n,m}$ be the left ideal of the algebra Z_{n+m} generated by elements z_{ia} with $i = 1,\dots,n$ and $a = n+1,\dots,n+m$; let $\mathrm{V}'_{n,m}$ be the right ideal of the algebra Z_{n+m} generated by elements z_{ai} with $i = 1,\dots,n$ and $a = n+1,\dots,n+m$.

Write any element $\lambda \in Q_+$ (the positive cone of the root lattice of $\mathfrak{gl}_{n+m}$) in the form $\lambda = \sum_{k=1}^{n+m} \lambda_k \varepsilon_k$. The element λ can be presented as a sum

$$\lambda = \lambda' + \lambda'', \tag{4.20}$$

where λ' is an element of the root lattice of $\mathfrak{gl}_n \oplus \mathfrak{gl}_m$, and λ'' is proportional to the simple root $\varepsilon_n - \varepsilon_{n+1}$: $\lambda' = \sum_{k=1}^{n+m} \lambda'_k \varepsilon_k$ with $\sum_{k=1}^{n} \lambda'_k = \sum_{k=n+1}^{n+m} \lambda'_k = 0$ and $\lambda'' = c(\varepsilon_n - \varepsilon_{n+1})$.

Lemma 4. *The left ideal $\mathrm{V}_{n,m} \subset \mathrm{Z}_{n+m}$ consists of images in Z_{n+m} of sums $\sum_{ia} X_{ia} E_{ia}$ with $X_{ia} \in \bar{\mathrm{A}}_{n+m}$, $i = 1,\dots,n$ and $a = n+1,\dots,n+m$.*

The right ideal $\mathrm{V}_{n,m} \subset \mathrm{Z}_{n+m}$ consists of images in Z_{n+m} of sums $\sum_{ai} E_{ai} Y_{ai}$ with $Y_{ai} \in \bar{\mathrm{A}}_{n+m}$, $i = 1,\dots,n$ and $a = n+1,\dots,n+m$.

Proof. Present the projector P for the Lie algebra $\mathfrak{gl}_{n+m}$ as a sum of terms

$$\xi e_{-\gamma_1} \cdots e_{-\gamma_t} e_{\gamma'_1} \cdots e_{\gamma'_{t'}}, \tag{4.21}$$

where $\xi \in \overline{\mathrm{U}}(\mathfrak{h})$, $\gamma_1,\dots,\gamma_t$ and $\gamma'_1,\dots,\gamma'_{t'}$ are positive roots of $\mathfrak{gl}_{n+m}$. For any $\lambda \in Q_+$ denote by P_λ the sum of above elements with $\gamma_1 + \cdots + \gamma_t = \gamma'_1 + \cdots + \gamma'_{t'} = \lambda$. Then $P = \sum_{\lambda \in Q_+} P_\lambda$. For any $X, Y \in \bar{\mathrm{A}}$ define the element $X \diamond_\lambda Y$ as the image of $X P_\lambda Y$ in the reduction algebra. We have $X \diamond Y = \sum_{\lambda \in Q_+} X \diamond_\lambda Y$.

For any $X \in \bar{\mathrm{A}}_{n+m}$, $i = 1,\dots,n$ and $a = n+1,\dots,n+m$ consider the product $X \diamond_\lambda z_{ia}$.

The product $X \diamond_\lambda z_{in}$ is zero if $\lambda'' \neq 0$ (the component λ'' is defined by (4.20)). Indeed, in this case in each summand of P_λ one of $e_{\gamma'_{k'}}$ is equal to some e_{jb}, $j = 1,\dots,n$ and $b = n+1,\dots,n+m$. Choose an ordered basis of $\mathfrak{n}_+$ which ends by all such e_{jb} (ordered arbitrarily); any element of $\mathrm{U}(\mathfrak{n}_+)$ can be written as a sum of ordered monomials, that is, monomials in which all such e_{jb} stand on the right. Since $[e_{jb}, E_{ia}] = 0$ for any $i,j = 1,\dots,n$ and $a,b = n+1,\dots,n+m$, the product $e_{\gamma'_{k'}} E_{ia}$ belongs to the left ideal $\bar{\mathrm{I}}_+$ and thus $X \diamond_\lambda z_{ia} = 0$ in Z_{n+m}.

If $\lambda'' = 0$ then generators of $\mathfrak{n}_+$ in monomials entering the decomposition of P_λ are among the elements e_{ij}, $1 \le i < j \le n$, and e_{ab}, $n+1 \le a < b \le n+m$ and thus their adjoint action leaves the space, spanned by all E_{ia}, $i = 1, \dots, n$, $a = n+1, \dots, n+m$ invariant, so $X \diamond_\lambda z_{ia}$ can be presented as an image of the sum $\sum_{jb} X_{jb} E_{jb}$ with $X_{jb} \in \bar{\mathrm{A}}_{n+m}$, $j = 1, \dots, n$, $b = n+1, \dots, n+m$. Thus, the left ideal, generated by all z_{ia} is contained in the vector space of images in Z_{n+m} of sums $\sum_{jb} X_{jb} E_{jb}$.

Moreover, for any $X \in \bar{\mathrm{A}}_{n+m}$ the element $X \diamond z_{ia}$ is the image of $X E_{ia} + \sum_{j,b:\, j<i,\, b>a} X^{(jb)} E_{jb}$ for some $X^{(jb)}$ and the double induction on i and a proves the inverse inclusion.

The second part of lemma is proved similarly. ■

Corollary 5. *We have the following decomposition of the free left (and right) $\overline{\mathrm{U}}(\mathfrak{h})$-modules:*

$$\mathrm{Z}_{n+m} = \mathrm{I}_{n,m} \oplus \overline{\mathrm{U}}(\mathfrak{h}) \cdot \iota_{n,m}(\mathrm{Z}_n \otimes \mathrm{Z}_m), \tag{4.22}$$

where $\mathrm{I}_{n,m} := \mathrm{V}_{n,m} + \mathrm{V}'_{n,m}$.

Proof. The double coset space Z_{n+m} is a free left and right $\overline{\mathrm{U}}(\mathfrak{h})$-module with a basis consisting of images of ordered monomials on elements E_{ij}, $i, j = 1, \dots, n+m$; recall that we always use orders compatible with the partial order $<$ on $\mathfrak{h}^*$, see (c) in Section 3, paragraph 2. We can choose an order for which all ordered monomials are of the form XYZ, where X is a monomial on E_{ai} with $i = 1, \dots, n$ and $a = n+1, \dots, n+m$, Z is a monomial on E_{ia} with $i = 1, \dots, n$ and $a = n+1, \dots, n+m$ while Y is a monomial on E_{ij} with $i, j = 1, \dots, n$ or $i, j = n+1, \dots, n+m$. Then we apply the lemma above. ■

For a moment denote for each $k > 0$ the multiplication map in Z_k by $\diamond_{(k)} : \mathrm{Z}_k \otimes \mathrm{Z}_k \to \mathrm{Z}_k$ (instead of the default notation $\diamond$, see (3.1)); denote also for each $k, l > 0$ by $\diamond_{(k,l)}$ the multiplication map $\diamond_{(k)} \otimes \diamond_{(l)}$ in $\mathrm{Z}_k \otimes \mathrm{Z}_l$.

Proposition 6. *For any $x, y \in \mathrm{Z}_n \otimes \mathrm{Z}_m$ we have*

$$\iota_{n,m}(x) \diamond_{(n+m)} \iota_{n,m}(y) = \iota_{n,m}(x \diamond_{(n,m)} y) + z,$$

where z is some element of $\mathrm{J}_{n,m} := \mathrm{V}_{n,m} \cap \mathrm{V}'_{n,m}$.

Let $\mathfrak{h}_n$ and $\mathfrak{h}_m$ be the Cartan subalgebras of $\mathfrak{gl}_n$ and $\mathfrak{gl}_m$, respectively. Denote the space $\mathrm{Z}_n \otimes_{\overline{\mathrm{U}}(\mathfrak{h}_n)} \overline{\mathrm{U}}(\mathfrak{h}) \otimes_{\overline{\mathrm{U}}(\mathfrak{h}_m)} \mathrm{Z}_m$ by $\overline{\mathrm{U}}(\mathfrak{h}) \cdot (\mathrm{Z}_n \otimes \mathrm{Z}_m)$. The composition law $\diamond_{(n,m)}$ naturally extends to the space $\overline{\mathrm{U}}(\mathfrak{h}) \cdot (\mathrm{Z}_n \otimes \mathrm{Z}_m)$ equipping it with an associative algebra structure (we keep the same symbol $\diamond_{(n,m)}$ for the extended composition law in $\overline{\mathrm{U}}(\mathfrak{h}) \cdot (\mathrm{Z}_n \otimes \mathrm{Z}_m)$). Also, the map $\iota_{n,m}$ admits a natural extension to a map $\iota_{n,m} : \overline{\mathrm{U}}(\mathfrak{h}) \cdot (\mathrm{Z}_n \otimes \mathrm{Z}_m) \to \mathrm{Z}_{n+m}$ denoted by the same symbol and defined by the rule $\iota_{n,m}(\varphi x) := \varphi\, \iota_{n,m}(x)$ for any $\varphi \in \overline{\mathrm{U}}(\mathfrak{h})$ and $x \in \mathrm{Z}_n \otimes \mathrm{Z}_m$. The statement of Proposition 6 remains valid for this extension as well, that is, one can take $x, y \in \overline{\mathrm{U}}(\mathfrak{h}) \cdot (\mathrm{Z}_n \otimes \mathrm{Z}_m)$ in the formulation.

Proof of Proposition 6. Denote by $P_{n,m} := P_n \otimes P_m$ the projector for the Lie algebra $\mathfrak{gl}_n \oplus \mathfrak{gl}_m$.

It is sufficient to prove the following statement. Suppose X and Y are (non-commutative) polynomials in E_{ij} with $i, j = 1, \dots, n$ Then the product of x and y in Z_{n+m} coincides with the image in Z_{n+m} of $X\, P_{n,m} Y$ modulo the left ideal $\mathrm{V}_{n,m}$ and modulo the right ideal $\mathrm{V}'_{n,m}$).

Due to the structure of the projector the condition $\lambda'' = 0$, see (4.20), implies that the product $X \diamond_\lambda Y$ related to $\mathfrak{gl}_n \oplus \mathfrak{gl}_m$ coincides with product $X \diamond_\lambda Y$ related to $\mathfrak{gl}_{n+m}$.

Let now $\lambda'' \neq 0$. Then each monomial $e_{\gamma'_1} \cdots e_{\gamma'_{t'}}$ in the decomposition of P_λ, see (4.21), contains generators e_{ia} with $i \in \{1, \dots, n\}$ and $a \in \{n+1, \dots, n+m\}$; these e_{ia} can be assumed to be right factors of the corresponding monomial (like in the proof of Lemma 4). The commutator of any such generator e_{ia} with every factor in Y is a linear combination of the elements E_{jb} with $j \in \{1,\dots,n\}$ and $b \in \{n+1,\dots,n+m\}$. Moving the resulting E_{jb} to the right we see that the product $X \diamond_\lambda Y$ is the image in Z_{n+m} of an element of the form $\sum_s X_s Y_s$ where each Y_s belongs to the left ideal of $\bar{\mathrm{A}}_{n+m}$ generated by E_{jb} with $j \in \{1,\dots,n\}$ and $b \in \{n+1,\dots,n+m\}$ (one can say more: each Y_s can be written in a form $\sum_{j,b} Y_s^{(jb)} E_{jb}$ where each $Y_s^{(jb)} \in \bar{\mathrm{A}}_{n+m}$ does not involve generators E_{ck} with $k \in \{1,\dots,n\}$ and $c \in \{n+1,\dots,n+m\}$; we don't need this stronger form). Thus, due to Lemma 4, $X \diamond_\lambda Y \in \mathrm{V}_{n,m}$.

Similarly, each X_s participating in the sum $\sum_s X_s Y_s$, see above, belongs to the right ideal of $\bar{\mathrm{A}}_{n+m}$ generated by the elements E_{bj} with $j \in \{1,\dots,n\}$ and $b \in \{n+1,\dots,n+m\}$. So, again by Lemma 4, $X \diamond_\lambda Y \in \mathrm{V}'_{n,m}$. ■

Suppose that we have a relation

$$\sum_k a_k \diamond_{(n,m)} b_k = 0, \tag{4.23}$$

where all a_k and b_k are elements of $Z_n \otimes Z_m$. Then, due to Proposition 6, we have the following relation in Z_{n+m}:

$$\sum_k \bar{a}_k \diamond_{(m+n)} \bar{b}_k = z, \tag{4.24}$$

where $\bar{a}_k = \iota_{n,m}(a_k)$, $\bar{b}_k = \iota_{n,m}(b_k)$ and $z \in \mathrm{J}_{n,m} = \mathrm{V}_{n,m} \cap \mathrm{V}'_{n,m}$.

On the other hand, suppose we have the following relation in Z_{n+m}:

$$\sum_k \bar{a}_k \diamond_{(m+n)} \bar{b}_k = u, \tag{4.25}$$

where all a_k and b_k are elements of $Z_n \otimes Z_m$, $\bar{a}_k = \iota_{n,m}(a_k)$, $\bar{b}_k = \iota_{n,m}(b_k)$, and $u \in \mathrm{I}_{n,m} = \mathrm{V}_{n,m} + \mathrm{V}'_{n,m}$. Then the elements a_k and b_k satisfy the relation (4.23) and $u \in \mathrm{J}_{n,m}$. Indeed, suppose that the relation (4.25) is satisfied and $\sum_k a_k \diamond_{(n,m)} b_k = v$ for some $v \in Z_n \otimes Z_m$. It follows from Proposition 6 that $\sum_k \bar{a}_k \diamond_{(m+n)} \bar{b}_k - \bar{v}$ belongs to $\mathrm{J}_{n,m}$; here $\bar{v} = \iota_{n,m}(v)$. Then (4.25) implies that $\bar{v} \in \mathrm{I}_{n,m}$ and thus $\bar{v} = 0$ due to Corollary 5. Thus $v = 0$, since the map $\iota_{n,m}$ is an inclusion, and $u \in \mathrm{J}_{n,m}$.

We refer to the implication (4.23) ⇒ (4.24) as *stabilization*. Call *cutting* the (almost inverse) implication (4.25) ⇒ (4.23) which can be understood as a procedure of getting relations in $\mathrm{Z}_n \otimes \mathrm{Z}_m$ from relations in Z_{n+m}; we say that (4.23) is the *cut* of (4.25). Clearly all relations in $\mathrm{Z}_n \otimes \mathrm{Z}_m$ can be obtained by cutting appropriate relations in Z_{n+m}.

Let $\pi_{n,m} : \mathrm{Z}_{n+m} \to \overline{\mathrm{U}}(\mathfrak{h}) \cdot (\mathrm{Z}_n \otimes \mathrm{Z}_m)$ be the composition of the projection $\bar{\pi}_{n,m}$ of Z_{n+m} onto $\iota_{n,m}(\overline{\mathrm{U}}(\mathfrak{h}) \cdot \mathrm{Z}_n \otimes \mathrm{Z}_m) = \overline{\mathrm{U}}(\mathfrak{h}) \cdot \iota_{n,m}(\mathrm{Z}_n \otimes \mathrm{Z}_m)$ along $\mathrm{I}_{n,m}$, see (4.22), and of the inverse to the inclusion $\iota_{n,m}$:

$$\pi_{n,m} = \iota_{n,m}^{-1} \circ \bar{\pi}_{n,m}.$$

We have the following consequence of Proposition 6 and Corollary 5.

Proposition 7. *Let x be a central element of Z_{n+m}. Then $\pi_{n,m}(x)$ is a central element of $\overline{\mathrm{U}}(\mathfrak{h}) \cdot (\mathrm{Z}_n \otimes \mathrm{Z}_m)$.*

Proof. Denote $X = \pi_{n,m}(x)$. Then, by definition, $x = \iota_{n,m}(X) + z$, where $z \in \mathrm{I}_{n,m}$. Since x is central, it is of zero weight; so X and z are of zero weight as well. Thus each monomial entering the decomposition of z contains both types of generators, E_{ai} and E_{ia}, where $i \in \{1,\dots,n\}$ and

$a \in \{n+1, \dots, n+m\}$, which implies that $z \in \mathrm{J}_{n,m} = \mathrm{V}_{n,m} \cap \mathrm{V}'_{n,m}$. Take any $Y \in \mathrm{Z}_n \otimes \mathrm{Z}_m$. We now prove that $X \diamond_{(n,m)} Y - Y \diamond_{(n,m)} X = 0$. Denote $y = \iota_{n,m}(Y)$. Due to Proposition 6,

$$\iota_{n,m}(X \diamond_{(n,m)} Y - Y \diamond_{(n,m)} X) = (x - z) \diamond_{(m+n)} y - y \diamond_{(m+n)} (x - z) + z', \tag{4.26}$$

where $z' \in \mathrm{J}_{n,m} = \mathrm{V}_{n,m} \cap \mathrm{V}'_{n,m}$. Since x is central in Z_{n+m}, the right hand side of (4.26) is equal to

$$y \diamond_{(m+n)} z - z \diamond_{(m+n)} y + z',$$

which is an element of $\mathrm{I}_{n,m} = \mathrm{V}_{n,m} \oplus \mathrm{V}'_{n,m}$ since $z, z' \in \mathrm{J}_{n,m}$. On the other hand, the left hand side of (4.26) belongs to $\overline{\mathrm{U}}(\mathfrak{h}) \cdot \iota_{n,m}(\mathrm{Z}_n \otimes \mathrm{Z}_m)$. Thus, by Corollary 5, both sides of (4.26) are equal to zero and $X \diamond_{(n,m)} Y - Y \diamond_{(n,m)} X = 0$ since the map $\iota_{n,m}$ is injective. ∎

The map $\pi_{n,m}$ obeys properties similar to those of the Harish-Chandra map $\mathrm{U}(\mathfrak{g})^{\mathfrak{h}} \to \mathrm{U}(\mathfrak{h})$ ($\mathrm{U}(\mathfrak{g})^{\mathfrak{h}}$ is the space of elements of zero weight). For instance, its restriction to the center of Z_{n+m} is a homomorphism. More precisely, if x is a central element of Z_{n+m}, then

$$\begin{aligned} &\pi_{n,m}(x \diamond_{(m+n)} y) = \pi_{n,m}(x) \diamond_{(n,m)} \pi_{n,m}(y) \qquad \text{and} \\ &\pi_{n,m}(y \diamond_{(m+n)} x) = \pi_{n,m}(y) \diamond_{(n,m)} \pi_{n,m}(x) \end{aligned} \tag{4.27}$$

for any $y \in \mathrm{Z}_{n+m}$. Indeed, let $X = \pi_{n,m}(x)$, $Y = \pi_{n,m}(y)$. Then

$$x = \iota_{n,m}(X) - z, \qquad y = \iota_{n,m}(Y) - u,$$

where $u \in \mathrm{I}_{n,m}$ while, as it was noted in the proof of Proposition 7, $z \in \mathrm{J}_{n,m}$. Moreover, it is clear that z can be written in the form $z = \sum_a z'_a z_a$, where $z_a \in \mathrm{V}_{n,m}$ and $z'_a \in \mathrm{V}'_{n,m}$ (for instance, use the order as in the proof of Corollary 5). Then (dropping for brevity the multiplication symbol $\diamond_{(m+n)}$) we have

$$\begin{aligned} \iota_{n,m}(X)\iota_{n,m}(Y) &= (x+z)(y+u) = \left(x + \sum_a z'_a z_a\right)(y + \tilde{z}' + \tilde{z}) \\ &= xy + \sum_a z'_a z_a (y + \tilde{z}' + \tilde{z}) + x\tilde{z} + \tilde{z}' x \equiv xy \mod \mathrm{I}_{n,m}. \end{aligned} \tag{4.28}$$

Here $\tilde{z} \in \mathrm{V}_{n,m}$ and $\tilde{z}' \in \mathrm{V}'_{n,m}$. In the last equality we used the centrality of x. Due to Proposition 6, (4.28) is precisely equivalent to the fist part of (4.27). The second part of (4.27) is proved similarly.

5 Proofs

5.1 Tensor J

The multiplication map $\diamond$ in Z_n (we return to the original notation) is given by the prescription (3.1), as in any reduction algebra. It can be formally expanded into a series over the root lattice of certain bilinear maps as follows. Set

$$\overline{\mathrm{U}}(\mathfrak{b}_\pm) := \overline{\mathrm{U}}(\mathfrak{h}) \otimes_{\mathrm{U}(\mathfrak{h})} \mathrm{U}(\mathfrak{b}_\pm), \qquad \overline{\mathrm{U}}^{12}(\mathfrak{b}) := \overline{\mathrm{U}}(\mathfrak{b}_-) \otimes_{\overline{\mathrm{U}}(\mathfrak{h})} \overline{\mathrm{U}}(\mathfrak{b}_+).$$

All these are associative algebras. Besides, both algebras $\overline{\mathrm{U}}(\mathfrak{b}_\pm)$ are $\overline{\mathrm{U}}(\mathfrak{h})$-bimodules. The algebra $\overline{\mathrm{U}}^{12}(\mathfrak{b})$ admits three commuting actions of $\overline{\mathrm{U}}(\mathfrak{h})$. Two of them are given by the assignments

$$X(Y \otimes Z) := XY \otimes Z, \qquad (Y \otimes Z)X := Y \otimes ZX,$$

for any $X \in \overline{\mathrm{U}}(\mathfrak{h})$, $Y \in \overline{\mathrm{U}}(\mathfrak{b}_-)$ and $Z \in \overline{\mathrm{U}}(\mathfrak{b}_+)$. The third action associates to any $X \in \overline{\mathrm{U}}(\mathfrak{h})$, $Y \otimes Z \in \overline{\mathrm{U}}^{12}(\mathfrak{b})$ the element $YX \otimes Z = Y \otimes XZ \in \overline{\mathrm{U}}^{12}(\mathfrak{b})$.

Present the projector P in an ordered form:

$$P = \sum_{\gamma,i} \grave{F}_{\gamma,i}\grave{E}_{\gamma,i}\grave{H}_{\gamma,i} = \sum_{\gamma,i} \grave{H}_{\gamma,i}\grave{F}_{\gamma,i}\grave{E}_{\gamma,i}, \tag{5.1}$$

the summation is over $\gamma \in \mathrm{Q}_+$ and $i \in \mathbb{Z}_{\geq 0}$; every $\grave{F}_{\gamma,i}$ is an element of $\mathrm{U}(\mathfrak{n}_-)$ of the weight $-\gamma$, every $\grave{E}_{\gamma,i}$ is an element of $\mathrm{U}(\mathfrak{n}_+)$ of the weight γ and $\grave{H}_{\gamma,i} \in \overline{\mathrm{U}}(\mathfrak{h})$. Let J be the following element of $\overline{\mathrm{U}}^{12}(\mathfrak{b})$:

$$\mathrm{J} := \sum_{\gamma,i} \grave{F}_{\gamma,i} \otimes \grave{E}_{\gamma,i}\grave{H}_{\gamma,i} = \sum_{\gamma,i} \grave{H}_{\gamma,i}\grave{F}_{\gamma,i} \otimes \grave{E}_{\gamma,i}, \qquad \gamma \in \mathrm{Q}_+, \quad i \in \mathbb{Z}_{\geq 0}.$$

Due to the PBW theorem in $\mathrm{U}(\mathfrak{gl}_n)$ the tensor J is uniquely defined by the projector P; it is of total weight zero: $h\mathrm{J} = \mathrm{J}h$ for any $h \in \mathfrak{h}$. We have the weight decomposition of J with respect to the adjoint action of $\mathfrak{h}$ in the second tensor factor of $\overline{\mathrm{U}}^{12}(\mathfrak{b})$:

$$\mathrm{J} = \bigoplus_{\lambda \in \mathrm{Q}_+} \mathrm{J}_\lambda,$$

where J_λ consists of all the terms, corresponding to $\grave{F}_{\lambda,i}\grave{E}_{\lambda,i}\grave{H}_{\lambda,i}$ in (5.1) (contributing to $\lambda \in \mathrm{Q}_+$ in the summation),

$$\mathrm{J}_\lambda := \sum_i \grave{F}_{\lambda,i} \otimes \grave{E}_{\lambda,i}\grave{H}_{\lambda,i}.$$

By definition of J, the multiplication $\diamond$ in the double coset space Z_n can be described by the relation

$$a \diamond b = m\left((a \otimes 1)\mathrm{J}(1 \otimes b)\right), \tag{5.2}$$

where $m(\sum_i c_i \otimes d_i)$ is the image in Z_n of the element $\sum_i c_i d_i$. Moreover, in (5.2) we can replace all products $\grave{E}_{\gamma,i}b$ in the second tensor factor by the adjoint action of $\grave{E}_{\gamma,i}$ on b (in fact, for $\grave{E}_{\gamma,i} = e_{\gamma_m} \cdots e_{\gamma_1}$, we can replace $\grave{E}_{\gamma,i}b$ by $[\grave{E}_{\gamma,i}, b]$ or by $\hat{e}_{\gamma_m} \cdots \hat{e}_{\gamma_1}(b)$, see (2.4)) and likewise all products $a\grave{F}_{\gamma,i}$ in the first tensor factor by the opposite adjoint action of $\grave{F}_{\gamma,i}$ on a. We have a decomposition of the product $\diamond$ into a sum over Q_+:

$$a \diamond b = \sum_{\lambda \in \mathrm{Q}_+} a \diamond_\lambda b, \qquad \text{where} \quad a \diamond_\lambda b := m\left((a \otimes 1)\mathrm{J}_\lambda(1 \otimes b)\right). \tag{5.3}$$

If a and b are weight elements of Z_n of weights $\nu(a)$ and $\nu(b)$, then the product $a \diamond_\lambda b$ is the image in Z_n of the sum $\sum_i a_i b_i$, where the weight of each b_i is $\nu(b) + \lambda$, and the weight of each a_i is $\nu(a) - \lambda$.

The tensor J satisfies the Arnaudon–Buffenoir–Ragoucy–Roche (ABRR) difference equation [1], see also [5] for the translation of the results of [1] to the language of reduction algebras. To describe the equation, let $\vartheta = \frac{1}{2}\sum_{k=1}^n \mathring{h}_k^2 \in \mathrm{U}(\mathfrak{h})$; for any positive root $\gamma \in \Delta_+$, denote by T_γ the following linear operator on the vector space $\overline{\mathrm{U}}^{12}(\mathfrak{b})$:

$$T_\gamma(X \otimes Y) := Xe_{-\gamma} \otimes e_\gamma Y.$$

The ABRR equation means the relation [1, 5]:

$$[1 \otimes \vartheta, \mathrm{J}] = -\sum_{\gamma \in \Delta_+} T_\gamma(\mathrm{J}).$$

This relation is equivalent to the following system of recurrence relations for the weight components J_λ:

$$\mathrm{J}_\lambda \cdot \left(\mathring{h}_\lambda + \frac{(\lambda,\lambda)}{2}\right) = -\sum_{\gamma\in\Delta_+} T_\gamma\left(\mathrm{J}_{\lambda-\gamma}\right), \tag{5.4}$$

where $\mathring{h}_\lambda := \sum_k \lambda_k \mathring{h}_k$ for $\lambda = \sum_k \lambda_k \varepsilon_k$. The recurrence relations (5.4) together with the initial condition $\mathrm{J}_0 = 1\otimes 1$ uniquely determine all weight components J_λ.

It should be noted that the recurrence relations (5.4) provides less information about the structure of the denominators (from $\mathrm{U}(\mathfrak{h})$) of the summands of the extremal projector P than the information implied by the product formula (see [3]) for the extremal projector.

Using (5.4) we get in particular:

$$\mathrm{J}_\alpha = -(\mathring{h}_\alpha + 1)^{-1} e_{-a}\otimes e_\alpha, \qquad \alpha = \varepsilon_i - \varepsilon_{i+1}, \tag{5.5}$$

$$\begin{aligned}\mathrm{J}_{\alpha+\beta} = {}& (\mathring{h}_{\alpha+\beta}+1)^{-1}\Big(-e_{-\alpha-\beta}\otimes e_{\alpha+\beta} + (\mathring{h}_\alpha+1)^{-1} e_{-\alpha}e_{-\beta}\otimes e_\beta e_\alpha \\ & + (\mathring{h}_\beta+1)^{-1} e_{-\beta}e_{-\alpha}\otimes e_\alpha e_\beta\Big), \qquad \alpha = \varepsilon_{i-1}-\varepsilon_i, \quad \beta = \varepsilon_i - \varepsilon_{i+1},\end{aligned} \tag{5.6}$$

$$\mathrm{J}_{\varepsilon_i-\varepsilon_j+\varepsilon_k-\varepsilon_l} = \mathrm{J}_{\varepsilon_i-\varepsilon_j}\cdot \mathrm{J}_{\varepsilon_k-\varepsilon_l}, \qquad i<j<k<l. \tag{5.7}$$

5.2 Braid group action

The proof of the relations (4.1) and (4.4) consists of the following arguments, valid for any reduction algebra. Let α be any simple root of $\mathfrak{gl}_n$, $\alpha = \varepsilon_i - \varepsilon_{i+1}$ and $\mathfrak{g}_\alpha$ the corresponding $\mathfrak{sl}_2$ subalgebra of $\mathfrak{gl}_n$. It is spanned by the elements $e_\alpha = e_{i,i+1}$, $e_{-\alpha} = e_{i+1,i}$ and $h_\alpha = h_i - h_{i+1}$. Let $\acute{\sigma}_\alpha = \acute{\sigma}_i$ be the corresponding automorphism of the algebra A and $\check{\mathrm{q}}_\alpha = \check{\mathrm{q}}_i$ the Zhelobenko automorphism of Z_n. Assume that $Y\in \mathrm{A}$ belongs, with respect to the adjoint action of $\mathfrak{g}_\alpha$, to an irreducible finite-dimensional $\mathfrak{g}_\alpha$-module of dimension $2j+1$, $j\in\{0,1/2,1,\dots\}$. Assume further that Y is homogeneous, of weight $2m$, $[h_\alpha, Y] = 2mY$. Identify Y with its image in Z_n. Then $\check{\mathrm{q}}_\alpha(Y)$ coincides with the image in Z_n of the element

$$\prod_{i=1}^{j+m}(h_\alpha + i + 1)\cdot \acute{\sigma}_\alpha(Y)\cdot \prod_{i=1}^{j+m}(h_\alpha - i + 1)^{-1}.$$

This can be checked directly using [6, Proposition 6.5].

In the realization of irreducible $\mathfrak{sl}_2$-modules as the spaces of homogeneous polynomials in two variables u and v,

$$e_\alpha \mapsto u\frac{\partial}{\partial v}, \qquad h_\alpha \mapsto u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} \qquad \text{and} \qquad e_{-\alpha}\mapsto v\frac{\partial}{\partial u},$$

the operator $\acute{\sigma}_\alpha$ becomes $(\acute{\sigma}_\alpha f)(u,v) = f(-v,u)$, or, in the basis $|j,k\rangle := x^{j+k}y^{j-k}$ (j labels the representation; $k = 0,1,\dots,2j$),

$$\acute{\sigma}_\alpha:\ |j,-j+k\rangle \mapsto (-1)^k |j,j-k\rangle.$$

Proof of Lemma 1, Subsection 4.2. To see this, write a reduced expression for $\check{\mathrm{q}}_{w_0}$, $\check{\mathrm{q}}_{w_0} = \check{\mathrm{q}}_{\alpha_{i_1}}\cdots\check{\mathrm{q}}_{\alpha_{i_M}}$ with $\alpha_{i_1},\dots,\alpha_{i_M}$ simple roots. Then $\check{\mathrm{q}}_{w_0} = \check{\mathrm{q}}_{\alpha_{i_M}}\cdots\check{\mathrm{q}}_{\alpha_{i_1}}$ as well. Writing, for $\check{\mathrm{q}}_{w_0}^2$, the second expression after the first one, we get squares of $\check{\mathrm{q}}_{\alpha_{i_s}}$'s (which are conjugations by $\mathring{h}_{\alpha_{i_s}}^{-1}$'s; they thus commute) one after another. Moving these conjugations to the left through the remaining $\check{\mathrm{q}}$'s, we produce, exactly like in the construction of a system of all positive roots from a reduced expression for the longest element of the Weyl group of a reductive Lie group, the conjugation by the product (4.8) over all positive roots. ■

Proof of Proposition 2, Subsection 4.2. Only formula (4.9) needs a proof (formula (4.10) is a particular case of (4.3)).

For a moment, denote the longest element of the symmetric group S_n by $\check{\mathrm{q}}_{w_0}^{(n)}$. Let $\psi_j := \check{\mathrm{q}}_j\check{\mathrm{q}}_{j-1}\cdots\check{\mathrm{q}}_1$ (the product in the descending order). We have $\check{\mathrm{q}}_{w_0}^{(n+1)} = \check{\mathrm{q}}_{w_0}^{(n)}\psi_n$ and $\check{\mathrm{q}}_{w_0}^{(n+1)} = \psi_1\psi_2\cdots\psi_n$ (the product in the ascending order).

For $j<n$ it follows from (4.4) that $\psi_j(z_{n+1,1}) = (-1)^j z_{n+1,j+1}$ (say, by induction on j). So,

$$\psi_n(z_{n+1,1}) = q_n\psi_{n-1}(z_{n+1,j+1}) = (-1)^{n-1}q_n(z_{n+1,n}) = (-1)^n z_{n,n+1},$$

again by (4.4). Next, $\psi_k\psi_{k+1}\cdots\psi_{n-1}(z_{n,n+1}) = z_{k,n+1}$ by induction on $n-k$ and again (4.4). Thus,

$$\check{\mathrm{q}}_{w_0}(z_{n+1,1}) = (-1)^n z_{1,n+1}, \tag{5.8}$$

establishing (4.9) for $i=n+1$ and $j=1$. We now prove (4.9) for $i>j$ (positions below the main diagonal) by induction backwards on the height $i-j$ of a negative root; the formula (5.8) serves as the induction base. Assume that (4.9) is verified for a given level $i-j$ and $i-j-1>0$ (so that the positions $(i,j+1)$ and $(i-1,j)$ are still under the main diagonal). By (4.4), $z_{i,j+1} = -\check{\mathrm{q}}_j(z_{ij})$, therefore

$$\begin{aligned}
\check{\mathrm{q}}_{w_0}(z_{i,j+1}) &= -\check{\mathrm{q}}_{w_0}(\check{\mathrm{q}}_j(z_{ij})) = -\check{\mathrm{q}}_{j'-1}(\check{\mathrm{q}}_{w_0}(z_{ij}))\\
&= (-1)^{i+j+1}\check{\mathrm{q}}_{j'-1}\left(z_{i'j'}\prod_{a:a<i'}A_{ai'}\prod_{b:b>j'}A_{j'b}\right)\\
&= (-1)^{i+j+1}z_{i',j'-1}A_{j'-1,j'}\prod_{a:a<i'}A_{ai'}\prod_{b:b>j'}A_{j'-1,b}\\
&= (-1)^{i+j+1}z_{i',(j+1)'}\prod_{a:a<i'}A_{ai'}\prod_{b:b>(j+1)'}A_{(j+1)',b}.
\end{aligned}$$

In the second equality we used the identity $\check{\mathrm{q}}_{w_0}\check{\mathrm{q}}_j = \check{\mathrm{q}}_{j'-1}\check{\mathrm{q}}_{w_0}$ in the braid group; the third equality is the induction assumption; in the fourth equality we used that $i' \neq j'-1$ (since $i-j-1>0$) and then (4.4); in the fifth equality we replaced $j'-1$ by $(j+1)'$. The calculation for $\check{\mathrm{q}}_{w_0}(z_{i-1,j})$ is similar; it uses $z_{i-1,j} = \check{\mathrm{q}}_{i-1}(z_{ij})$. The proof of the formula (4.9) for positions below the main diagonal is finished.

The proof of (4.9) for $i<j$ (positions above the main diagonal) follows now from Lemma 1. ■

5.3 Derivation of relations

The set of defining relations in Z_n divides into several different types, see Section 4.3. We prove the necessary amount of relations of each type and get the rest by applying the transformations from the braid group as well as the anti-involution ϵ, see (3.7).

We never use the automorphism ω, defined in (3.8), in the derivation of relations. However, the involution ω is compatible with our set of relations in the sense explained in Section 5.4.

In the following we denote by the symbol $\equiv$ the equalities of elements from $\bar{\mathrm{A}}$ modulo the sum $(\bar{\mathrm{I}}_- + \bar{\mathrm{I}}_+)$ of two ideals $\bar{\mathrm{I}}_-$ and $\bar{\mathrm{I}}_+$ defined in the beginning of Section 3. Moreover, for any two elements X and Y of the algebra $\bar{\mathrm{A}}$ we may regard the expressions $X\diamond Y$ and $X\diamond_\lambda Y$ as the sums of elements from $\bar{\mathrm{A}}$ defined in (5.2) and (5.3). The sum $X\diamond_\lambda Y$ is finite. By the construction, all but a finite number of terms in the product $X\diamond Y$ belong to $(\bar{\mathrm{I}}_- + \bar{\mathrm{I}}_+)$. Unlike to the system of notation adopted in Section 3, our proof of each relation in Z_n will use equalities in $\bar{\mathrm{A}}$ taken modulo $(\bar{\mathrm{I}}_- + \bar{\mathrm{I}}_+)$.

We also use the notation H_i for the element $E_{ii} \in \mathrm{A}$ and $H_{ij} = H_i - H_j = E_{ii} - E_{jj}$.

1. We first prove in Z_n the relation

$$z_{12} \diamond z_{13} = z_{13} \diamond z_{12} \frac{\mathring{h}_{23}}{\mathring{h}_{23}+1}. \tag{5.9}$$

Elements z_{12} and z_{13} are images in Z_n of E_{12} and E_{13}. Consider the product $E_{12} \diamond_\lambda E_{13}$. Since the adjoint action of $\mathfrak{gl}_n$ preserves the space $\mathfrak{p}$, see Section 2, this product is the sum of such monomials $E_{ij}E_{kl}$, with coefficients in $\overline{\mathrm{U}}(\mathfrak{h})$, that (i): the weight $\varepsilon_k - \varepsilon_l$ of E_{kl} is equal to the weight $\varepsilon_1 - \varepsilon_3$ of E_{13} plus $\lambda \in \mathrm{Q}_+$, while (ii): the weight $\varepsilon_i - \varepsilon_j$ of E_{ij} is equal to the weight $\varepsilon_1 - \varepsilon_2$ of E_{12} minus λ. Assume that $E_{12} \diamond_\lambda E_{13} \neq 0$. By (i), $\lambda = -\varepsilon_1 + \varepsilon_3 + \varepsilon_k - \varepsilon_l$ and it can be positive only if $k = 1$ and $l \geq 3$. So, the condition (i) implies that either $\lambda = 0$ or $\lambda = \varepsilon_3 - \varepsilon_l$ with $l > 3$. The possibility $\lambda = \varepsilon_3 - \varepsilon_l$, $l > 3$, is excluded by the condition (ii). Therefore, $\lambda = 0$ and

$$E_{12} \diamond E_{13} \equiv E_{12}E_{13}. \tag{5.10}$$

Similarly, for $\lambda \in \mathrm{Q}_+$, which can non-trivially contribute to the product $E_{13} \diamond E_{12}$, the analogue of the condition (i) on the weight λ gives the restriction $\lambda = 0$ or $\lambda = \varepsilon_2 - \varepsilon_k$, $k > 2$; the analogue of the condition (ii) further restricts λ: $\lambda = 0$ or $\lambda = \varepsilon_2 - \varepsilon_3$, so we have

$$E_{13} \diamond_{\varepsilon_2-\varepsilon_3} E_{12} \equiv -E_{13}e_{32}e_{23}\frac{1}{\mathring{h}_{23}+1}E_{12} \equiv -E_{13}e_{32}e_{23}E_{12}\frac{1}{\mathring{h}_{23}} \equiv E_{12}E_{13}\frac{1}{\mathring{h}_{23}},$$

since $\mathrm{J}_{\varepsilon_2-\varepsilon_3} = -e_{32} \otimes e_{23}(\mathring{h}_{23}+1)^{-1}$ as it follows from the ABRR equation, see (5.5), or from the precise explicit expression for the projector P, see [3]. Thus, since E_{12} and E_{13} commute in the universal enveloping algebra

$$E_{13} \diamond E_{12} \equiv E_{13}E_{12} + E_{13} \diamond_{\varepsilon_2-\varepsilon_3} E_{12} = E_{12}E_{13}\left(1 + \frac{1}{\mathring{h}_{23}}\right), \tag{5.11}$$

Comparing (5.10) and (5.11) we find (5.9).

Applying to (5.9) the anti-involution ϵ, see (3.7), we get the relation

$$z_{21} \diamond z_{31} = z_{31} \diamond z_{21} \frac{\mathring{h}_{23}+1}{\mathring{h}_{23}}. \tag{5.12}$$

The rest of the relations (4.12) are obtained from (5.9) and (5.12) by applying different transformations $\check{\mathrm{q}}_w$ from the Weyl group.

2. Now we prove in Z_n the relation

$$z_{13} \diamond z_{24} - z_{24} \diamond z_{13} = \left(\frac{1}{\mathring{h}_{12}} - \frac{1}{\mathring{h}_{34}}\right) z_{23} \diamond z_{14}. \tag{5.13}$$

We begin by the proof of this relation in Z_4. We proceed in the same manner as for the derivation of the relation (5.9),

$$\begin{aligned}
E_{13} \diamond E_{24} &\equiv E_{13}E_{24} + E_{13} \diamond_{\varepsilon_1-\varepsilon_2} E_{24} \\
&\equiv E_{13}E_{24} - E_{13}e_{21}e_{12}\frac{1}{\mathring{h}_{12}+1}E_{24} \equiv E_{13}E_{24} + E_{23}E_{14}\frac{1}{\mathring{h}_{12}}, \\
E_{24} \diamond E_{13} &\equiv E_{24}E_{13} + E_{24} \diamond_{\varepsilon_3-\varepsilon_4} E_{13} \\
&\equiv E_{24}E_{13} - E_{24}e_{43}e_{34}\frac{1}{\mathring{h}_{34}+1}E_{13} \equiv E_{13}E_{24} + E_{23}E_{14}\frac{1}{\mathring{h}_{34}},
\end{aligned}$$

$$E_{23}\diamond E_{14}\equiv E_{23}E_{14}.$$

Combining the three latter equalities we obtain (5.13) in Z_4.

The difference of the left and right hand sides of (5.13) in Z_n is a linear combination of monomials in z_{ij} of the total weight $\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4$. The weight is non-trivial, so the monomials can be only quadratic. Due to the stabilization phenomenon, each monomial should contain z_{ij} with $i>4$ or $j>4$, but, by the weight arguments, there is no such non-zero possibility, which completes the proof of the relation (5.13) in Z_n.

The rest of relations (4.13) is then obtained by applications of the transformations from the braid group.

3a. We continue and derive in Z_4 the relation (we remind the notation $t_{ij}:=z_{ii}-z_{jj}$, see Section 3, and $H_{ij}=E_{ii}-E_{jj}$):

$$z_{23}\diamond z_{12}-z_{12}\diamond z_{23}=t_{12}\diamond z_{13}\frac{1}{\mathring{h}_{12}}+t_{23}\diamond z_{13}\frac{1}{\mathring{h}_{23}}-z_{43}\diamond z_{14}\frac{\mathring{h}_{34}+1}{\mathring{h}_{34}\mathring{h}_{24}}. \tag{5.14}$$

Using (5.5)–(5.7), we calculate, to obtain the result for Z_4:

$$\begin{aligned} E_{12}\diamond E_{23} &\equiv E_{12}E_{23}+E_{12}\diamond_{\varepsilon_1-\varepsilon_2}E_{23}+E_{12}\diamond_{\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4}E_{23}\\ &\equiv E_{12}E_{23}-H_{12}E_{13}\frac{1}{\mathring{h}_{12}}, \end{aligned} \tag{5.15}$$

$$\begin{aligned} E_{23}\diamond E_{12} &\equiv E_{23}E_{12}+E_{23}\diamond_{\varepsilon_2-\varepsilon_3}E_{12}+E_{23}\diamond_{\varepsilon_2-\varepsilon_4}E_{12}\\ &\equiv E_{23}E_{12}+H_{23}E_{13}\frac{1}{\mathring{h}_{23}}-E_{43}E_{14}\frac{(\mathring{h}_{23}-1)}{\mathring{h}_{23}\mathring{h}_{24}}, \end{aligned} \tag{5.16}$$

$$H_{12}\diamond E_{13}\equiv H_{12}E_{13}+H_{12}\diamond_{\varepsilon_3-\varepsilon_4}E_{13}\equiv H_{12}E_{13}, \tag{5.17}$$

$$H_{23}\diamond E_{13}\equiv H_{23}E_{13}+H_{23}\diamond_{\varepsilon_3-\varepsilon_4}E_{13}\equiv H_{23}E_{13}+E_{43}E_{14}\frac{1}{\mathring{h}_{34}}, \tag{5.18}$$

$$E_{43}\diamond E_{14}\equiv E_{43}E_{14}. \tag{5.19}$$

Combining the above equalities and taking into account that $[E_{12},E_{23}]=e_{13}\equiv 0$, we get (5.14) in Z_4. We could apply here the stability arguments (as we shall do in the sequel) but we give some more details at this point to give a flavor of how such derivations of relations work. For the same, as (5.15)–(5.19), calculations for Z_n, the analogues of the conditions (i) and (ii), see paragraph 1 of this subsection, restrict λ to be of the form $\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_k$, $k\geq 3$ for (5.15); $\varepsilon_2-\varepsilon_k$, $k\geq 2$ for (5.16); $\varepsilon_3-\varepsilon_k$, $k\geq 3$ for (5.17) and (5.18); $\varepsilon_4-\varepsilon_k$, $k\geq 4$ for (5.19). It follows, for, say, $n=5$, that the right hand sides of (5.15)–(5.19) might be modified only by an addition of the term proportional to $E_{53}E_{15}$; and this will be compensated by an addition of the term, proportional to $z_{53}\diamond z_{15}$ to the right hand side of (5.14), since $E_{53}\diamond E_{15}\equiv E_{53}E_{15}$; the proportionality coefficient is uniquely defined. This pattern clearly continues and we conclude that there is a relation in Z_n of the form

$$z_{23}\diamond z_{12}-z_{12}\diamond z_{23}=t_{12}\diamond z_{13}\frac{1}{\mathring{h}_{12}}+t_{23}\diamond z_{13}\frac{1}{\mathring{h}_{23}}-\sum_{k>3}z_{k3}\diamond z_{1k}X_k, \tag{5.20}$$

with certain, uniquely defined, coefficients $X_k\in\overline{\mathrm{U}}(\mathfrak{h})$, $k=4,\dots,n$, and already known $X_4=(\mathring{h}_{34}+1)\mathring{h}_{34}^{-1}\mathring{h}_{24}^{-1}$. To find $X_5,\dots,X_n$, we apply to (5.20) the automorphisms $\check{\mathrm{q}}_k$, $k=4,\dots,n-1$, which leave invariant the left hand side and the first two terms in the right hand side of (5.20). The uniqueness of the relation of the form (5.20), together with the equality $\check{\mathrm{q}}_k(z_{k3}\diamond z_{1k})=z_{k+1,3}\diamond z_{1,k+1}(\mathring{h}_{k,k+1}+1)\mathring{h}_{k,k+1}^{-1}$, imply the recurrence relation $X_{k+1}=\check{\mathrm{q}}_k(X_k)\cdot(\mathring{h}_{k,k+1}+1)\mathring{h}_{k,k+1}^{-1}$

and we find

$$X_k = \frac{1}{\mathring{h}_{2k}} \prod_{j=3}^{k-1} \frac{\mathring{h}_{jk}+1}{\mathring{h}_{jk}}.$$

After the renormalization (4.11) and the change of variables (4.2), the derived relation becomes one of the relations in the first line of (4.14).

Applying the transformations from the braid group, we obtain the rest of the relations from the list (4.14).

3b. We have the following equalities in Z_3:

$$z_{12} \diamond t_1 = t_1 \diamond z_{12} \frac{\mathring{h}_{12}+2}{\mathring{h}_{12}+1} - t_2 \diamond z_{12} \frac{1}{\mathring{h}_{12}+1} - z_{32} \diamond z_{13} \frac{\mathring{h}_{23}+1}{\mathring{h}_{23}(\mathring{h}_{13}+1)}, \tag{5.21}$$

$$z_{12} \diamond t_2 = -t_1 z_{12} \frac{1}{\mathring{h}_{12}+1} + t_2 \diamond z_{12} \frac{\mathring{h}_{12}+2}{\mathring{h}_{12}+1} + z_{32} \diamond z_{13} \frac{\mathring{h}_{13}+2}{\mathring{h}_{23}(\mathring{h}_{13}+1)}, \tag{5.22}$$

and the equality in Z_4:

$$z_{12} \diamond t_4 = t_4 \diamond z_{12} - z_{42} \diamond z_{14} \frac{\mathring{h}_{12}+1}{(\mathring{h}_{14}+1)\mathring{h}_{24}}. \tag{5.23}$$

Equalities (5.21) and (5.22) are the results of the following calculations for Z_3, using (5.5)–(5.7), and of the commutativity $[H_1, E_{12}] = e_{12} \equiv 0$, $[H_2, E_{12}] = -e_{12} \equiv 0$:

$$E_{12} \diamond H_1 \equiv E_{12}H_1 + H_{12}E_{12} \frac{1}{\mathring{h}_{12}+1} - E_{32}E_{13} \frac{\mathring{h}_{12}}{(\mathring{h}_{12}+1)(\mathring{h}_{13}+1)},$$

$$E_{12} \diamond H_2 \equiv E_{12}H_2 - H_{12}E_{12} \frac{1}{\mathring{h}_{12}+1} - E_{32}E_{13} \frac{1}{(\mathring{h}_{12}+1)(\mathring{h}_{13}+1)},$$

$$H_1 \diamond E_{12} \equiv H_1 E_{12}, \qquad H_2 \diamond E_{12} \equiv H_2 E_{12} - E_{32}E_{13} \frac{1}{\mathring{h}_{23}}, \qquad E_{32} \diamond E_{13} \equiv E_{32}E_{13}.$$

The derivation of (5.23) can be done with the help of the following calculations for Z_4:

$$E_{12} \diamond H_4 \equiv E_{12}H_4 + E_{42}E_{14} \frac{1}{\mathring{h}_{14}+1}, \tag{5.24}$$

$$H_4 \diamond E_{12} \equiv H_4 E_{12} + E_{42}E_{14} \frac{1}{\mathring{h}_{24}}, \qquad E_{42} \diamond E_{14} \equiv E_{42}E_{14}.$$

We shall make a comment about the line (5.24) only. Here one might expect, by the analogues of the conditions (i) and (ii), see paragraph 1 of this subsection, non-trivial contributions to $E_{12} \diamond H_4$ from the weights 0, $\varepsilon_1 - \varepsilon_2$, $\varepsilon_1 - \varepsilon_3$ and $\varepsilon_1 - \varepsilon_4$. So we need, in addition to (5.5)–(5.7), some information about $\mathrm{J}_{\varepsilon_1-\varepsilon_4}$. It follows from the ABRR equation that $\mathrm{J}_{\varepsilon_1-\varepsilon_4}(\mathring{h}_{14}+1) = -T_{\varepsilon_1-\varepsilon_2}(\mathrm{J}_{\varepsilon_2-\varepsilon_4}) - T_{\varepsilon_1-\varepsilon_3}(\mathrm{J}_{\varepsilon_3-\varepsilon_4}) - T_{\varepsilon_1-\varepsilon_4}(\mathrm{J}_0) - T_{\varepsilon_2-\varepsilon_3}(\mathrm{J}_{\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4}) - T_{\varepsilon_2-\varepsilon_4}(\mathrm{J}_{\varepsilon_1-\varepsilon_2}) - T_{\varepsilon_3-\varepsilon_4}(\mathrm{J}_{\varepsilon_1-\varepsilon_3})$. Since e_{13} and e_{12} commute with H_4, the parts $T_{\varepsilon_2-\varepsilon_4}(\mathrm{J}_{\varepsilon_1-\varepsilon_2})$ and $T_{\varepsilon_3-\varepsilon_4}(\mathrm{J}_{\varepsilon_1-\varepsilon_3})$ do not contribute; e_{42} and e_{43} commute with E_{12}, so the parts $T_{\varepsilon_1-\varepsilon_2}(\mathrm{J}_{\varepsilon_2-\varepsilon_4})$ and $T_{\varepsilon_1-\varepsilon_3}(\mathrm{J}_{\varepsilon_3-\varepsilon_4})$ do not contribute either; $\mathrm{J}_{\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4} = \mathrm{J}_{\varepsilon_3-\varepsilon_4}\mathrm{J}_{\varepsilon_1-\varepsilon_2}$ does not contribute again since e_{12} commute with H_4. Thus the only contribution is from $T_{\varepsilon_1-\varepsilon_4}(\mathrm{J}_0)$ and we quickly arrive at (5.24).

Applying the automorphism $\check{q}_3$ of the algebra Z_4 to the relation (5.23), see (4.1) and (4.4), we find

$$z_{12} \diamond \big(t_3\mathring{h}_{34} - t_4\big) = \big(t_3\mathring{h}_{34} - t_4\big) \diamond z_{12} - z_{32} \diamond z_{13} \frac{\mathring{h}_{34}(\mathring{h}_{12}+1)}{(\mathring{h}_{13}+1)\mathring{h}_{23}}.$$

We then add (5.23) to this relation and obtain the following relation in Z_4:

$$z_{12}\diamond t_3=t_3\diamond z_{12}-z_{32}\diamond z_{13}\frac{(\mathring{h}_{12}+1)}{(\mathring{h}_{13}+1)\mathring{h}_{23}}-z_{42}\diamond z_{14}\frac{\mathring{h}_{12}+1}{(\mathring{h}_{14}+1)\mathring{h}_{24}\mathring{h}_{34}}. \tag{5.25}$$

The stabilization arguments for (5.21), (5.22) and (5.25) imply the existence of the following relations in Z_n:

$$z_{12}\diamond t_1=t_1\diamond z_{12}\frac{\mathring{h}_{12}+2}{\mathring{h}_{12}+1}-t_2\diamond z_{12}\frac{1}{\mathring{h}_{12}+1}+\sum_{k>2}z_{k2}\diamond z_{1k}X_k^{(1)}, \tag{5.26}$$

$$z_{12}\diamond t_2=-t_1\diamond z_{12}\frac{1}{\mathring{h}_{12}+1}+t_2\diamond z_{12}\frac{\mathring{h}_{12}+2}{\mathring{h}_{12}+1}+\sum_{k>2}z_{k2}\diamond z_{1k}X_k^{(2)}, \tag{5.27}$$

$$z_{12}\diamond t_3=t_3\diamond z_{12}-z_{32}\diamond z_{13}\frac{(\mathring{h}_{12}+1)}{(\mathring{h}_{13}+1)\mathring{h}_{23}}+\sum_{k>3}z_{k2}\diamond z_{1k}X_k^{(3)}, \tag{5.28}$$

where all $X_k^{(i)}$ belong to $\overline{\mathrm{U}}(\mathfrak{h})$ and the initial $X_k^{(i)}$ are known:

$$X_3^{(1)}=-\frac{\mathring{h}_{23}+1}{\mathring{h}_{23}(\mathring{h}_{13}+1)},\qquad X_3^{(2)}=\frac{\mathring{h}_{13}+2}{\mathring{h}_{23}(\mathring{h}_{13}+1)},\qquad X_4^{(3)}=-\frac{\mathring{h}_{12}+1}{(\mathring{h}_{14}+1)\mathring{h}_{24}\mathring{h}_{34}}.$$

By the braid group transformation laws, $X_{k+1}^{(i)}=\check{\mathrm{q}}_k\big(X_k^{(i)}\big)\cdot(\mathring{h}_{k,k+1}+1)\mathring{h}_{k,k+1}^{-1}$ with $k>2$ for $i=1,2$ and $k>3$ for $i=3$, so that

$$X_k^{(1)}=-\frac{1}{\mathring{h}_{1k}+1}\prod_{j=2}^{k-1}\frac{\mathring{h}_{jk}+1}{\mathring{h}_{jk}},\qquad X_k^{(2)}=\frac{\mathring{h}_{1k}+2}{(\mathring{h}_{1k}+1)\mathring{h}_{2k}}\prod_{j=3}^{k-1}\frac{\mathring{h}_{jk}+1}{\mathring{h}_{jk}},$$

$$X_k^{(3)}=-\frac{\mathring{h}_{12}+1}{(\mathring{h}_{1k}+1)\mathring{h}_{2k}\mathring{h}_{3k}}\prod_{j=4}^{k-1}\frac{\mathring{h}_{jk}+1}{\mathring{h}_{jk}}.$$

After the renormalization (4.11) and the change of variables (4.2), the relations (5.26)–(5.28) turn into the relations (4.16) for $i=1$, $j=2$ and $k=3$.

Applying the transformations from the braid group, we obtain the rest of the relations from the list (4.16).

4a. We now prove the relations (4.17) using the arguments similar to [10, Subsection 6.1.2]. Consider the products $H_k\diamond_\lambda H_l$ and $H_l\diamond_\lambda H_k$ with $\lambda\neq 0$. These products are linear combinations, over $\overline{\mathrm{U}}(\mathfrak{h})$, of monomials

$$a_{kl;\vec{\gamma}}:=H_ke_{-\gamma_1}\cdots e_{-\gamma_m}e_{\gamma_m}\cdots e_{\gamma_1}H_l\qquad\text{and}\qquad a_{lk;\vec{\gamma}}:=H_le_{-\gamma_1}\cdots e_{-\gamma_m}e_{\gamma_m}\cdots e_{\gamma_1}H_k,$$

respectively; here $m\geq 0$ and $\vec{\gamma}:=\{\gamma_1,\dots,\gamma_m\}$. By construction, the coefficient, from $\overline{\mathrm{U}}(\mathfrak{h})$, of the monomial $a_{kl;\vec{\gamma}}$ in $H_k\diamond_\lambda H_l$ equals the coefficient of $a_{lk;\vec{\gamma}}$ in $H_l\diamond_\lambda H_k$. The expressions $a_{kl;\vec{\gamma}}$ and $a_{lk;\vec{\gamma}}$ are both equal in Z_n to

$$(\gamma_1,\varepsilon_k)(\gamma_1,\varepsilon_l)E_{-\gamma_1}e_{-\gamma_2}\cdots e_{-\gamma_m}e_{\gamma_m}\cdots e_{\gamma_2}E_{\gamma_1}.$$

Thus $H_k\diamond_\lambda H_l\equiv H_l\diamond_\lambda H_k$ for any $\lambda\neq 0$. In the zero weight part $\diamond_0$ of the product $\diamond$ we have the equality $H_kH_l=H_lH_k$ as well. Therefore, $H_k\diamond H_l\equiv H_l\diamond H_k$.

4b. The last group (4.18) of relations is left. Like above, we first explicitly derive the following relation in Z_3:

$$z_{12}\diamond z_{21}=h_{12}-t_{12}\diamond t_{12}\frac{1}{\mathring{h}_{12}-1}+z_{21}\diamond z_{12}\frac{(\mathring{h}_{12}-1)(\mathring{h}_{12}+2)}{\mathring{h}_{12}(\mathring{h}_{12}+1)}$$

$$+ z_{31} \diamond z_{13} \frac{(\mathring{h}_{12}-1)(\mathring{h}_{13}+2)}{\mathring{h}_{12}\mathring{h}_{23}(\mathring{h}_{13}+1)} - z_{32} \diamond z_{23} \frac{\mathring{h}_{23}+2}{(\mathring{h}_{23}+1)\mathring{h}_{13}} \qquad (5.29)$$

(the first term in the right hand side is h_{12}, without hat). The relation (5.29) is a corollary of the following calculations for Z_3, together with the commutation relation $[E_{12}, E_{21}] = h_{12}$,

$$E_{12} \diamond E_{21} \equiv E_{12}E_{21} - H_{12}^2 \frac{1}{(\mathring{h}_{12}-1)} + E_{21}E_{12} \frac{2}{(\mathring{h}_{12}-1)\mathring{h}_{12}} - E_{32}E_{23} \frac{\mathring{h}_{12}-2}{(\mathring{h}_{12}-1)\mathring{h}_{13}} + E_{31}E_{13} \frac{\mathring{h}_{12}-2}{(\mathring{h}_{12}-1)\mathring{h}_{12}\mathring{h}_{13}}, \qquad (5.30)$$

$$H_{12} \diamond H_{12} \equiv H_{12}^2 - E_{21}E_{12} \frac{4}{\mathring{h}_{12}+1} - E_{32}E_{23} \frac{1}{\mathring{h}_{23}+1} + E_{31}E_{13} \left(-1 + \frac{1}{\mathring{h}_{23}+1} + \frac{4}{\mathring{h}_{12}+1} \right) \frac{1}{\mathring{h}_{13}+1}, \qquad (5.31)$$

$$E_{21} \diamond E_{12} \equiv E_{21}E_{12} - E_{31}E_{13} \frac{1}{\mathring{h}_{23}}, \qquad (5.32)$$

$$E_{32} \diamond E_{23} \equiv E_{32}E_{23} - E_{31}E_{13} \frac{1}{\mathring{h}_{12}}, \qquad (5.33)$$

$$E_{31} \diamond E_{13} \equiv E_{31}E_{13}. \qquad (5.34)$$

Here only the calculation of $E_{12} \diamond E_{21}$ deserves a little explanation; by the analogues of the conditions (i) and (ii), see paragraph 1 of this subsection, non-trivial contributions to $E_{12} \diamond E_{21}$ from the weights 0, $\varepsilon_1 - \varepsilon_2$, $2(\varepsilon_1 - \varepsilon_2)$, $\varepsilon_1 - \varepsilon_3$ and $2\varepsilon_1 - \varepsilon_2 - \varepsilon_3$ are possible. By the ABRR equation, $\mathrm{J}_{2(\varepsilon_1-\varepsilon_2)}(2\mathring{h}_{12}+4) = -T_{\varepsilon_1-\varepsilon_2}(\mathrm{J}_{\varepsilon_1-\varepsilon_2})$ and $\mathrm{J}_{2\varepsilon_1-\varepsilon_2-\varepsilon_3}(2\mathring{h}_1 - \mathring{h}_2 - \mathring{h}_3 + 3) = -T_{\varepsilon_1-\varepsilon_2}(\mathrm{J}_{\varepsilon_1-\varepsilon_3}) - T_{\varepsilon_1-\varepsilon_3}(\mathrm{J}_{\varepsilon_1-\varepsilon_2}) - T_{\varepsilon_2-\varepsilon_3}(\mathrm{J}_{2(\varepsilon_1-\varepsilon_2)})$. We leave further details to the reader.

By the stabilization law in Z_4 we have a relation

$$z_{12} \diamond z_{21} = h_{12} - t_{12} \diamond t_{12} \frac{1}{\mathring{h}_{12}-1} + \sum_{1 \le i < j \le n} z_{ji} \diamond z_{ij} X_{ij}, \qquad X_{ij} \in \overline{\mathrm{U}}(\mathfrak{h}) \qquad (5.35)$$

with $n = 4$, which differs from (5.29) by a presence of terms

$$z_{43} \diamond z_{34}, \qquad z_{42} \diamond z_{24}, \qquad z_{41} \diamond z_{14},$$

with coefficients in $\overline{\mathrm{U}}(\mathfrak{h})$. Consider in Z_4 the products $z_{12} \diamond z_{21}$, $t_{12} \diamond t_{12}$ and $z_{ji} \diamond z_{ij}$, $1 \le i < j \le 4$. The weights $(\varepsilon_3 - \varepsilon_4) - (\varepsilon_i - \varepsilon_j)$ do not belong to the cone Q_+ if $1 \le i < j < 4$. Thus in the decomposition

$$E_{ji} \diamond E_{ij} \equiv \sum_{k<l} E_{lk}E_{kl}a_{kl}, \qquad a_{kl} \in \overline{\mathrm{U}}(\mathfrak{h}), \quad 1 \le i < j < 4,$$

the term with $E_{43}E_{34}$ has a zero coefficient, $a_{34} = 0$. The same statement holds for the products $E_{41} \diamond E_{14}$ and $E_{42} \diamond E_{24}$ since the weights $(\varepsilon_3 - \varepsilon_4) - (\varepsilon_i - \varepsilon_4)$ do not belong to Q_+ for $i = 1, 2$. Consider the product $E_{12} \diamond E_{21}$. Here the term with $E_{43}E_{34}$ is equal to $E_{12} \diamond_{\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4} E_{21}$. By (5.7), $\mathrm{J}_{\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4} = e_{43}e_{21} \otimes e_{12}e_{34}(\mathring{h}_{12}+1)^{-1}(\mathring{h}_{34}+1)^{-1}$ and

$$E_{12} \diamond_{\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4} E_{21} = E_{12}e_{43}e_{21}e_{12}e_{34}E_{21} \frac{1}{(\mathring{h}_{12}-1)(\mathring{h}_{34}+1)} \equiv 0,$$

since $[e_{34}, E_{21}] = 0$ (and $[E_{12}, e_{43}] = 0$). In the similar manner, the term with $E_{43}E_{34}$ in $H_{12} \diamond H_{12}$ equals $H_{12} \diamond_{\varepsilon_3-\varepsilon_4} H_{12}$ and vanishes since $[e_{34}, H_{12}] = 0$.

On the other hand, the product $E_{43} \diamond E_{34}$ definitely contains $E_{43}E_{34} = E_{43} \diamond_0 E_{34}$. We thus conclude that the term $z_{43} \diamond z_{34}$ is absent in (5.35), that is $X_{34} = 0$.

For $n > 4$, again by the stabilization law, we have a unique relation of the form (5.35). By uniqueness, it is invariant with respect to the transformations $\check{q}_3, \check{q}_4, \dots, \check{q}_{n-1}$ which do not change the product $z_{12} \diamond z_{21}$. Since $X_{34} = 0$, we find, applying $\check{q}_4, \check{q}_5, \dots, \check{q}_{n-1}$, that $X_{3j} = 0$, $j > 3$, wherefrom we further conclude, applying $\check{q}_3, \check{q}_4, \dots, \check{q}_{j-2}$, that $X_{ij} = 0$, $2 < i < j$. We get finally the following relation in Z_n:

$$z_{12} \diamond z_{21} = h_{12} - t_{12} \diamond t_{12} \frac{1}{\mathring{h}_{12} - 1} + \sum_{k=2,\dots,n} z_{k1} \diamond z_{1k} X_{1k} + \sum_{k=3,\dots,n} z_{k2} \diamond z_{2k} X_{2k} \tag{5.36}$$

with known

$$X_{12} = \frac{(\mathring{h}_{12} - 1)(\mathring{h}_{12} + 2)}{\mathring{h}_{12}(\mathring{h}_{12} + 1)}, \qquad X_{13} = \frac{(\mathring{h}_{12} - 1)(\mathring{h}_{13} + 2)}{\mathring{h}_{12}\mathring{h}_{23}(\mathring{h}_{13} + 1)}, \qquad X_{23} = -\frac{\mathring{h}_{23} + 2}{(\mathring{h}_{23} + 1)\mathring{h}_{13}}.$$

Applying to (5.36) the transformations $\check{q}_3, \check{q}_4, \dots, \check{q}_{n-1}$ we find by uniqueness

$$X_{i,k+1} = \frac{\mathring{h}_{k,k+1} + 1}{\mathring{h}_{k,k+1}} \cdot \check{q}_k(X_{ik}), \qquad i = 1, 2; \quad k = 3, 4, \dots, n-1,$$

and thus

$$X_{1k} = \frac{(\mathring{h}_{12} - 1)(\mathring{h}_{1k} + 2)}{\mathring{h}_{12}\mathring{h}_{2k}(\mathring{h}_{1k} + 1)} \cdot \prod_{a=3}^{k-1} \frac{\mathring{h}_{ak} + 1}{\mathring{h}_{ak}}, \qquad X_{2k} = -\frac{\mathring{h}_{2k} + 2}{(\mathring{h}_{2k} + 1)\mathring{h}_{1k}} \cdot \prod_{a=3}^{k-1} \frac{\mathring{h}_{ak} + 1}{\mathring{h}_{ak}}$$

for $k > 2$.

After the renormalization (4.11) and the change of variables (4.2), the relations (5.36) with the obtained X_{1k} and X_{2k} turns into the relation (4.18) for $i = 1$ and $j = 2$.

Applying the transformations from the braid group, we obtain the rest of the relations from the list (4.18).

5.4 Proof of Theorem 3

For the proof of Theorem 3 we just apply the results of [7], which state that the system $\mathfrak{R}$ is the system of defining relations once it is satisfied in the algebra Z_n.

Remark 1. An attentive look shows that the system $\mathfrak{R}$ is closed under the anti-involution ϵ; that is, ϵ transforms any relation from $\mathfrak{R}$ into a linear over $\overline{\mathrm{U}}(\mathfrak{h})$ combination of relations from $\mathfrak{R}$. Moreover, $\mathfrak{R}$ and $\epsilon(\mathfrak{R})$ are equivalent over $\overline{\mathrm{U}}(\mathfrak{h})$. Indeed, all relations in Section (5.3) were derived in three steps: first we derive a relation in Z_n with $n \le 4$; next by the stabilization principle we extend the derived relation to Z_n with arbitrary n; and then we find the whole list of relations of a given (sub)type by applying the braid group transformations (products of the generators $\check{q}_i$). Due to (3.6) one could use $\check{q}_i^{-1}$ instead of $\check{q}_i$ equivalently over $\overline{\mathrm{U}}(\mathfrak{h})$. A straightforward calculation establishes the equivalence of the extended to arbitrary n lists $\mathfrak{R}$ and $\epsilon(\mathfrak{R})$ over $\overline{\mathrm{U}}(\mathfrak{h})$ for Z_n with $n \le 4$ (this verification is lengthy for some relations). Then with the help of (4.6) we finish the check of the equivalence of $\mathfrak{R}$ and $\epsilon(\mathfrak{R})$ over $\overline{\mathrm{U}}(\mathfrak{h})$ for Z_n with arbitrary n.

Similar arguments establish the equivalence of $\mathfrak{R}$ and $\omega(\mathfrak{R})$ over $\overline{\mathrm{U}}(\mathfrak{h})$; here ω is the involution defined in (3.8). In [7] this equivalence was obtained differently, as a by-product of the equivalence, over $\overline{\mathrm{U}}(\mathfrak{h})$, of the system $\mathfrak{R}$ and the system (3.3) of ordering relations.

6 Examples: $\mathfrak{sl}_3$ and $\mathfrak{sl}_2$

In this section we write down the complete list of ordering relations for the diagonal reduction algebras $\mathrm{DR}(\mathfrak{sl}_3)$ and $\mathrm{DR}(\mathfrak{sl}_2)$. For completeness we include the formulas for the action of the braid group generators and the expressions for the central elements.

We first give the list of relations for $\mathfrak{sl}_3$. It is straightforward to give the list for $\mathfrak{sl}_2$ directly; we comment however on how the list of relations and the expressions for the central elements for $\mathfrak{sl}(2)$ can be obtained by the cut procedure.

The list of relations for $\mathfrak{gl}_3$ follows immediately from the list for $\mathfrak{sl}_3$.

1. Relations for DR($\mathfrak{sl}_3$). We write the ordering relations for the natural set of generators z_{ij}, without redefinitions. We use here the following notation for $\mathfrak{sl}_3$:

$$z_\alpha := z_{12}, \quad z_\beta := z_{23}, \quad z_{\alpha+\beta} := z_{13}, \quad z_{-\alpha} := z_{21}, \quad z_{-\beta} := z_{23}, \quad z_{-\alpha-\beta} := z_{31},$$
$$t_\alpha := t_{12}, \quad t_\beta := t_{23}, \quad h_\alpha := h_{12}, \quad h_\beta := h_{23}.$$

The relations are given for the following order $\dot{\succ}$ (this order was used in the proof in [7] of the completeness of the set of relations):

$$z_{\alpha+\beta} \dot{\succ} z_\alpha \dot{\succ} z_\beta \dot{\succ} t_\beta \dot{\succ} t_\alpha \dot{\succ} z_{-\beta} \dot{\succ} z_{-\alpha} \dot{\succ} z_{-\alpha-\beta}. \tag{6.1}$$

Due to the established in Theorem 3 and remarks in Section 4.4 bijection between the set $\mathfrak{R}$ and the set $\mathfrak{R}^{\prec}$ of defining relations, one can divide the ordering relations into the types, in the same way as we divided the defining relations from the list $\mathfrak{R}$.

The relations of type 1 are immediately rewritten as ordering relations:

$$z_{\alpha+\beta} \diamond z_\alpha = z_\alpha \diamond z_{\alpha+\beta} \frac{h_\beta+2}{h_\beta+1}, \tag{6.2}$$
$$z_{\alpha+\beta} \diamond z_\beta = z_\beta \diamond z_{\alpha+\beta} \frac{h_\alpha+2}{h_\alpha+1}, \tag{6.3}$$
$$z_\alpha \diamond z_{-\beta} = z_{-\beta} \diamond z_\alpha \frac{h_\alpha+h_\beta+3}{h_\alpha+h_\beta+2}, \tag{6.4}$$
$$z_\beta \diamond z_{-\alpha} = z_{-\alpha} \diamond z_\beta \frac{h_\alpha+h_\beta+3}{h_\alpha+h_\beta+2}, \tag{6.5}$$
$$z_{-\alpha} \diamond z_{-\alpha-\beta} = z_{-\alpha-\beta} \diamond z_{-\alpha} \frac{h_\beta+2}{h_\beta+1}, \tag{6.6}$$
$$z_{-\beta} \diamond z_{-\alpha-\beta} = z_{-\alpha-\beta} \diamond z_{-\beta} \frac{h_\alpha+2}{h_\alpha+1}. \tag{6.7}$$

The relations of type 2 are absent for $\mathfrak{sl}_3$.

The ordering relations corresponding to the relations of type 3 we collect according to their weights. For each weight there is one relation of subtype (3**a**) and two relations of subtype (3**b**).

Weight $\alpha+\beta$:

$$z_\alpha \diamond z_\beta = -t_\alpha \diamond z_{\alpha+\beta} \frac{1}{h_\alpha+1} - t_\beta \diamond z_{\alpha+\beta} \frac{1}{h_\beta+1} + z_\beta \diamond z_\alpha, \tag{6.8}$$
$$\begin{aligned} z_{\alpha+\beta} \diamond t_\alpha = {} & t_\alpha \diamond z_{\alpha+\beta} \frac{h_\alpha h_\beta + h_\beta^2 + 2h_\alpha + 6h_\beta + 9}{(h_\beta+2)(h_\alpha+h_\beta+3)} \\ & + t_\beta \diamond z_{\alpha+\beta} \frac{h_\beta^2 + h_\alpha + 6h_\beta + 9}{(h_\beta+1)(h_\beta+2)(h_\alpha+h_\beta+3)} - z_\beta \diamond z_\alpha \frac{h_\alpha + 2h_\beta + 6}{(h_\alpha+2)(h_\beta+2)}, \end{aligned} \tag{6.9}$$
$$z_{\alpha+\beta} \diamond t_\beta = t_\alpha \diamond z_{\alpha+\beta} \frac{h_\beta}{(h_\beta+2)(h_\alpha+h_\beta+3)} + t_\beta \diamond z_{\alpha+\beta} \frac{h_\beta\big(h_\alpha h_\beta + h_\beta^2 + 3h_\alpha + 7h_\beta + 11\big)}{(h_\beta+1)(h_\beta+2)(h_\alpha+h_\beta+3)}$$

$$+ z_\beta \diamond z_\alpha \frac{2h_\alpha + h_\beta + 6}{(h_\alpha+2)(h_\beta+2)}. \tag{6.10}$$

Weight α:

$$z_\alpha \diamond t_\alpha = t_\alpha \diamond z_\alpha \frac{h_\alpha+4}{h_\alpha+2} - z_{-\beta} \diamond z_{\alpha+\beta} \frac{h_\alpha + 2h_\beta + 6}{(h_\beta+1)(h_\alpha+h_\beta+3)}, \tag{6.11}$$

$$z_\alpha \diamond t_\beta = -t_\alpha \diamond z_\alpha \frac{1}{h_\alpha+2} + t_\beta \diamond z_\alpha + z_{-\beta} \diamond z_{\alpha+\beta} \frac{2h_\alpha + h_\beta + 6}{(h_\beta+1)(h_\alpha+h_\beta+3)}, \tag{6.12}$$

$$z_{\alpha+\beta} \diamond z_{-\beta} = -t_\alpha \diamond z_\alpha \frac{h_\beta - 1}{h_\beta(h_\alpha+h_\beta+2)} - t_\beta \diamond z_\alpha \frac{h_\alpha + 2h_\beta + 2}{h_\beta(h_\alpha+h_\beta+2)}$$
$$+ z_{-\beta} \diamond z_{\alpha+\beta} \frac{(h_\beta+2)(h_\beta-1)}{h_\beta(h_\beta+1)}. \tag{6.13}$$

Weight β:

$$z_\beta \diamond t_\alpha = t_\alpha \diamond z_\beta - t_\beta \diamond z_\beta \frac{1}{h_\beta+2} - z_{-\alpha} \diamond z_{\alpha+\beta} \frac{h_\alpha + 2h_\beta + 6}{(h_\alpha+1)(h_\alpha+h_\beta+3)}, \tag{6.14}$$

$$z_\beta \diamond t_\beta = t_\beta \diamond z_\beta \frac{h_\beta+4}{h_\beta+2} + z_{-\alpha} \diamond z_{\alpha+\beta} \frac{2h_\alpha + h_\beta + 6}{(h_\alpha+1)(h_\alpha+h_\beta+3)}, \tag{6.15}$$

$$z_{\alpha+\beta} \diamond z_{-\alpha} = t_\alpha \diamond z_\beta \frac{2h_\alpha + h_\beta + 2}{h_\alpha(h_\alpha+h_\beta+2)} + t_\beta \diamond z_\beta \frac{h_\alpha - 1}{h_\alpha(h_\alpha+h_\beta+2)}$$
$$+ z_{-\alpha} \diamond z_{\alpha+\beta} \frac{(h_\alpha+2)(h_\alpha-1)}{h_\alpha(h_\alpha+1)}. \tag{6.16}$$

Weight $-\beta$:

$$t_\alpha \diamond z_{-\beta} = z_{-\beta} \diamond t_\alpha - z_{-\beta} \diamond t_\beta \frac{1}{h_\beta} - z_{-\alpha-\beta} \diamond z_\alpha \frac{h_\alpha + 2h_\beta + 3}{(h_\alpha+2)(h_\alpha+h_\beta+2)}, \tag{6.17}$$

$$t_\beta \diamond z_{-\beta} = z_{-\beta} \diamond t_\beta \frac{h_\beta+2}{h_\beta} + z_{-\alpha-\beta} \diamond z_\alpha \frac{2h_\alpha + h_\beta + 6}{(h_\alpha+2)(h_\alpha+h_\beta+2)}, \tag{6.18}$$

$$z_\alpha \diamond z_{-\alpha-\beta} = z_{-\beta} \diamond t_\alpha \frac{2h_\alpha + h_\beta + 2}{(h_\alpha+1)(h_\alpha+h_\beta+1)} + z_{-\beta} \diamond t_\beta \frac{h_\alpha}{(h_\alpha+1)(h_\alpha+h_\beta+1)}$$
$$+ z_{-\alpha-\beta} \diamond z_\alpha \frac{h_\alpha(h_\alpha+3)}{(h_\alpha+1)(h_\alpha+2)}. \tag{6.19}$$

Weight $-\alpha$:

$$t_\alpha \diamond z_{-\alpha} = z_{-\alpha} \diamond t_\alpha \frac{h_\alpha+2}{h_\alpha} - z_{-\alpha-\beta} \diamond z_\beta \frac{h_\alpha + 2h_\beta + 6}{(h_\beta+2)(h_\alpha+h_\beta+2)}, \tag{6.20}$$

$$t_\beta \diamond z_{-\alpha} = -z_{-\alpha} \diamond t_\alpha \frac{1}{h_\alpha} + z_{-\alpha} \diamond t_\beta + z_{-\alpha-\beta} \diamond z_\beta \frac{2h_\alpha + h_\beta + 3}{(h_\beta+2)(h_\alpha+h_\beta+2)}, \tag{6.21}$$

$$z_\beta \diamond z_{-\alpha-\beta} = -z_{-\alpha} \diamond t_\alpha \frac{h_\beta}{(h_\beta+1)(h_\alpha+h_\beta+1)} - z_{-\alpha} \diamond t_\beta \frac{h_\alpha + 2h_\beta + 2}{(h_\beta+1)(h_\alpha+h_\beta+1)}$$
$$+ z_{-\alpha-\beta} \diamond z_\beta \frac{h_\beta(h_\beta+3)}{(h_\beta+1)(h_\beta+2)}. \tag{6.22}$$

Weight $-\alpha-\beta$:

$$z_{-\beta} \diamond z_{-\alpha} = -z_{-\alpha-\beta} \diamond t_\alpha \frac{1}{h_\alpha} - z_{-\alpha-\beta} \diamond t_\beta \frac{1}{h_\beta} + z_{-\alpha} \diamond z_{-\beta}, \tag{6.23}$$

$$t_\alpha \diamond z_{-\alpha-\beta} = z_{-\alpha-\beta} \diamond t_\alpha \frac{h_\alpha h_\beta + h_\beta^2 + h_\alpha + 3h_\beta + 3}{(h_\beta+1)(h_\alpha+h_\beta+1)} + z_{-\alpha-\beta} \diamond t_\beta \frac{h_\beta^2 + h_\alpha + 4h_\beta + 3}{h_\beta(h_\beta+1)(h_\alpha+h_\beta+1)} - z_{-\alpha} \diamond z_{-\beta} \frac{h_\alpha + 2h_\beta + 3}{(h_\alpha+1)(h_\beta+1)}, \tag{6.24}$$

$$t_\beta \diamond z_{-\alpha-\beta} = z_{-\alpha-\beta} \diamond t_\alpha \frac{h_\beta - 1}{(h_\beta+1)(h_\alpha+h_\beta+1)} + z_{-\alpha-\beta} \diamond t_\beta \frac{(h_\beta-1)(h_\alpha h_\beta + h_\beta^2 + 2h_\alpha + 4h_\beta + 3)}{h_\beta(h_\beta+1)(h_\alpha+h_\beta+1)} + z_{-\alpha} \diamond z_{-\beta} \frac{2h_\alpha + h_\beta + 3}{(h_\alpha+1)(h_\beta+1)}. \tag{6.25}$$

Finally, we rewrite the relations of the type 4, that is, of weight 0, in the form of ordering relations. In addition to the general commutativity relation (subtype (4**a**))

$$t_\beta \diamond t_\alpha = t_\alpha \diamond t_\beta, \tag{6.26}$$

we have three relations of subtype (4**b**):

$$z_\alpha \diamond z_{-\alpha} = h_\alpha - t_\alpha \diamond t_\alpha \frac{1}{h_\alpha} + z_{-\alpha} \diamond z_\alpha \frac{h_\alpha(h_\alpha+3)}{(h_\alpha+1)(h_\alpha+2)} - z_{-\beta} \diamond z_\beta \frac{h_\beta + 3}{(h_\beta+2)(h_\alpha+h_\beta+2)} + z_{-\alpha-\beta} \diamond z_{\alpha+\beta} \frac{h_\alpha(h_\alpha+h_\beta+4)}{(h_\alpha+1)(h_\beta+1)(h_\alpha+h_\beta+3)}, \tag{6.27}$$

$$z_\beta \diamond z_{-\beta} = h_\beta - t_\beta \diamond t_\beta \frac{1}{h_\beta} - z_{-\alpha} \diamond z_\alpha \frac{h_\alpha + 3}{(h_\alpha+2)(h_\alpha+h_\beta+2)} + z_{-\beta} \diamond z_\beta \frac{h_\beta(h_\beta+3)}{(h_\beta+1)(h_\beta+2)} + z_{-\alpha-\beta} \diamond z_{\alpha+\beta} \frac{h_\beta(h_\alpha+h_\beta+4)}{(h_\alpha+1)(h_\beta+1)(h_\alpha+h_\beta+3)}, \tag{6.28}$$

$$\begin{aligned} z_{\alpha+\beta} \diamond z_{-\alpha-\beta} = {} & \frac{h_\alpha h_\beta (h_\alpha+h_\beta+2)}{(h_\alpha+1)(h_\beta+1)} \\ & - \left(t_\alpha \diamond t_\alpha \frac{h_\beta}{h_\beta+1} + 2t_\alpha \diamond t_\beta + t_\beta \diamond t_\beta \frac{h_\alpha}{h_\alpha+1} \right) \frac{1}{h_\alpha+h_\beta+1} \\ & - z_{-\alpha} \diamond z_\alpha \frac{h_\alpha(h_\alpha+3)}{(h_\alpha+1)(h_\alpha+2)(h_\beta+1)} - z_{-\beta} \diamond z_\beta \frac{h_\beta(h_\beta+3)}{(h_\beta+1)(h_\beta+2)(h_\alpha+1)} \\ & + z_{-\alpha-\beta} \diamond z_{\alpha+\beta} \frac{h_\alpha h_\beta (h_\alpha+h_\beta+4)(\mathring{h}_\alpha^2 \mathring{h}_\beta + \mathring{h}_\alpha \mathring{h}_\beta^2 + \mathring{h}_\alpha^2 + \mathring{h}_\alpha \mathring{h}_\beta + \mathring{h}_\beta^2)}{(h_\alpha+1)^2(h_\beta+1)^2(h_\alpha+h_\beta+2)(h_\alpha+h_\beta+3)}, \end{aligned} \tag{6.29}$$

where in one factor in the numerator of the last coefficient we returned to the notation $\mathring{h}_\alpha = h_\alpha + 1$ and $\mathring{h}_\beta = h_\beta + 1$ to make the expression fit into the line.

2. Relations for DR($\mathfrak{gl}_3$). The ordering relations for the reduction algebra DR($\mathfrak{gl}_3$) are easily restored from the list (6.2)–(6.29): the $\mathfrak{gl}(3)$ generators t_1, t_2 and t_3, with $t_\alpha = t_1 - t_2$ and $t_\beta = t_2 - t_3$, can be written as

$$t_1 = \frac{1}{3}(2t_\alpha + t_\beta + I^{(3,t)}), \qquad t_2 = \frac{1}{3}(-t_\alpha + t_\beta + I^{(3,t)}), \qquad t_1 = \frac{1}{3}(-t_\alpha - 2t_\beta + I^{(3,t)}),$$

where $I^{(3,t)}$ is the image of the central generator of $\mathfrak{gl}(3)$, $I^{(3,t)} = t_1 + t_2 + t_3$. Since $I^{(3,t)}$ is central, one immediately writes relations for DR($\mathfrak{gl}_3$). We illustrate it on the example of relations between the generator z_α and the $\mathfrak{gl}(3)$ generators t_1, t_2 and t_3:

$$z_\alpha t_1 = t_1 z_\alpha \frac{h_\alpha + 3}{h_\alpha + 2} - t_2 z_\alpha \frac{1}{h_\alpha + 2} - z_{-\beta} z_{\alpha+\beta} \frac{h_\beta + 2}{(h_\beta+1)(h_\alpha+h_\beta+3)},$$

$$z_\alpha t_2 = -t_1 z_\alpha \frac{1}{h_\alpha+2} + t_2 z_\alpha \frac{h_\alpha+3}{h_\alpha+2} + z_{-\beta} z_{\alpha+\beta} \frac{h_\alpha+h_\beta+4}{(h_\beta+1)(h_\alpha+h_\beta+3)},$$
$$z_\alpha t_3 = t_3 z_\alpha - z_{-\beta} z_{\alpha+\beta} \frac{h_\alpha+2}{(h_\beta+1)(h_\alpha+h_\beta+3)}.$$

3. Braid group action. There are two braid group generators, $\check{\mathrm{q}}_\alpha$ and $\check{\mathrm{q}}_\beta$, for the diagonal reduction algebra $\mathrm{DR}(\mathfrak{sl}_3)$. Given the action of $\check{\mathrm{q}}_\alpha$, the action of $\check{\mathrm{q}}_\beta$ on $\mathrm{DR}(\mathfrak{sl}_3)$ can be reconstructed by using the automorphism ω, see (3.8), arising from the outer automorphism of the root system of $\mathfrak{sl}_3$, which exchanges the roots α and β,

$$\check{\mathrm{q}}_\beta = \omega \check{\mathrm{q}}_\alpha \omega^{-1}.$$

The action of the automorphism ω on the Cartan subalgebra $\langle h_\alpha, h_\beta\rangle$ of the diagonal Lie algebra $\mathfrak{sl}_3$ and on the generators of the reduction algebra $\mathrm{DR}(\mathfrak{sl}_3)$ reads

$$h_\alpha \leftrightarrow h_\beta, \qquad t_\alpha \leftrightarrow t_\beta,$$
$$z_\alpha \leftrightarrow z_\beta, \qquad z_{-\alpha} \leftrightarrow z_{-\beta},$$
$$z_{\alpha+\beta} \leftrightarrow -z_{\alpha+\beta}, \qquad z_{-\alpha-\beta} \leftrightarrow -z_{-\alpha-\beta}.$$

The action of the braid group generator $\check{\mathrm{q}}_\alpha$ on the Cartan subalgebra $\langle h_\alpha, h_\beta\rangle$ of the diagonal Lie algebra $\mathfrak{sl}_3$ reads:

$$\check{\mathrm{q}}_\alpha(h_\alpha) = -h_\alpha - 2, \qquad \check{\mathrm{q}}_\alpha(h_\beta) = h_\alpha + h_\beta + 1. \tag{6.30}$$

This action reduces to the standard action of the Weyl group for the shifted generators $\mathring{h}_\alpha = h_\alpha + 1$ and $\mathring{h}_\beta = h_\beta + 1$.

The action of $\check{\mathrm{q}}_\alpha$ on the zero weight generators $\{t_\alpha, t_\beta\}$ of the diagonal reduction algebra $\mathrm{DR}(\mathfrak{sl}_3)$ is given by:

$$\check{\mathrm{q}}_\alpha(t_\alpha) = -t_\alpha \frac{h_\alpha+2}{h_\alpha}, \qquad \check{\mathrm{q}}_\alpha(t_\beta) = t_\alpha \frac{h_\alpha+1}{h_\alpha} + t_\beta. \tag{6.31}$$

Finally, the action of $\check{\mathrm{q}}_\alpha$ on the rest of the generators is

$$\begin{aligned} &\check{\mathrm{q}}_\alpha(z_\alpha) = -z_{-\alpha} \frac{h_\alpha+1}{h_\alpha-1}, && \check{\mathrm{q}}_\alpha(z_{-\alpha}) = -z_\alpha, \\ &\check{\mathrm{q}}_\alpha(z_\beta) = z_{\alpha+\beta}, && \check{\mathrm{q}}_\alpha(z_{\alpha+\beta}) = -z_\beta \frac{h_\alpha+1}{h_\alpha}, \\ &\check{\mathrm{q}}_\alpha(z_{-\alpha-\beta}) = -z_{-\beta}, && \check{\mathrm{q}}_\alpha(z_{-\beta}) = z_{-\alpha-\beta} \frac{h_\alpha+1}{h_\alpha}. \end{aligned} \tag{6.32}$$

The set of ordering relations (6.2)–(6.29) is covariant with respect to the braid group generated by $\check{\mathrm{q}}_\alpha$ and $\check{\mathrm{q}}_\beta$. "Covariant" means that the elements of the braid group map a relation to a linear over $\overline{\mathrm{U}}(\mathfrak{h})$ combination of relations. For example, the operator $\check{\mathrm{q}}_\alpha$, up to multiplicative factors from $\overline{\mathrm{U}}(\mathfrak{h})$, transforms the relation (6.27) into itself and permutes the relations (6.28) and (6.29). Due to the choice (6.1) of the order, the set of relations (6.2)–(6.29) is invariant with respect to the anti-involution ϵ. The set of relations (6.2)–(6.29) is covariant under the involution ω as well.

4. Central elements of $\mathbf{DR}(\mathfrak{sl}_3)$. The degree 1 and degree 2 (in generators z_{ij}) central elements of the reduction algebra $\mathrm{DR}(\mathfrak{sl}_3)$ are:

$$\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_3),1\}} = t_\alpha(2h_\alpha + h_\beta + 6) + t_\beta(h_\alpha + 2h_\beta + 6),$$
$$\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_3),2\}} = \frac{1}{3}(t_\alpha \diamond t_\alpha + t_\beta \diamond t_\beta + t_\alpha \diamond t_\beta + h_\alpha^2 + h_\beta^2 + h_\alpha h_\beta)$$

$$+ z_{-\alpha} \diamond z_\alpha \frac{h_\alpha + 3}{h_\alpha + 2} + z_{-\beta} \diamond z_\beta \frac{h_\beta + 3}{h_\beta + 2}$$
$$+ z_{-\alpha-\beta} \diamond z_{\alpha+\beta} \frac{h_\alpha + h_\beta + 4}{h_\alpha + h_\beta + 3} \left(1 + \frac{1}{h_\alpha + 1} + \frac{1}{h_\beta + 1}\right) + 2(h_\alpha + h_\beta).$$

Both Casimir operators, $\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_3),1\}}$ and $\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_3),2\}}$ arise from the quadratic Casimir operator $\mathcal{C}^{\{\mathfrak{sl}_3,2\}}$ of the Lie algebra $\mathfrak{sl}_3$, whose ordered form is

$$\mathcal{C}^{\{\mathfrak{sl}_3,2\}} = (\mathcal{E}_{-\alpha}\mathcal{E}_\alpha + \mathcal{E}_{-\beta}\mathcal{E}_\beta + \mathcal{E}_{-\alpha-\beta}\mathcal{E}_{\alpha+\beta}) + \frac{1}{3}(\mathcal{H}_\alpha^2 + \mathcal{H}_\beta^2 + \mathcal{H}_\alpha\mathcal{H}_\beta) + \mathcal{H}_\alpha + \mathcal{H}_\beta.$$

The operator $\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_3),1\}}$ is the image of $\mathcal{C}^{\{\mathfrak{sl}_3,2\}} \otimes 1 - 1 \otimes \mathcal{C}^{\{\mathfrak{sl}_3,2\}}$ and the operator $\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_3),2\}}$ is the image of $\mathcal{C}^{\{\mathfrak{sl}_3,2\}} \otimes 1 + 1 \otimes \mathcal{C}^{\{\mathfrak{sl}_3,2\}}$. We calculate $\mathcal{C}^{\{\mathfrak{sl}_3,2\}} \otimes 1 + 1 \otimes \mathcal{C}^{\{\mathfrak{sl}_3,2\}}$ and replace the multiplication by the product $\diamond$. Here one needs, in addition to (5.30)–(5.34), the expression for $H_{23} \diamond H_{23}$ which is obtained by applying the involution ω to (5.31) and the equality (in the notation of Section 5.3):

$$H_{12} \diamond H_{23} \equiv H_{12}H_{23} + E_{21}E_{12}\frac{2}{\mathring{h}_{12} + 1} + E_{32}E_{23}\frac{2}{\mathring{h}_{23} + 1}$$
$$- E_{31}E_{13}\left(1 + \frac{2}{\mathring{h}_{12} + 1} + \frac{2}{\mathring{h}_{23} + 1}\right)\frac{1}{\mathring{h}_{13} + 1}.$$

The central elements $\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_3),1\}}$ and $\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_3),2\}}$ are invariant with respect to the braid group:

$$\check{\mathrm{q}}_\alpha\big(\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_3),i\}}\big) = \mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_3),i\}}, \qquad \check{\mathrm{q}}_\beta\big(\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_3),i\}}\big) = \mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_3),i\}}, \qquad i = 1, 2.$$

The central elements $\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_3),1\}}$ and $\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_3),2\}}$ are invariant with respect to the anti-involution ϵ and the involution ω as well.

5. Diagonal reduction algebra $\mathrm{DR}(\mathfrak{sl}_2)$**.** For the diagonal reduction algebra of $\mathfrak{sl}_2$ we use the following notation:

$$z_+ := z_\alpha, \qquad z_- := z_{-\alpha}, \qquad t := t_\alpha, \qquad h := h_\alpha.$$

The cut provides the following description of the algebra Z_2 with generators z_+, z_- and t:

$$z_+ \diamond t = t \diamond z_+ \frac{h+4}{h+2}, \tag{6.33}$$
$$z_+ \diamond z_- = h - t \diamond t\frac{1}{h} + z_- \diamond z_+ \frac{h(h+3)}{(h+1)(h+2)}, \tag{6.34}$$
$$t \diamond z_- = z_- \diamond t\frac{h+2}{h}. \tag{6.35}$$

The Casimir operators for $\mathrm{DR}(\mathfrak{sl}_2)$ are

$$\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_2),1\}} := (h+2)t, \tag{6.36}$$
$$\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_2),2\}} := z_- \diamond z_+ \frac{(h+3)}{(h+2)} + t \diamond t\frac{1}{4} + \frac{h(h+4)}{4}. \tag{6.37}$$

Both operators, $\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_2),1\}}$ and $\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_2),2\}}$ arise from the quadratic Casimir operator $\mathcal{C}^{\{\mathfrak{sl}_2,2\}}$ of the Lie algebra $\mathfrak{sl}_2$,

$$\mathcal{C}^{\{\mathfrak{sl}_2,2\}} = \mathcal{E}_-\mathcal{E}_+ + \frac{1}{4}\mathcal{H}(\mathcal{H}+2),$$

$\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_2),1\}}$ is the image of $\mathcal{C}^{\{\mathfrak{sl}_2,2\}}\otimes 1-1\otimes\mathcal{C}^{\{\mathfrak{sl}_2,2\}}$ and $\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_2),2\}}$ is the image of $\mathcal{C}^{\{\mathfrak{sl}_2,2\}}\otimes 1+1\otimes\mathcal{C}^{\{\mathfrak{sl}_2,2\}}$.

The Casimir operators can be obtained by the cutting also, as explained in Subsection 4.6, see Proposition 7. One replaces the $\mathfrak{sl}_3$ generators by the $\mathfrak{gl}_3$ generators in the Casimir operators for $\mathfrak{sl}_3$ then cuts and rewrites, using the notation (3.9) and (3.10), the result according to the $\mathfrak{gl}_2$ formulas

$$t_1^{(2)}=\frac{1}{2}\big(t+I^{(2,t)}\big),\qquad t_2^{(2)}=\frac{1}{2}\big(-t+I^{(2,t)}\big),$$
$$t=t_1^{(2)}-t_2^{(2)},\qquad I^{(2,t)}=t_1^{(2)}+t_2^{(2)}.$$

The cut of $\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_3),1\}}$ is

$$\frac{3}{2}\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_2),1\}}+\frac{1}{2}I^{(2,t)}\diamond\big(I^{(2,h)}+6\big)-t_3^{(3)}\diamond\big(I^{(2,h)}+6\big)-I^{(2,t)}h_3+2t_3^{(3)}h_3 \tag{6.38}$$

and the cut of $\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_3),2\}}$ is

$$\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_2),2\}}+\frac{1}{12}I^{(2,t)}\diamond I^{(2,t)}+\frac{1}{12}I^{(2,h)}\diamond\big(I^{(2,h)}+12\big)$$
$$-\frac{1}{3}I^{(2,t)}\diamond t_3^{(3)}-\left(\frac{1}{3}I^{(2,h)}+2\right)h_3+\frac{1}{3}\big(t_3^{(3)}\diamond t_3^{(3)}+h_3^2\big). \tag{6.39}$$

As expected, the coefficients of $(t_3^{(3)})^{\diamond i}h_3^j$ for all i and j in the expressions (6.38) and (6.39) are central elements of the algebra Z_2.

Due to (6.30), (6.31) and (6.32), the action of the braid group generator reads

$$\check{\mathrm{q}}(h)=-h-2,\qquad \check{\mathrm{q}}(t)=-t\frac{h+2}{h},\qquad \check{\mathrm{q}}(z_+)=-z_-\frac{h+1}{h-1},\qquad \check{\mathrm{q}}(z_-)=-z_+. \tag{6.40}$$

It preserves the commutation relations of $\mathrm{DR}(\mathfrak{sl}_2)$. The Casimir operators (6.36) and (6.37) are invariant under the transformation (6.40) and under the anti-involution ϵ.

It should be noted that $\check{\mathrm{q}}$ can be included in a family of more general automorphisms of the reduction algebra $\mathrm{DR}(\mathfrak{sl}_2)$.

Lemma 8. *The most general automorphism of the reduction algebra* $\mathrm{DR}(\mathfrak{sl}_2)$ *transforming the weights of elements in the same way as the operator* $\check{\mathrm{q}}$ *and linear over* $\overline{\mathrm{U}}(\mathfrak{h})$ *in the generators* z_+, z_- *and* t *is*

$$h\mapsto -h-2,\qquad t\mapsto\beta t\frac{h+2}{h},\qquad z_+\mapsto z_-\frac{1}{(h-1)\gamma(h)},$$
$$z_-\mapsto z_+(h+3)\gamma(h+2), \tag{6.41}$$

where $\beta=\pm 1$ *is a constant and* $\gamma(h)$ *is an arbitrary function.*

Proof. We are looking for an invertible transformation which preserves the relations (6.33)–(6.35) and has the form

$$h\mapsto f_1(h),\qquad t\mapsto tf_2(h),\qquad z_+\mapsto z_-f_3(h),\qquad z_-\mapsto z_+f_4(h) \tag{6.42}$$

with $f_1(h),f_2(h),f_3(h),f_4(h)\in\overline{\mathrm{U}}(\mathfrak{h})$. Applying the transformation (6.42) to the relations (6.33) and (6.35), we find (after simplifications) the conditions:

$$(h+2)\big(f_1(h)+4\big)f_2(h-2)-x\big(f_1(h)+2\big)f_2(h)=0, \tag{6.43}$$
$$(h+4)\big(f_1(h)+2\big)f_2(h)-(x+2)f_1(h)f_2(h+2)=0. \tag{6.44}$$

Replacing h by $h-2$ in the second equation and then excluding f_2 from the system (6.43), (6.44), we obtain the difference equation

$$f_1(h) - f_1(h-2) + 2 = 0,$$

whose general solution in $\overline{\mathrm{U}}(\mathfrak{h})$ is

$$f_1(h) = -h + c, \tag{6.45}$$

where c is a constant.

Applying the transformation (6.42) to the relation (6.34) and collecting the free term and the terms with $t \diamond t$ and $z_- \diamond z_+$, we find (after simplifications)

$$1 + \frac{h\big(f_1(h)+3\big)G(h)}{\big(f_1(h)+2\big)\big(f_1(h)+1\big)} = 0, \tag{6.46}$$

$$\frac{f_2(h)^2}{f_1(h)} + \frac{f_1(h)\big(f_1(h)+3\big)G(h)}{h\big(f_1(h)+2\big)\big(f_1(h)+1\big)} = 0, \tag{6.47}$$

$$G(h+2) + \frac{h(h+3)f_1(h)\big(f_1(h)+3\big)}{(h+1)(h+2)\big(f_1(h)+2\big)\big(f_1(h)+1\big)}G(h) = 0, \tag{6.48}$$

where $G(h) := f_3(h)f_4(h-2)$. Excluding G from the system (6.46), (6.47), we obtain

$$f_1(h)^2 = h^2 f_2(h)^2 \qquad \text{or} \qquad f_2(h) = \beta\frac{f_1(h)}{h} \tag{6.49}$$

with $\beta^2 = 1$.

The substitution of (6.45) and (6.49) into (6.43) leads to

$$c = -2$$

and it then follows from (6.46) that

$$G(h) = \frac{h+1}{h-1}.$$

The remaining relation (6.48) is now automatically satisfied. The proof is finished. ■

The Casimir operator $\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_2),2\}}$ is invariant under the general automorphism (6.41). The Casimir operator $\mathcal{C}^{\{\mathrm{DR}(\mathfrak{sl}_2),1\}}$ is invariant under the automorphism (6.41) iff $\beta = -1$.

The map $\check{\mathrm{q}}$ defined by (6.40) is a particular choice of (6.41), corresponding to $\beta = -1$ and $\gamma(h) = -\frac{1}{h+1}$.

The map (6.40) is not an involution (but it squares to the identity on the weight zero subspace of the algebra). However, the general map (6.41) squares to the identity on the whole algebra iff the function γ is odd,

$$\gamma(-h) = -\gamma(h).$$

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