First draft

The Geometric Phase as Analog of Fractional Exponential Function

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Abstract

The notion of geometric phase arises in connection with parallel transport in differential geometry and the formulation of gauge transformation in field theory. Here we show that the geometric phase is locally equivalent to the action of fractional exponential, which is applicable to manifolds having minimal fractal topology or for modeling complex phenomena using fractional calculus.

Key words: Geometric phase, gauge transformation, metric connection, fractional exponential, fractional calculus, minimal fractal manifold.

1. Introduction

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2. Berry phase in quantum physics

A quantum system adiabatically transported around a closed path C in the space of external parameters acquires a non-integrable phase (*Berry phase*, BP in short). Since BP depends exclusively on the geometry of the path, it provides key insights into the geometric structure of quantum mechanics and Quantum Field Theory. The BP concept is closely tied to *holonomy*, that is, the extent to which some of variables change as other variables or parameters defining a system return to their initial values [1, 2].

Consider a quantum system described by the time-independent Hamiltonian $H(t)$, whose associated eigenstate is $|\psi(t)\rangle$ and which is embedded in a slowly changing environment. After a periodic evolution of the environmental parameters ($t \to t+T$), the eigenstate returns to itself apart from a phase angle,

$$|\psi(t)\rangle = e^{i\alpha}|\psi(0)\rangle \tag{1}$$

If ω denotes the eigenvalue of $|\psi(t)\rangle$, a generalization of the phase angle $\alpha = \omega T$ in units of $\hbar = 1$ is given by the "dynamical phase"

$$\gamma_d = \int_0^T \omega(t)\,dt = \int_0^T \langle\psi(t)|H(t)|\psi(t)\rangle\,dt \tag{2}$$

Berry has shown that there is a time-independent (but contour dependent) supplemental "geometric phase" entering the phase angle, namely,

$$\alpha = \gamma_d + \gamma(C) \tag{3}$$

where

$$\gamma(C) = \oint_C \langle\psi\,|\,i\nabla\psi\rangle \cdot d\boldsymbol{x} \tag{4}$$

The dynamical phase γ_d encodes information about the *duration* associated with the cyclic evolution of the complex vector $|\psi(t)\rangle$. By contrast, since (4) follows from the parallel transport around the closed loop *C*, it implicitly encodes the geometry of the environment where the transport takes place.

3. The geometry of gauge and gravitational fields

The gauge field concept is known to have a deep geometric foundation [3, 4]. In particular, all gauge fields A_μ define parallel transport in internal (charge) space with the field strength $F_{\mu\nu}$ playing the role of curvature tensor. The key point is that, in any spacetime or internal space where the coordinates are position-dependent, comparing two vectors at different points along a path is meaningless unless one makes use of parallel transport or affine connection. Changes in the local frame are compensated by gauge fields in internal space *or* Christoffel symbols in non-Euclidean spacetime. The geometric analogy between gauge theory and General Relativity is captured in the table below.

Gauge Theory	**General Relativity**
Gauge transformation	Coordinate transformation
Gauge group	Group of coordinate transformations
Gauge potential A_μ	Connection coefficient $\Gamma^{\kappa}_{\mu\nu}$
Field strength $F_{\mu\nu}$	Curvature tensor $R^{\kappa}_{\lambda\mu\nu}$

Comparison between the geometry of gauge and gravitational fields.

A helpful illustration of this analogy is offered by the parallel transport of a complex vector $|\psi\rangle$ round a closed rectangular loop using covariant operators [3, 4]. The difference between the value of $|\psi\rangle$ at the starting point ($|\psi\rangle_0$) and at the end point $|\psi\rangle_0 \rightarrow |\psi\rangle_f$ is given by

$$\Delta\psi = \psi_f - \psi_0 = -ig\,\Delta S^{\mu\nu} F_{\mu\nu}\psi \qquad (5)$$

in which $\Delta S^{\mu\nu}$ denotes the area subtended by the rectangle and the strength of the gauge field is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right] \tag{6}$$

Echoing the formation of the Berry phase, the effect of parallel transport is to induce a non-vanishing rotation of $|\psi\rangle$ in internal space proportional to the strength of the gauge field.

4. Fractional exponential function

For a function $u \in \Phi(\mathbf{R})$, the fractional Fourier transform of order $0 < \beta \leq 1$ is defined as [5]

$$\hat{u}_\beta(\omega) = \int_{-\infty}^{+\infty} u(t) e_\beta(\omega, t) dt, \quad \omega \in \mathbf{R} \tag{7}$$

where the kernel functions are

$$e_\beta(\omega, t) = \begin{cases} \exp(-i|\omega|^{1/\beta} t), \omega \leq 0 \\ \exp(i|\omega|^{1/\beta} t), \omega \geq 0 \end{cases} \tag{8}$$

When $\beta = 1 - \varepsilon$, $\varepsilon << 1$ the frequency entering (6) takes the form

$$|\omega|^{1/\beta} = |\omega|^{1/(1-\varepsilon)} \approx |\omega|^{(1+\varepsilon)} \tag{9}$$

leading to a non-vanishing correction to the conventional phase angle given by

$$\alpha_F(\varepsilon) = |\omega|^\varepsilon t = e^{\varepsilon \ln|\omega|} t \approx (1 + \varepsilon \ln|\omega|) t \tag{10}$$

The adiabatic condition $\omega << 1$ yields an undefined phase (10), which signal unbounded phase fluctuations on all scales and the onset of critical behavior.

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5. Further implications of fractional exponential in field theory

Significant consequences for Beyond the Standard Model (BSM) physics and ultraviolet completion programs.

- *Lie groups and Lie algebra* [6]

Linear representation of a group of elements $a(\theta)$

$$a \mapsto D_R(a) \tag{11}$$

$$D_R(a(\theta)) = \exp(i\theta_\alpha T_R^\alpha) \tag{12}$$

- *Spin-statistics theorem* [7]

As the wavefunction of a system of n identical particles stays invariant in modulus, the interchange of quantum numbers between the i-th and the j- th particle picks up a non-vanishing phase according to

$$\psi(q_i, q_j) = \exp(2\pi i\sigma)\psi(q_j, q_i) \tag{13}$$

The parameter σ is defined only modulo integers and fixes the statistics of particles i, j.

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6. Conclusions and outlook.

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References

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