Unimodular bimode gravity and the coherent scalar-graviton field as galaxy dark matter

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Abstract

The explicit violation of the general gauge invariance/relativity is adopted as the origin of dark matter and dark energy of the gravitational nature. The violation of the local scale invariance alone, with the residual unimodular one, is considered. Besides the four-volume preserving deformation mode – the transverse-tensor graviton – the metric comprises a compression mode – the scalar graviton, or the *systolon*. A unimodular invariant and general covariant metric theory of the bimode/scalar-tensor gravity is consistently worked out. To reduce the primordial ambiguity of the theory a dynamical global symmetry is imposed, with its subsequent spontaneous breaking revealed. The static spherically symmetric case in the empty, but possibly for the origin, space is studied. A three-parameter solution describing a new static space structure – the dark *lacuna* – is constructed. It enjoys the property of gravitational confinement, with the logarithmic potential of gravitational attraction at the periphery, and results in the asymptoticaally flat rotation curves. Comprising a super-massive dark *fracture* (a scalar-modified black hole) at the origin surrounded by a cored dark *halo*, the dark lacunas are proposed as a prototype model of galaxies, implying an ultimate account for the distributed non-gravitational matter and a putative asphericity or rotation.

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1   Introduction and motivations

The General Relativity (GR) is well-known to be the viable theory of gravity perfectly consistent (modulo some additional assumptions) with all the available observations. Nevertheless, it hardly is an ultimate theory being rather an effective field theory produced by a more fundamental one. The latter well may enjoy a wider "low-energy" remnant, with GR being just a principle part of it. The basic requirements to such a GR extension may be that it should safely retain all the well-established theoretical and observational properties of GR, but at the same time should encompass some new phenomena beyond GR (or v.v.). Consider these items in more detail.

Gauge invariance: unimodular vs. general   The essence of GR may be expressed by saying that it is a gauge theory of a massless tensor field/graviton. As the respective gauge group it is conventionally taken the group of the general diffeomorphisms. For consistency with GR, an extension to the latter should conceivably be based on a gauge principle, too. Following the original description of the irreducible (one-dimensional) unitary representations of the (non-compact) Poincare group due to Wigner [1], it was later found by van der Bij, van Dam and Ng [2] that the necessary and sufficient gauge group admitting a massless tensor field is the group of transverse diffeomorphisms, with the group of general diffeomorphisms being thus excessive to this purpose. For this reason, a theory of gravity based on such a minimally violated gauge invariance might well be the most natural candidate to supersede GR. To retain general covariance the transverse diffeomorphisms are to be substituted though by the unimodular ones (see Sec. 2.1).

To clarify the structure of such a GR extension decompose the group of the general diffeomorphisms into the commuting subgroups of the transverse diffeomorphisms and that of the local scale transformations. By construction, the determinant of metric, g, changes only with respect to the latter subgroup. Under restriction by the transverse subgroup, g is a scalar and can be treated as an independent field variable to be implement in the gravity Lagrangian threefold.

(*i*) *General invariant Lagrangian*   First, one may retain g in the gravity Lagrangian implicitly through metric, but made it unphysical by means of the local scale transformations as in GR (or in its modifications preserving general gauge invariance). As a physical degree of freedom in metric there is left just the (massless) tensor graviton.

(*ii*) *Restricted Lagrangian*   Second, one may restrict the Lagrangian by fixing g a priori (as it was originally proposed in other context by Anderson and Finkelstein [3]), so that the respective degree of freedom in metric is absent, leaving only the massless tensor graviton (see also, e.g., [2], [4]–[17]). Being a close counterpart of GR such a theory of tensor gravity is conventionally referred to as the Unimodular Relativity (UR). A possible advantage of UR compared to GR is that the cosmological term is no more a Lagrangian parameter as in GR itself, but appears as an integration constant. This may help in solving the so-called naturalness problem of cosmological term.

(*iii*) *Extended Lagrangian*   And v.v., one may extend the gravity Lagrangian in the wake of Buchmüller, Dragon and Kreuzer [18]–[20] by adding to it the terms explicitly containing the derivatives of g. Due to violation of the general gauge invariance, the degree of freedom corresponding to g can not be eliminated any more and gets physical. Thus, in addition to the massless tensor graviton the metric comprises a (light) gravitationally interacting scalar particle. Studying such a scalar-tensor extension to GR is of considerable interest by itself from the theoretical point of view.

Unimodular invariance and dark matter   However, the main motivation to adhere to the theory of the latter type may lie in the problems of the so-called dark matter (DM) and dark energy (DE).[1] These substances, being crucial for the structure of the Universe, present perhaps the greatest challenge facing the modern physics. Basically, while DE should be "spilled" all over the Universe on the cosmological scale, DM should cluster on the galactic and galaxy cluster scales. At that, DM proves to be one of the most important building-blocks of the Universe. Its energy yield largely exceeds that of

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[1]For a review on these topics see, e.g., [21].

luminous matter. While DE might already be anticipated as given by the well-known cosmological term, DM appears, surprisingly enough, absolutely ad hoc from the modern theory point of view. Although the evidence for DM is overwhelming, its nature remains still obscure.

The crucial points concerning DM are, first, that it should interact with the luminous matter very faintly and, second, its clusterization in gravity field should be weaker compared to the luminous matter. Such an elusive kind of matter may well have an unconventional nature, e.g., the gravitational one. With this in mind, one can put forward the hypothesis that as an origin of such a DM there may serve the explicit violation of the general gauge invariance/relativity [22].[2] The reason is that the ensuing in this case extra terms in gravity equations may be treated as an effective energy-momentum tensor of some additional physical degree(s) of freedom. Interacting only gravitationally, but otherwise than the non-gravitational matter, this new substance may well suit as DM. Thus due to violation of general relativity the metric field itself may store a kind of the gravitational DM.[3]

As a paradigm one may consider the minimal violation of general relativity to the unimodular one [22, 23]. In this case, on the one hand, the residual invariance suffices to justify the masslessness of tensor graviton. On the other hand, due to violation of the local scale invariance alone, the metric comprises no more than one extra physical component besides the massless tensor graviton. Originating from metric and interacting only gravitationally, the respective particle may naturally be associated with the scalar graviton. To have sense in the arbitrary observer's coordinates the theory should be cast to the general covariant form by introducing a non-dynamical scalar density of the same weight as $\sqrt{-g}$. By this token, it was found appropriate to treat the scalar-graviton field as an independent variable, embedding within it (virtually) the unknown scalar density. In the static spherically symmetric case this made it possible to find an exact solution to the extended gravity equations in the empty, except for the origin, space [24]. Such a solution describes a two-parameter generalization of the one-parameter black holes (BHs) and is valid also in GR with a (free massless) scalar field. Beyond GR, an approximate, regular at the origin one-parameter solution in the empty space was then found analytically [25] and refined numerically [26]. This solution presents a halo-type static space structure built entirely of scalar gravitons. Such a structure was shown to possess a soft-core energy density profile, qualitatively compatible with that for the galaxy DM halos. This makes it urgent to investigate the more general solutions to the aforementioned equations, as well as consequences thereof.

Content In this article the metric theory of the scalar-tensor gravity built on principles of the unimodular relativity and general covariance is consistently worked out. In Section 2 the theoretical background is developed. In Sec. 2.1 the explicit violation of general relativity is discussed in toto. The violation of the local scale invariance alone, with the residual unimodular invariance, is then investigated in more detail. In Sec. 2.2 the effective field theory of metric, enjoying the residual unimodular invariance as well as the general covariance, is considered. To terminate a priori allowed effective Lagrangians, a dynamical global symmetry at the classical level is imposed. The ensuing classical

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[2]The term "relativity/invariance violation" is more appropriate (see Sec. 2.1) than the "covariance violation" used previously in [22].

[3] This does not exclude, of course, a fraction of a more conventional particle DM, so that the direct searches for DM are by no means meaningless.

equations are then written down, and the spontaneous breaking of the global symmetry is displayed. Section 3 deals with the ensuing classical equations and their solutions in the static spherically symmetric case. In Sec. 3.1 the gravity equations in the empty, but possibly for the origin, space are presented in a specific gauge. In Sec. 3.2 an exact two-parameter solution of the BH-type, valid also in GR with a (free massless) scalar field, is rebuilt in the most transparent fashion, and then compared in several gauges. In Sec. 3.3 an approximate, regular at the origin one-parameter halo-type solution, missing in GR with a free scalar field, is exposed. In Sec. 3.4 an irregular at the origin three-parameter solution interpolating between the two preceding extreme cases is constructed. The property of the gravitational confinement for the respective extended static space structures is found. Section 4 is devoted to interpretation and application of the found solutions/structures in the context of galaxy DM. In Sec. 4.1 the energy content of the structures is revealed. In Sec. 4.2 the asymptotic flat rotation curves (RCs) ensuing due to these structures are exposed, with their DM interpretation presented. Finally, the relevance of the structures to galaxies is discussed. In Conclusion the state of affairs of the theory and its future prospects are outlined.

2 Theoretical frameworks

2.1 General relativity violation

Relativity/invariance vs. covariance To go beyond GR let us refine the paradigm of general relativity and/or general covariance adopted in GR. Let $I = \int \mathcal{L} d^4 x$ be the action of a field theory, with $\mathcal{L}$ being the density of its effective Lagrangian. To be precise, under the (classical) dynamics there will be understood the principle of the minimal action, with the emerging classical equations. In this respect, $\mathcal{L}$ may depend on two kinds of fields: the *dynamical/relative* and *non-dynamical/absolute* ones, generically, φ's and φ_*'s. The non-dynamical fields are given a priori. The dynamical ones are those, classical equations for which are obtained by extremizing I, under frozen φ_*'s. The φ's include metric $g_{\mu\nu}$ (or its restricted part), matter fields and, optionally, the undetermined Lagrange multiplier which is a kind of the dynamical scalar field. The classical solutions for φ's should depend on φ_*'s as on the external functional parameters. An arbitrary effective field theory, in contrast to GR, may thus be said to be the theory of "restricted relativity", with φ's implemented in the relative fashion and φ_*'s in the absolute one.

Accordingly, there are envisaged two types of space-time properties of the field theory.

(*i*) *Covariance* This is a kinematic property describing a maximal kinematically allowed set G of the simultaneous diffeomorphisms of φ's and φ_*'s, under which I remains invariant. For consistency, φ_*'s should transform under G through themselves, whereas the transformed φ's may depend on φ_*'s as well. To have sense in the arbitrary observer's coordinates (chosen a priori) $\mathcal{L}$ should be a general covariant scalar density due to a required number of φ_*'s (if any). In what follows, G will be supposed to be the group of general diffeomorphisms.

(*ii*) *Relativity* This is a dynamic property describing a maximal subset $H \subseteq G$ of the diffeomorphisms of φ's alone (with φ_*'s transforming as scalars), under which I remains invariant. At that, in the process of variation any $\delta\varphi$ within the configuration space are admitted, whereas φ_*'s are to be frozen ($\delta\varphi_* = 0$). By the very nature, it is H which serves as a space-time gauge group for the theory. So, as a synonym of relativity there will also be used the term (gauge) *invariance*.

In GR the covariance and relativity/invariance coincide, both being the general ones. Beyond GR one should distinguish them. In particular, the non-trivial quotient G/H may result in appearance of the extra physical degree(s) of freedom. In what follows we consider the unimodular invariant but general covariant metric theory of gravity, with a dynamical scalar-graviton field in metric and a non-dynamical scalar-density field. This is the maximal allowed explicit violation of general relativity/invariance consistent with the masslessness of transverse-tensor graviton.

Unimodularity and bimodality   To be precise, under the theory of gravity at the energies lower than the Plank mass there will be understood the effective field theory of metric. Ultimately, the latter theory is to be considered as a "low-energy" remnant of a more fundamental theory. This remnant may basically be characterized by the two ingredients: a residual invariance group $H \subseteq G$ and a set of the low-energy fields. As a paradigm, let H be the maximal subgroup of G given by the unimodular group U. Let $g_{\mu\nu}$ be a dynamical metric field, with $\det(g_{\mu\nu}) = g$, and let $\sqrt{-g_*}$ be a non-dynamical scalar density of the same weight as $\sqrt{-g}$. Related to DM and associated with a non-dynamical measure, $\sqrt{-g_*}$ will be, in the wake of [3], referred to as the (dark) *modulus*. A priori there are envisaged arbitrary (including singular) moduli. The dark moduli are to be considered as a kind of a new substance, the nature of which in the framework of the effective field theory is unspecified and should ultimately be revealed by a more fundamental theory.

Without loss of generality, the Lagrangian density $\mathcal{L}$ for a unimodular invariant effective field theory of metric may be expressed through the general covariant scalar Lagrangian L as $\mathcal{L} = L(g_{\mu\nu}, g, g_*)\sqrt{-g}$. The gauge properties of a field theory under the infinitesimal diffeomorphisms $x^\mu \to \hat{x}^\mu = x^\mu + \Delta_\xi x^\mu$, $\Delta_\xi x^\mu = -\xi^\mu$, are expressed through a Lie derivative. The latter accounts for the net variation of a field due to both its tensor and argument variations. For the metric field it is as follows:

$$\begin{aligned}\Delta_\xi g_{\mu\nu} &= \partial_\mu \xi^\lambda g_{\lambda\nu} + \partial_\nu \xi^\lambda g_{\mu\lambda} + \xi^\lambda \partial_\lambda g_{\mu\nu} = \\ &= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu,\end{aligned} \tag{1}$$

with $\xi_\mu = g_{\mu\lambda}\xi^\lambda$ and ∇_μ being a covariant derivative with respect to $g_{\mu\nu}$. It follows hereof that

$$\Delta_\xi \sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\Delta_\xi g_{\mu\nu} = \partial_\mu(\sqrt{-g}\xi^\mu). \tag{2}$$

By definition, the same should fulfill for $\Delta_\xi \sqrt{-g_*}$. Thus

$$\Delta_\xi \ln\sqrt{g/g_*} = \xi^\mu \partial_\mu \ln\sqrt{g/g_*}, \tag{3}$$

i.e., g/g_* transforms under G as a scalar.

Now, restrict the set of ξ^μ by a subset ξ_u^μ defined by requirement of invariance of a dark modulus (the *unimodularity* condition):[4]

$$\Delta_{\xi_u}\sqrt{-g_*} = \partial_\mu(\sqrt{-g_*}\xi_u^\mu) = 0. \tag{4}$$

This singles out in a covariant fashion the residual unimodular invariance subgroup $H = U$. At that

$$\Delta_{\xi_u} \ln\sqrt{-g} = \xi_u^\mu \partial_\mu \ln\sqrt{g/g_*} \tag{5}$$

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[4]Restricted ab initio to coordinates where $g_* = -1$ (if possible), the covariant unimodularity condition reduces to the non-covariant transversality condition: $\partial_\mu \xi_t^\mu = 0$, with $\xi_t^\mu = \xi_u^\mu|_{g_*=-1}$. For the "transverse/TDiff gravity" based on the latter condition see, e.g., [27]–[29].

is not bound to vanish. The general covariance forbids dependence of L on g_* and g separately, leaving just

$$\mathcal{L} = L(g_{\mu\nu}, g/g_*)\sqrt{-g}. \tag{6}$$

An arbitrary g_* is unavoidable in the general covariant framework of the unimodular invariant field theory, despite the fact that it might superficially seem irrelevant. Besides, it is urgent in practice. In addition to the implicit manifestations it may conceivably result in some direct ones. The terms violating general invariance and containing derivatives of g/g_* result ultimately in the scalar propagating mode in metric. In the absence of derivatives, g/g_* becomes just an auxiliary field. In what follows the general covariant and unimodular invariant metric theory of the bimode/scalar-tensor gravity – the Unimodular Bimodal Gravity (UBG) – is consistently exposed.[5]

2.2 Unimodular Bimodal Gravity

Extended gravity Lagrangian In the framework of the effective field theory any choice $X = X(g/g_*)$ is a priori allowed as a field variable instead of g/g_*. The different choices are related through the field redefinition. Thus let us take without loss of generality as the (dimensionless) scalar-graviton field

$$X = \ln \sqrt{-g}/\sqrt{-g_*}. \tag{7}$$

This choice proves to be advantageous from the symmetry considerations (see below). In this terms, one has for the UBG Lagrangian generically

$$L_{\text{UBG}} = L_g + L_s + \Delta L_{gs} + L_m, \tag{8}$$

where

$$L_g = -\frac{1}{2}\kappa_g^2 R, \quad L_s = \frac{1}{2}\kappa_s^2 \nabla X \cdot \nabla X \tag{9}$$

present the kinetic terms, respectively, for the tensor and scalar modes, with $\nabla X \cdot \nabla X = g^{\mu\nu}\nabla_\mu X \nabla_\nu X$, etc., and $\nabla_\mu X \equiv \partial_\mu X$. The general invariant L_g is chosen for simplicity as in GR (cf. though Sec. 4.2). Here R is the Ricci scalar and $\kappa_g = 1/\sqrt{8\pi G}$ is the mass scale for tensor gravity, with G being the Newton's constant. In the unimodular invariant L_s, the parameter κ_s is a mass scale for the scalar gravity. For tensor dominance of the bimode gravity (generically $|L_g| > |L_s|$) one would a priori expect $\kappa_s/\kappa_g < \mathcal{O}(1)$. Moreover, to comply with astronomic observations for galaxies it proves that $\kappa_s/\kappa_g \sim 10^{-3}$ (see Sec. 4.2). Further, $\Delta L_{gs}(R, X, \nabla_\mu X, \ldots)$ is the rest of the gravity Lagrangian depending, generally, on R and X. In particular, ΔL_{gs} may include some scalar potential, $\Delta L_{gs} = -V_s(X) + \ldots$. At that, the non-vanishing asymptotically constant part of the potential, $V_s|_\infty$, may be attributed to the cosmological term. The Lagrangian for gravity is to be supplemented by that for the non-gravitational matter, $L_m(\psi, R, X, \nabla_\mu X, \ldots)$, with ψ designating a generic matter field. In principle the latter may correspond both to the luminous matter and a putative non-gravitational DM. Finally, instead of being imposed explicitly, the restriction (7) may be enforced classically with the help of the undetermined Lagrange multiplier λ by adding to L the term

$$L_\lambda = \lambda\left(\sqrt{-g_*}/\sqrt{-g} - e^{-X}\right). \tag{10}$$

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[5]To be distinguished from the "plain" (by default, monomode/tensor) UG sometimes used as a synonym of UR. Under the phenomenon of unimodular gravity we understand generically both the unimodular bimode and monomode/tensor gravities, the latter of them treated as a marginal case. The term bimode gravity refines the more sophisticated one – "metagravity" – used previously in [22, 23].

This allows one to treat X as an independent variable.

Global compression symmetry First of all, in the spirit of the effective field theory, the extra terms in Lagrangian containing derivatives of X are to be suppressed compared to the kinetic term for X by powers of $1/\kappa_s$, and thus may be neglected. To further terminate the Lagrangian, enhance the residual unimodular invariance by a dynamical global symmetry defined in the fixed coordinates through the field substitutions as follows:

$$
\begin{aligned}
g_{\mu\nu}(x) &\rightarrow \hat{g}_{\mu\nu}(x) = k_0^2 g_{\mu\nu}(k_0 x), \\
g(x) &\rightarrow \hat{g}(x) = k_0^8 g(k_0 x), \\
g_*(x) &\rightarrow \hat{g}_*(x) = g_*(k_0 x),
\end{aligned}
\tag{11}
$$

with $k_0 > 0$ being an arbitrary constant. This is a generic symmetry distinguishing the dynamical and non-dynamical fields. For the former ones it coincides (modulo the coordinate redefinition) with the conventional global scale symmetry, being a part of the general coordinate transformations. For this reason, the general invariant part of L_{UBG} (L_g and, supposedly, L_m) is global symmetric. A common multiplicative factor appearing in $\mathcal{L}$ due to $\sqrt{-g}$ does not influence the classical equations. Eq. (11) will be referred to as the *compression* transformations. In these terms, the moduli are "incompressible" in contrast to the dynamical metric. Further, it follows from (11) that X transforms inhomogeneously under the compressions:

$$
X(x) \rightarrow \hat{X}(x) = X(k_0 x) + 4 \ln k_0.
\tag{12}
$$

The emergence of the respective (approximate) global symmetry may serve as a natural reason for suppression of the derivativeless couplings of the scalar graviton as violating the global symmetry (e.g., $V_s(X)$, etc.). Due to (12) the scalar graviton may be treated as a (pseudo-)Goldstone boson for a hidden/non-linear realization of the compression symmetry. Associated with the scale invariance, such a scalar graviton/(pseudo-)Goldstone boson may more specifically be termed the *systolon*,[6] with the tensor graviton being conventionally just the graviton. The systolon presents a scalar compression mode in metric in addition to the transverse-tensor, four-volume preserving deformation mode presented by the (massless) graviton. Finally, imposing

$$
\lambda(x) \rightarrow \hat{\lambda}(x) = k_0^4 \lambda(k_0 x),
\tag{13}
$$

one gets that L_λ does not violate global symmetry, too.

The (approximate) global compression symmetry may be used as the third basic ingredient of UBG, in addition to general covariance and unimodular invariance/relativity. To such a reduced version of the theory we will adhere in what follows. This is a minimal bimode extension to UR. Namely, from (9) and (10) in a formal limit of "switching-off" the scalar mode, $X \rightarrow 0$, one recovers at the classical level UR, the latter, in turn, being classically equivalent to GR with a cosmological term. For more generality, we still retain the terms $V_s(X)$ and $L_m(X)$ assuming that they are the leading corrections to the otherwise global symmetric Lagrangian.[7]

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[6]From the Greek $\sigma\upsilon\sigma\tau o\lambda\acute{\eta}$ meaning the compression, contraction.

[7]UBG may be considered as a general covariant counterpart of a theory of gravity [18]–[20] comprising in metric the (tensor) graviton and dilaton, but restricted ab initio (hereof the term "restricted gravity") exclusively to coordinates corresponding in UBG to $g_* = -1$. The general covariance proves though to be crucial for real treating the theory and associating it with DM. In passing, it is the gauge invariance which is in fact restricted, the gravity itself being rather extended.

Extended gravity equations Varying the total action (assuming $\Delta L_{gs} = -V_s$) with respect to $g^{\mu\nu}$ and X independently one gets the tensor and scalar gravity equations, respectively, as follows:

$$\kappa_g^2\Big(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\Big) - \lambda e^{-X}g_{\mu\nu} = T_{m\mu\nu} + t_{s\mu\nu},$$
$$\kappa_s^2\nabla\cdot\nabla X + \partial V_s/\partial X - \lambda e^{-X} = \partial L_m/\partial X, \qquad (14)$$

with

$$\nabla\cdot\nabla X = g^{\mu\nu}\nabla_\mu\nabla_\nu X = \partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu X)/\sqrt{-g} \qquad (15)$$

being the general covariant d'Alambertian operator. Conventionally,

$$T_{m\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}L_m)}{\partial g^{\mu\nu}} = 2\partial L_m/\partial g^{\mu\nu} - L_m g_{\mu\nu} \qquad (16)$$

is the canonical energy-momentum tensor of the non-gravitational matter, and $t_{s\mu\nu}$ is the similar scalar-field tensor given by the unconstrained L_s:

$$t_{s\mu\nu} = \nabla_\mu\sigma\nabla_\nu\sigma - \Big(\frac{1}{2}\nabla\sigma\cdot\nabla\sigma - V_s\Big)g_{\mu\nu}. \qquad (17)$$

In the above we have conventionally introduced the dimensionfull scalar field $\sigma = \kappa_s X$. Introducing W_s as κ_s times the canonical inhomogeneous scalar-field wave operator:

$$W_s = \kappa_s(\nabla\cdot\nabla\sigma + \partial V_s/\partial\sigma - \partial L_m/\partial\sigma) \qquad (18)$$

one may present the scalar gravity equation as follows

$$W_s = \lambda e^{-\sigma/\kappa_s}, \qquad (19)$$

with the r.h.s. playing the role of a (virtual) source density.

Excluding λ from (14) we finally arrive at the extended gravity equations in the superficially conventional form

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{\kappa_g^2}T_{\mu\nu}, \quad T_{\mu\nu} = T_{m\mu\nu} + T_{s\mu\nu}. \qquad (20)$$

Here $T_{\mu\nu}$ is the total "bare" energy-momentum tensor, with the tensor gravity included only in the minimal fashion through metric. At that, the bare effective tensor for systolons

$$T_{s\mu\nu} = t_{s\mu\nu} + W_s g_{\mu\nu} \qquad (21)$$

looks like the canonical one, $t_{s\mu\nu}$, except for the Lagrangian potential V_s substituted by the effective one

$$\Lambda_s = V_s + W_s, \qquad (22)$$

with W_s given by (18). Otherwise, $T_{s\mu\nu}$ may formally be brought to the form:

$$T_{s\mu\nu} = (\rho_s + p_s)n_\mu n_\nu + \rho_s g_{\mu\nu}, \qquad (23)$$

where $n_\mu \equiv \nabla_\mu\sigma/(-\nabla\sigma\cdot\nabla\sigma)^{1/2}$ ($n\cdot n = -1$) and

$$\rho_s = -\nabla\sigma\cdot\nabla\sigma/2 + \Lambda_s, \quad p_s = -\nabla\sigma\cdot\nabla\sigma/2 - \Lambda_s. \qquad (24)$$

Superficially, this has little to do with a continuous medium, the account for the proper gravity contribution being to this end in order (see Sec. 4.1).

Finally, varying L_λ with respect to λ one recovers the constraint (7) which is to be understood in (20). The non-canonical contribution to $T_{s\mu\nu}$ appears ultimately from the scalar-field kinetic term due to such a constraint, missing in GR with a scalar field. This may be inferred directly without Lagrange multiplier, with $T_{s\mu\nu}$ appearing as a canonical energy-momentum tensor from the Lagrangian (9) with the explicit constraint (7) [22].

Presenting (20) in the equivalent form

$$R_{\mu\nu} = \frac{1}{\kappa_g^2}\Big(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\Big), \tag{25}$$

$T \equiv T_\lambda^\lambda = T_0^0 + T_l^l$, $l = 1, 2, 3$, we find in particular that

$$R_0^0 = (T_0^0 - T_l^l)/(2\kappa_g^2). \tag{26}$$

This is of special importance for revealing the energy content of a static space structures (see Sec. 4.1). In the empty space, with $L_m = 0$ except possibly for a point, Eq. (25) reads

$$R_{\mu\nu} = \frac{1}{\kappa_g^2}\Big(\nabla_\mu\sigma\nabla_\nu\sigma - \Lambda_s g_{\mu\nu}\Big), \tag{27}$$

where Λ_s is given in general by (22) and (18). Studying this equation and consequences thereof in the static spherically symmetric case is the main concern of the following. Under the quasi-harmonicity condition, $W_s = 0$ except possibly for a point, Eq. (27) reduces to the conventional Einstein-Klein-Gordon system in GR for a self-interacting scalar field minimally coupled to gravity, the solutions for both theories thus coinciding in this case.

Spontaneous global symmetry breaking   Due to the contracted Bianchi identity, $\nabla_\mu(R_\nu^\mu - R/2\,\delta_\nu^\mu) = 0$, the energy-momentum tensor of the non-gravitational matter and systolons collectively, not generally by parts, is bound to covariantly conserve, $\nabla_\mu T_\nu^\mu = 0$. In view of (18) this results in the third-order differential equation

$$\partial_\nu W_s + W_s \partial_\nu\sigma/\kappa_s = -\nabla_\mu T_{m\nu}^\mu, \tag{28}$$

which as it stands is hardly of use. However, in the case if L_m is independent of σ (in particular, in empty space, $L_m = 0$), the emerging general invariance of the matter action implies that $\nabla_\mu T_{m\nu}^\mu = 0$. Thus $\partial_\nu(\ln|W_s| + \sigma/\kappa_s) = 0$, and there appears the first integral of motion

$$W_s = \Lambda_0 e^{-\sigma/\kappa_s}, \tag{29}$$

with Λ_0 being an integration constant. It follows then from (19) that now $\lambda = \Lambda_0$. With account for (13) this signifies spontaneous breaking of the global symmetry $\sigma \to \sigma + \sigma_0$, with σ_0 being a constant. In view of (18), Eq. (29) becomes nothing but the bona fide second-order scalar-field equation:

$$\nabla \cdot \nabla\sigma + \partial\Lambda_s/\partial\sigma = 0, \tag{30}$$

with the effective potential looking like the conventional one plus the exponential contribution

$$\Lambda_s = V_s + \Lambda_0 e^{-\sigma/\kappa_s}. \tag{31}$$

In this case UBG is classically equivalent to GR with a scalar field supplemented by the exponential potential in Lagrangian (with an alien for GR scale κ_s).[8] At $\Lambda_0 = 0$ the theories moreover classically coincide. For physical reason, there should fulfil $\Lambda_0 \leq 0$ (see Sec. 4.1). This means that due to a potential well produced by a systolon condensate, the local vacuum with the spontaneously broken global symmetry lies lower than the symmetric one ($\Lambda_0 = 0$), ensuring ultimately spontaneous breaking of the symmetry.

UBG and beyond Let us make some remarks posing UBG among the related metric theories of gravity with the additional degree(s) of freedom, which may have bearing to DM.

(i) *Unimodular scalar-tensor gravity* First of all, putting $g_{\mu\nu} \equiv (g/g_*)^{1/4} g_{u\mu\nu}$, with $\det g_{u\mu\nu} \equiv g_u = g_*$, one may equivalently present (6) as $\mathcal{L} = L(g_{u\mu\nu}, X)\sqrt{-g_*}$, where $X = X(g/g_*)$ may be chosen, e.g., as in (7). Under the global compression symmetry one has $g_{u\mu\nu}(x) \to \hat{g}_{u\mu\nu}(x) = g_{u\mu\nu}(k_0 x)$, while g_* and X transform as before. Now choosing $\sigma \equiv \kappa_s X$ as an independent scalar field one may reduce the original theory to the 10-field unimodular scalar-tensor gravity, which proves to be nothing but the unimodular bimode gravity in disguise. The inverse transition is achieved by putting $g_{u\mu\nu} \equiv e^{-\sigma/2\kappa_s} g_{\mu\nu}$ and, respectively, $\sigma/\kappa_s = (1/2) \ln g/g_*$. The two theories may thus be treated just as the generic UBG in the two different frames: the bimode and scalar-tensor ones.[9]

(ii) *General scalar-tensor gravity* More generally, one may confront UBG, containing 10 dynamical fields, with the general invariant theory defined by the arbitrary $\mathcal{L} = L(g_{\mu\nu}, \sigma)\sqrt{-g}$ containing additionally an independent scalar field σ, altogether 11 dynamical fields.[10,11] As it was stated earlier, under the separate conservation of the energy-momentum tensor of the non-gravitational matter (in particular, in the matterless vacuum) UBG is classically equivalent to the general scalar-tensor gravity, with a supplementary exponential contribution to the Lagrangian potential of the latter emulating the proper spontaneously emerging term in the classical equations of the former. Nevertheless, in the context of DM there are two important differences. First, to match with astronomic observations the sign of the extra contribution is to be negative (see Sec. 4.2). Being harmless in the classical equations such an unbounded from below term in the Lagrangian could result in a quantum inconsistency. Second, appearing as an integration constant in the classical equations, Λ_0 may vary for different solutions, whereas in the Lagrangian potential it should be fixed ad hoc ones forever. Thus in the DM context UBG seems to be more safe and flexible compared to the general scalar-tensor gravity.

(iii) *Bimetric multimode gravity* In a wider perspective of the general relativity violation one may consider the multimode gravity, with the general covariant Lagrangian density $\mathcal{L} = L(g_{\mu\nu}, g_{*\mu\nu})\sqrt{-g}$ depending on the dynamical and non-dynamical metrics $g_{\mu\nu}$ and $g_{*\mu\nu}$, respectively, as well as their determinants. Basically, such a theory is characterized by its residual gauge invariance $H \subseteq G$. Containing 10 independent dynamical fields, the theory with the trivial $H = I$ may thus encounter up to 10 physical degrees of freedom (including, in general, the ghost ones) and present potentially a lot of problems

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[8]This statement resembles a similar one about the classical equivalence between UR and GR supplemented by the cosmological term in Lagrangian. This can be seen at $V_s = 0$ in a formal limit $\sigma \to 0$.

[9]In the latter frame the unimodularity literally means the invariance of g_u.

[10]For a confrontation between the transverse/TDiff gravity and GR with a scalar field see [30].

[11]As a particular implementation of such a theory there may be mentioned the well-known Brans-Dicke model [31]. In the Einstein frame the scalar field in this version of the theory acquires the direct derivativeless coupling with matter being thus strongly restricted observationally (see, e.g., [32]).

and ambiguities. To eliminate/reduce them the residual unimodular invariance, $H = U$, may, e.g., be imposed. In conjunction with general covariance this would retain the dependence of L just on $g_{\mu\nu}$ and g/g_*, with $g_* = \det(g_{*\mu\nu})$, and no more than a single extra graviton. Nevertheless, in the wake of the developed approach to DM one could envisage in this direction a multimode gravitational DM.[12]

3 Equations and solutions

3.1 Static spherical symmetry

Quasi-Galilean coordinates   Consider the static spherically symmetric field configurations. Start with the quasi-Galilean coordinates $x^\mu = (x^0, x^m) \equiv (t, \mathbf{x})$ $(m = 1, 2, 3)$, where the general covariant line element is as follows:

$$ds^2 = a\,dt^2 - (b - c)(\mathbf{n}d\mathbf{x})^2 - c\,d\mathbf{x}^2. \tag{32}$$

Here $\mathbf{n}$ $(\mathbf{n}^2 = 1)$ is given by $n^m = x^m/|\mathbf{x}|$, with $|\mathbf{x}|^2 = \mathbf{x}^2 \equiv \delta_{mn}x^m x^n$. The metric variables a, b and c are arbitrary functions of $|\mathbf{x}|$ alone. The same concerns g_* (and thus σ). The respective metric looks like

$$g_{00} = a, \quad g_{mn} = -b\bar{n}_m\bar{n}_n - c(\delta_{mn} - \bar{n}_m\bar{n}_n). \tag{33}$$

where for short we designated $\bar{n}_m \equiv \delta_{mn}n^n$ (vs. conventional $n_m = g_{mn}n^n$). The rest of the metric elements is zero. Respectively, the inverse metric is

$$g^{00} = \frac{1}{a}, \quad g^{mn} = -\frac{1}{b}n^m n^n - \frac{1}{c}(\delta^{mn} - n^m n^n). \tag{34}$$

Due to the rotation invariance one can choose the spatial coordinates in a point $\mathbf{x}$ so that $\bar{\mathbf{n}} = (1, 0, 0)$, bringing the metric in this point to the diagonal form $g_{\mu\nu} = \mathrm{diag}\,(a, -b, -c, -c)$. Thus $\sqrt{-g} = \sqrt{abc}$, and the scalar field looks like

$$\sigma = \kappa_s \ln(\sqrt{abc}/\sqrt{-g_*}). \tag{35}$$

The quasi-Galilean coordinates are appropriate for revealing the energy content of the static space structures (see Sec. 4.1).

Polar coordinates   For real dealing with the spherical symmetry better suit the polar coordinates $x^\mu = (x^0, x^m) = (t, r, \theta, \varphi)$, $m = r, \theta, \varphi$ (with a unit of length l_0 tacitly understood where it is necessary). Conventionally, $\mathbf{x} = r\mathbf{n} = r(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. The line element now reads

$$ds^2 = a\,dt^2 - b\,dr^2 - cr^2 d\Omega^2, \quad d\Omega^2 = d\theta^2 + \sin^2\theta\,d\varphi^2, \tag{36}$$

with the metric being $g_{\mu\nu} = \mathrm{diag}(a, -b, -cr^2, -cr^2\sin^2\theta)$. The scalar field looks as before, with the transformation Jacobian cancelled out in the ratio of the two scalar densities of the same weight. The functions a, b, c and g_* depend on r alone.

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[12]For the bimetric gravity in the context of the massive tensor graviton, with a priori unspecified $g_{*\mu\nu}$ and extra terms only in the Lagrangian potential, cf., e.g., [33, 34].

The systolon bare effective energy-momentum tensor (17), (21) and (22) is then $T_{s\nu}^{\mu} \equiv \text{diag}(p_0, -p_r, -p_\theta, -p_\varphi)$, where

$$p_0 = -p_\theta = -p_\varphi = \frac{1}{2b}\sigma'^2 + \Lambda_s,$$

$$p_r = \frac{1}{2b}\sigma'^2 - \Lambda_s, \tag{37}$$

with p_0 being the bare energy density, p_n ($n = r, \theta, \varphi$) the bare pressure and prime indicating a radial derivative.[13] There follows hereof, in particular, an important spatial invariant (see Sec. 4.1):

$$T_{s0}^{\ 0} - T_{sn}^{\ n} = p_0 + \Sigma p_n = -2\Lambda_s. \tag{38}$$

With R_ν^μ being diagonal and $R_\varphi^\varphi = R_\theta^\theta$, there is left three independent gravity equations which may be brought to the form:

$$R_0^0 = \frac{1}{2\kappa_g^2}\left(T_{m0}^{\ 0} - T_{mn}^{\ n} - 2\Lambda_s\right),$$

$$R_0^0 - R_r^r = \frac{1}{\kappa_g^2}\left(T_{m0}^{\ 0} - T_{mr}^{\ r} + \frac{1}{b}\sigma'^2\right),$$

$$R_0^0 - R_\theta^\theta = \frac{1}{\kappa_g^2}\left(T_{m0}^{\ 0} - T_{m\theta}^{\ \theta}\right). \tag{39}$$

In the above the curvature elements are expressed through a, b and c. The contracted Bianchi identity reads

$$W_s' + W_s\sigma'/\kappa_s = -T_{mr}'^{\ r}. \tag{40}$$

Under the general invariance (σ-independence) of L_m (in particular, at $L_m = 0$) one has $T_{mr}'^{\ r} = 0$, and there appears the first integral of motion $W_s = \Lambda_0 e^{-\sigma/\kappa_s}$, with Λ_0 being an integration constant. The third-order equation (40) reduces in this case to the second-order scalar-field equation (30) which looks now like

$$\nabla \cdot \nabla \sigma = -\left(\sqrt{a/b}cr^2\sigma'\right)'/\sqrt{ab}cr^2 =$$

$$= (\Lambda_0/\kappa_s)e^{-\sigma/\kappa_s} - \partial V_s/\partial\sigma. \tag{41}$$

Radial coordinate/gauge fixing   The choice of the radial coordinate r is not unique. Under the local radial rescaling $r \to \hat{r} = \hat{r}(r)$, the general covariant line element is moreover formal invariant: $ds^2 = d\hat{s}^2 = \hat{a}dt^2 - \hat{b}d\hat{r}^2 - \hat{c}\hat{r}^2d\Omega^2$, with the relation

$$a(r) = \hat{a}(\hat{r}(r)),$$
$$b(r) = (d\hat{r}/dr)^2\hat{b}(\hat{r}(r)),$$
$$c(r) = (\hat{r}/r)^2\hat{c}(\hat{r}(r)). \tag{42}$$

The quasi-Galilean $\sqrt{-g} = \sqrt{abc}$ transforms under rescaling as

$$\sqrt{-g(r)} = \sqrt{-\hat{g}(\hat{r}(r))}\,\frac{\hat{r}^2}{r^2}\frac{d\hat{r}}{dr} \tag{43}$$

(and so does $\sqrt{-g_*}$), with σ transforming as a scalar, $\sigma(r) = \hat{\sigma}(\hat{r}(r))$. The three-volume element $\sqrt{-g}d^3x = \sqrt{abc}r^2drd\Omega^2$, being (spatial) invariant, is also formal invariant. It

---

[13]Here and in what follows, we define for simplicity the energy densities without $\sqrt{-g}$.

follows from (42) that metric, asymptotic Minkowskian with respect to r, retains this property with respect to $\hat{r}$ if there fulfills asymptotically $\hat{r} = r(1 + \mathcal{O}(1/r))$. Thus, the ansatz (36) does not allow to fix the metric uniquely.

The aforesaid ambiguity can be removed by the radial coordinate/gauge fixing. Namely, the three gravity equations (39) contain superficially four variables: a, b, c and σ, only three of them being independent due to constraint (35). To account for the latter, two opposite routes of dealing with a priori unknown g_* are envisaged.

(i) *Explicit* g_* The direct route is to assume g_* in some starting coordinates and solve the equations with the proper boundary conditions. Then postulating a relation between the starting and observer's coordinates, transform the solution to the latter coordinates, including boundary conditions. In practice, one may start from the transverse coordinates defined by $g_* = -1$, what is equivalent to imposing the scalar-field dependent gauge $\sqrt{-g} = e^X$. In the arbitrary observer's coordinates the solution will thus explicitly comprise $g_* \neq -1$.

(ii) *Implicit* g_* The inverse route is to get (virtually) rid of g_* in favour of σ. Imposing then a gauge $F(a, b, c) = 0$ as a definition of observer's coordinates, solve the equations, with the proper boundary conditions imposed already in the latter coordinates. The dependence on g_* remains implicit. Having found the solution one can extract hereof the required g_* solving, in a sense, an inverse problem. Inserted back into the gravity equations this g_* is bound to produce in the direct route in the same observer's coordinates precisely the given solution. Having a priori no knowledge about g_* we will adopt in what follows such an inverse route, with several particular gauges compared (see Sec. 3.2).

Reciprocal gauge It proves to be convenient to impose the gauge $ab = 1$ (refer to it as the reciprocal one).[14] Note that as it stands this gauge is not invariant under the dynamical global symmetry (11). Designate $A \equiv a = 1/b$, $C \equiv r^2 c$ (some element of length l_0 is put to unity here).

With account for

$$
\begin{aligned}
R_0^0 &= \frac{1}{2}\frac{(CA')'}{C}, \\
R_r^r &= \frac{1}{2}\frac{(CA')'}{C} + A\Big(\frac{C''}{C} - \frac{1}{2}\frac{C'^2}{C^2}\Big), \\
R_\theta^\theta &= \frac{1}{2}\frac{(AC')'}{C} - \frac{1}{C}
\end{aligned}
\tag{44}
$$

and $\nabla \cdot \nabla X = -(ACX')'/C$, the gravity equations look like

$$
\begin{aligned}
CA'' + C'A' &= v_s^2(ACX')' - \\
-\frac{2C}{\kappa_g^2}\Big(V_s + \frac{\partial V_s}{\partial X} &- \frac{\partial L_m}{\partial X}\Big) + \frac{C}{\kappa_g^2}\Big(T_{m0}^{\,0} - T_{mn}^{\,n}\Big), \\
(\ln C)'' + \frac{1}{2}(\ln C)'^2 &= -\frac{v_s^2}{2}X'^2 - \frac{1}{\kappa_g^2}\frac{1}{A}\Big(T_{m0}^{\,0} - T_{mr}^{\,r}\Big), \\
CA'' - AC'' &= -2 + \frac{2C}{\kappa_g^2}\Big(T_{m0}^{\,0} - T_{m\theta}^{\,\theta}\Big),
\end{aligned}
\tag{45}
$$

---

[14]The ultimate reason is the appearance in this gauge of an explicit exact solution valid also in GR with a scalar field (see Sec. 3.2).

where $\upsilon_s^2 \equiv 2\kappa_s^2/\kappa_g^2$. With κ_g taken as an overall mass scale, υ_s is the single free parameter of the vacuum gravity Lagrangian (under $V_s = 0$). According to (45) υ_s^2 plays the role of a coupling between the scalar and tensor gravities. (For its observational meaning see Sec. 4.2.) With $\upsilon_s \sim 10^{-3}$ the coupling proves to be weak.

Lagrange multiplier ansätze   In the equations above, the Lagrange multiplier is excluded. It may be revealed through (19) as follows:

$$- \kappa_s^2 (ACX')'/C + \partial V_s/\partial X - \partial L_m/\partial X = \lambda e^{-X}. \qquad (46)$$

Given a solution to (45), Eq. (46) defines λ. V.v., making an ansatz for λ (to be confirmed) one can look for a respective solution to (45) (if any). This allows one to (partly) disentangle the scalar and tensor gravity equations.

Being interested in the case of the empty, but possibly for the origin, space we envisage at $L_m = 0$ the three following ansätze.

(*i*) *Singular ansatz*   Here $\lambda = \Delta_0$, with Δ_0 being a δ-type function concentrated at the origin and determined implicitly through self-consistency.

(*ii*) *Regular ansatz*   Here $\lambda = \Lambda_0$ everywhere (including the origin), with Λ_0 being an arbitrary constant.

(*iii*) *Interpolating ansatz*   Here

$$\lambda \simeq \begin{cases} \Delta_0, & r < r_0, \\ \Lambda_0, & r > r_0, \end{cases} \qquad (47)$$

with r_0 being some matching distance. In reality λ should be smoothed around r_0. Consider these ansätze in turn (neglecting by V_s).

3.2   Quasi-harmonic solution

Reciprocal gauge   The singular ansatz $\lambda = \Delta_0$ reduces to the quasi-harmonicity condition $\nabla \cdot \nabla X = 0$, or $(ACX')' = 0$ (except for the origin). The exact solution to the proper equations was found previously in [24]. Here we reproduce it in the nutshell. Combining the first and the last gravity equations (45) one gets $(AC)'' = 2$ with the ensuing relation

$$AC \equiv \Delta = (r - r_1)(r - r_2), \qquad (48)$$

where the roots r_1 and r_2 are a priori either real or complex-conjugate of each other. Introduce a new radial coordinate

$$\chi = - \int \frac{dr}{\Delta}, \qquad (49)$$

with $\chi = 1/r + \mathcal{O}(1/r^2)$ at $r \to \infty$ (a unit of length l_0 is understood here and in what follows). In these terms the first gravity equation (45) and the scalar-field equation (46) reduce to the free harmonic form

$$d^2 \ln A/d\chi^2 = d^2 X/d\chi^2 = 0, \qquad (50)$$

with general solution

$$\ln A = \ln A_0 - \nu_0 \chi, \quad X = X_0 - \varsigma_0 \chi, \qquad (51)$$

depending on the integration constants A_0, X_0, ν_0 and ς_0. Conventionally, impose the asymptotic free boundary conditions $A = 1$ and $X = 0$ at $\chi = 0$ ($r \to \infty$), so that $A_0 = 1$

and $X_0 = 0$. Substituting $\ln C = \ln \Delta - \ln A$ into the second gravity equation (45) we finally find the integration condition

$$\nu_0^2 + \upsilon_s^2 \varsigma_0^2 = (r_1 - r_2)^2. \tag{52}$$

It follows that for solution to be real it is necessary that roots be real $((r_1 - r_2)^2 \geq 0)$.[15] In this case we get from (49) at $r_1 \neq r_2$ (let by default $r_1 \geq r_2$):

$$\chi = -\frac{1}{r_1 - r_2} \ln \frac{r - r_1}{r - r_2} \tag{53}$$

(modulo a constant), or inversely

$$r = \frac{r_1 + r_2}{2} + \frac{r_1 - r_2}{2} \operatorname{cth} \frac{(r_1 - r_2)\chi}{2}. \tag{54}$$

At $r_1 = r_2 \equiv r_0$ one has $\chi = 1/(r - r_0)$ and, respectively, $r = r_0 + 1/\chi$. Altogether, the respective line element and the scalar field look like

$$\begin{aligned}
ds^2 &= q^\nu dt^2 - q^{-\nu}(dr^2 + \Delta d\Omega^2), \\
X &= \varsigma \ln q,
\end{aligned} \tag{55}$$

where

$$q = (r - r_1)/(r - r_2), \tag{56}$$

with $\nu = \nu_0/(r_1 - r_2)$, $\varsigma = \varsigma_0/(r_1 - r_2)$ and $\nu^2 + \upsilon_s^2 \varsigma^2 = 1$.

The gravity and scalar-field equations in the reciprocal gauge, (45) and (46), still possess in the vacuum a residual invariance under the global shifts of r. Due to the invariance one can redefine r as $r \to r + r_2$, so that $q = 1 - r_f/r$ and $\Delta = r^2 q$, with $r_f \equiv r_1 - r_2 \geq 0$. The looked-for solution (designate it with a subscript f) acquires the following standard form:

$$\begin{aligned}
a_f = 1/b_f &= (1 - r_f/r)^{\nu_f}, \\
c_f &= (1 - r_f/r)^{1-\nu_f}, \\
\sqrt{2}\sigma_f/\kappa_g \equiv \Sigma_f = \upsilon_s X_f &= \eta_f \sqrt{1 - \nu_f^2} \ln(1 - r_f/r),
\end{aligned} \tag{57}$$

with $\nu_f = \nu_0/r_f$ and a signature factor $\eta_f = \pm 1$. The solution depends on two canonical parameters r_f and ν_f.[16]

For consistency there should fulfil $|\nu_f| \leq 1$. The solution is unique (up to constants A_0 and X_0) and bound to be singular, with the regular solution being necessarily trivial ($r_f = 0$). Note that the particular case $\eta_f = 1$, $\nu_f = \upsilon_s/\sqrt{1 + \upsilon_s^2}$ results in the relation $\ln a_f = \upsilon_s^2 X_f$, which proves to correspond to a matterless static space structure (see Sec. 4.1). The two-parameter solution above supersedes the one-parameter Schwarzschild solution for BHs ($\nu_f = 1$, $X_f = 0$). In GR it describes BHs dressed with a (free massless) scalar field.[17] Reflecting a singularity in space, with the coherent scalar field treated as

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[15]In the case of complex-conjugate roots $((r_1 - r_2)^2 < 0)$, the reality of metric ($\nu_0^2 \geq 0$) implies $\varsigma_0^2 < 0$ meaning a ghost scalar-field solution ($X'^2 < 0$).

[16]A more symmetric form of the solution (still in the gauge $ab = 1$) would correspond to the shift $r \to r + (r_1 + r_2)/2$, so that $\chi = -(1/r_f) \ln(r - r_f/2)/(r + r_f/2)$, $r = (r_f/2)\operatorname{cth}(r_f\chi/2)$ and $\Delta = r^2 - (r_f/2)^2$.

[17]In GR the respective solution was first obtained in an implicit form in the gauge $c = 1$ by Bergmann and Leipnik [35] (see also [36]). In the explicit form (57) it was discovered somewhat later in a different context by Buchdahl [37] and was then extensively studied in literature including modification by the scalar-field self-interaction (see, e.g., [38]–[45]).

DM, such a scalar-modified BH may conveniently be referred to as the dark *fracture*. Accounting for (35) one finds the respective dark modulus as follows:

$$\sqrt{-g_{*f}} = (1 - r_f/r)^{1-\mu_f}, \tag{58}$$

where

$$\mu_f = \nu_f + \eta_f\sqrt{1 - \nu_f^2}/v_s. \tag{59}$$

Now, it is possible to reverse the line of reasoning by saying that the dark fracture is the static space structure which originates from the dark modulus (58) and is given by (57), with $\nu_f = \nu_f(\mu_f)$. In particular, to the constant modulus ($\mu_f = 1$) there corresponds BH ($\nu_f = 1$). At $v_s \ll 1$ there fulfills $\nu_f \simeq 1 - v_s^2(\mu_f - 1)^2/2$, with ν_f being close to unity in a wide range of μ_f. Thus, the appearance of quasi-BHs ($|\nu_f - 1| \ll 1$) is parametrically enhanced. For an arbitrary fracture the modulus, being complex singular, can not be brought to $g_{*f} = -1$ by any real coordinate transformations.

In general, the solution is complex interior to r_1 and, as such, should be treated here as an analytical continuation from the exterior region $r > r_1$ into the complex r-plane with the cut $(-\infty, r_1)$. Moreover, the event horizon at $r = r_1$ proves to be point-like singular [38]. To avoid the complexity one could perform the admitted shift $r \to r + r_1$, so that $q = 1/(1 + r_f/r)$ and $\Delta = r^2/q$. At $r > 0$ the solution becomes now real and regular, with a singularity only at $r = 0$. The same concerns the modulus, which may now be brought to unity (but for $r = 0$). The asymptotic of the solution being unchanged under the shift, the latter may be considered as a kind of "realization". Being mathematically equivalent, such a "truncated" real form seems physically more reasonable. Nevertheless, ultimate treating the complex singularity would be of considerable interest.[18]

Until stated otherwise, we pursue the standard form of the solution. Decomposing the latter in $1/r$ one gets in the leading approximation:

$$\begin{aligned}
a_f = 1/b_f &= 1 - r_g/r, \\
c_f &= 1 - r_c/r, \\
\Sigma_f = v_s X_f &= -r_s/r,
\end{aligned} \tag{60}$$

where $r_c = \sqrt{r_g^2 + r_s^2} - r_g$. Here r_g and r_s are two independent phenomenological parameters

$$r_g = \nu_f r_f, \quad r_s = \eta_f\sqrt{1 - \nu_f^2}\,r_f, \tag{61}$$

or inversely

$$r_f = \sqrt{r_g^2 + r_s^2}, \quad \nu_f = r_g\big/\sqrt{r_g^2 + r_s^2}. \tag{62}$$

Taking as independent r_f and r_g one has

$$\nu_f = r_g/r_f, \quad r_c = r_f - r_g, \quad r_s = \eta_f\sqrt{r_f^2 - r_g^2}. \tag{63}$$

For definiteness, we consider the case $r_f \geq r_g \geq 0$, with r_s remaining sign-indefinite ($\eta_f = \pm 1$). The parameters r_g and r_s fix the Newtonian-Coulombic approximation and have the meaning, respectively, of the gravitational and scalar radii of a fracture. Under the radial rescaling $r \to \hat{r}(r)$, with $\hat{r} = r(1 + \mathcal{O}(1/r))$ asymptotically, these parameters

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[18]For treating the interior of BH as unattainable due to its event horizon shrinking to a point singularity by dressing with a massless scalar field minimally coupled to gravity cf. [38].

are invariant and may serve as a substitute to the canonical ones, r_f and ν_f, with r_f being the radius of a fracture and ν_f its "nudity". (BHs, $\nu_f = 1$, are the "nude" fractures.)

Now we may reverse the approach. Namely, in an exterior region where $L_m = 0$ let us look for an asymptotic free solution to (45), with $V_s = 0$ and $\nabla \cdot \nabla X = -(ACX')'/C = 0$. With $\Sigma = v_s X$, decompose the solution in a power series of $1/r$. Starting from (60) with a priori arbitrary parameters r_g, r_c and r_s, we get in the second order the restriction $r_c(r_c + 2r_g) = r_s^2$. Afterwards we can step-by-step uniquely reconstruct the quasi-harmonic solution (57) in the empty space within the series convergence region $r > r_f$. Because we have nowhere used any assumptions about the distribution of matter interior to r_f, but for spherical symmetry, the exterior vacuum solution is to be uniquely determined by the two interior integral characteristics corresponding to r_g and r_s, independent of the details of the interior distribution. Physically, r_g reflects the net gravitating energy of a fracture and r_s its net systolon energy (see Sec. 4.1). If some "reasonable" matter distributions exist, but for the point-like one, this would extend to UBG the Birkhoff uniqueness theorem in GR. The Buchdahl solution may thus be to dark fractures as the Schwarzschild solution is to BHs.

In the above we have in fact shown that the gauge most appropriate to the problem at hand from the point of view of its physical content is the harmonic gauge. It may be given by requirement $\nabla \cdot \nabla \hat{X} \sim d^2 \hat{X}/d\hat{r}^2$, or in view of (41) by $(\hat{a}/\hat{b})^{1/2}\hat{c}\hat{r}^2 = 1$, with $\hat{r}$ substituting χ. (For completeness though, we should have found $\hat{b}$ and $\hat{c}$.) The original reciprocal gauge ($b = 1/a$) may, in turn, be more appropriate for studying the quasi-harmonic solution in the exterior region. Below we shortly compare the exterior solution in several radial coordinates/gauges peculiar from supplementary points of view.

Astronomic gauge The coordinates where $\hat{c} = 1$ are peculiar in astronomy, with the surface of a concentric sphere being conventionally $\hat{S} = 4\pi\hat{r}^2$. In view of (42) such a gauge results in the exterior coordinate transformation

$$\hat{r} = \sqrt{c(r)}\, r = r(1 - r_f/r)^{(1-\nu_f)/2}, \quad r > r_f. \tag{64}$$

The inverse to this relation being given only implicitly, the exact solution in these coordinates [35] can not, unfortunately, be presented in an explicit form (but for $\nu_f = 1$).

Unimodular gauge The gauge $\sqrt{-\hat{g}} = (\hat{a}\hat{b})^{1/2}\hat{c} = 1$ is peculiar by the fact that here the modulus is directly reflected by scalar field, $\sqrt{-\hat{g}_*} = e^{-\hat{X}}$. In view of (43) this gauge results in the exterior coordinate transformation

$$\hat{r} = \left(3 \int c(r)r^2 dr\right)^{1/3} =$$
$$= \left(3 \int (1 - r_f/r)^{1-\nu_f} r^2 dr\right)^{1/3}, \quad r > r_f, \tag{65}$$

with the integration constant set to zero to ensure $\hat{r} = r\left(1 + \mathcal{O}(r_f/r)\right)$ at $r > r_f$.

Transverse gauge This gauge is peculiar from the theoretical considerations. It is given by $\hat{g}_* = -1$, with the group of diffeomorphisms being the transverse one ($\partial_\mu \hat{\xi}^\mu = 0$) in the respective coordinates. Otherwise, this is equivalent to the gauge $\sqrt{-\hat{g}} = e^{\hat{X}}$

(which may also be imposed in GR with a scalar field). Applying (43) to g_* one gets the equation

$$\frac{\hat{r}^2}{r^2}\frac{d\hat{r}}{dr} = \sqrt{-g_*(r)} = \sqrt{-g(r)}e^{-X(r)}, \tag{66}$$

so that in the exterior region

$$\begin{aligned}
\hat{r} &= \left(3\int e^{-X(r)}c(r)r^2 dr\right)^{1/3} = \\
&= \left(3\int (1 - r_f/r)^{1-\mu_f} r^2 dr\right)^{1/3}, \quad r > r_f.
\end{aligned} \tag{67}$$

with the asymptotic relation $\hat{r} = r\big(1 + \mathcal{O}(r_f/r)\big)$. In general, the exact solution can not be presented in this case in the explicit form (including GR with a scalar field). For BHs ($\nu_f = \mu_f = 1$) all the aforementioned gauges/coordinates (but for harmonic) coincide identically implying in particular $g_* = -1$.

Isotropic gauge   This gauge is peculiar from the phenomenological considerations. In view of (33), under the gauge $\hat{c} = \hat{b}$ one has in the quasi-Galilean coordinates $\hat{g}_{mn} = -\hat{b}\delta_{mn}$ ($m, n = 1, 2, 3$), with $ds^2 = a dt^2 - \hat{b}d\hat{\mathbf{x}}^2$. Such a conformal flat spatial metric is natural in confronting with Newton's dynamics. With account for (42) this results in the exterior coordinate transformation

$$\ln \hat{r} = \int \left(\frac{b(r)}{c(r)}\right)^{1/2}\frac{dr}{r} = \int \frac{1}{\sqrt{1 - r_f/r}}\frac{dr}{r}, \tag{68}$$

so that

$$\hat{r} = \frac{r}{4}\left(1 + \sqrt{1 - r_f/r}\right)^2, \quad r > r_f, \tag{69}$$

or inversely

$$r = \hat{r}\left(1 + \frac{r_f}{4\hat{r}}\right)^2, \quad \hat{r} > r_f/4. \tag{70}$$

This gives

$$\begin{aligned}
\hat{a}_f &= \left(1 - \frac{r_f}{4\hat{r}}\right)^{2\nu_f}\bigg/\left(1 + \frac{r_f}{4\hat{r}}\right)^{2\nu_f}, \\
\hat{b}_f = \hat{c}_f &= \left(1 - \frac{r_f}{4\hat{r}}\right)^{2(1-\nu_f)}\left(1 + \frac{r_f}{4\hat{r}}\right)^{2(1+\nu_f)}, \\
\hat{\Sigma}_f = \upsilon_s \hat{X}_f &= 2\eta_f\sqrt{1 - \nu_f^2}\ln\left(\left(1 - \frac{r_f}{4\hat{r}}\right)\bigg/\left(1 + \frac{r_f}{4\hat{r}}\right)\right),
\end{aligned} \tag{71}$$

with $\hat{\Sigma}_f$ being the odd function of $\hat{r}/r_f$. Now the solution preserves the dynamical global symmetry, as well as possesses additionally a hidden symmetry: $r_f \to -r_f$, $\nu_f \to -\nu_f$ and $\eta_f \to -\eta_f$. The same concerns $\hat{g}_*$.

Post-Newtonian approximation   Without loss of generality present the fracture solution in the isotropic polar coordinates (with the hat-sign omitted) as follows:

$$\begin{aligned}
a_f &= 1 - \frac{r_g}{r} + \frac{\beta_f}{2}\frac{r_g^2}{r^2}, \\
b_f = c_f &= 1 + \gamma_f\frac{r_g}{r}, \\
\Sigma_f = \upsilon_s X_f &= -\xi_f\frac{r_s}{r}.
\end{aligned} \tag{72}$$

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Here β_f and γ_f are the effective parameters for a dark fracture in the parametrized post-Newtonian (PPN) formalism. It may be said in general that $(\beta - 1)$ reflects empirically the degree of non-linearity in the superposition law for gravity, while $(\gamma - 1)$ corresponds to the amount of space curvature produced by the unit rest mass [32]. The parameter ξ_f may be treated as an effective form-factor in the scalar-field Coulomb law. Decomposing (71) in $1/r$ and using (62) one gets in the leading approximation

$$
\begin{aligned}
\beta_f &= 1 - \left(1 + \frac{\epsilon_f^2}{9}\right)\frac{3}{8}\frac{r_g}{r}, \\
\gamma_f &= 1 + \left(1 - \frac{\epsilon_f^2}{3}\right)\frac{3}{8}\frac{r_g}{r}, \\
\xi_f &= 1 + \frac{1 + \epsilon_f^2}{48}\frac{r_g^2}{r^2},
\end{aligned}
\tag{73}
$$

with

$$
\epsilon_f \equiv r_s/r_g = \eta_f\sqrt{1/\nu_f^2 - 1}.
\tag{74}
$$

At small ϵ_f a dark fracture closely reproduces BH ($r_s = 0$, $\nu_f = 1$, $\epsilon_f = 0$), with the scalar dressing being thus very neatly hidden. The precision local tests of GR in Solar System result typically in $|\beta - 1| \leq 10^{-4}$ and $|\gamma - 1| \leq 10^{-5}$ [32]. With a value near the Earth $r_S/r \simeq 2 \times 10^{-8}$, r_S being the Sun gravitational radius, this implies for the Sun as a dark fracture just $|\epsilon_S| \leq 10^2$, or $\nu_S \geq 10^{-2}$, the Sun scalar form-factor ξ_S remaining practically unity. At face value this gives very loose restriction.[19] Other observational manifestations are conceivably needed to test the theory (see Sec. 4.2).[20]

3.3 Regular non-harmonic solution

Vacuum scaled distance   To begin with, take $\ln A \equiv \alpha$ and $\ln c \equiv \zeta$ as the new metric variables and present the gravity and the scalar-field equations, respectively, (45) and (46) in the empty, but possibly for the origin, space equivalently as follows:

$$
\begin{aligned}
\frac{d}{dr}\left(e^{\alpha+\zeta}r^2\frac{d}{dr}\left(\alpha - \upsilon_s^2 X\right)\right) &= -\frac{2}{\kappa_g^2}\left(V_s + \frac{\partial V_s}{\partial X}\right), \\
\frac{d}{dr}\left(r^2\frac{d\zeta}{dr}\right) + \frac{1}{2}\left(r\frac{d\zeta}{dr}\right)^2 &= -\frac{\upsilon_s^2}{2}\left(r\frac{dX}{dr}\right)^2, \\
\frac{d}{dr}\left(e^{\alpha+\zeta}r^2\frac{d(\alpha - \zeta)}{dr} + 2r\left(1 - e^{\alpha+\zeta}\right)\right) &= 0, \\
e^{-\zeta}\frac{d}{dr}\left(e^{\alpha+\zeta}r^2\frac{dX}{dr}\right) &= \frac{r^2}{\kappa_s^2}\left(\frac{\partial V_s}{\partial X} - \lambda e^{-X}\right).
\end{aligned}
\tag{75}
$$

It becomes now evident that the second-order system above always has one first integrals, with one more appearing at $V_s = 0$. Moreover, in the latter case the first equation above may always be satisfied with $\alpha = \upsilon_s^2 X$ (modulo a constant), with the set of solutions to the remaining equations being definitely not empty (cf., e.g., Sec. 3.2). Such a solution proves to correspond to a matterless static space structure (see Sec. 4.1).

---

[19]The validity of decomposition in $1/r$ implies that at the distances at hand there should take place $|\epsilon_f| \ll (r/r_g)^{1/2} \simeq 10^4$, being still safely fulfilled.

[20]For discussion of the transverse gravity vs. observations cf. also [28].

Now, under the regular ansatz $\lambda = \Lambda_0 < 0$ everywhere (including the origin) let us introduce in the empty space a characteristic length scale R_0 through

$$-\Lambda_0 = 6\kappa_s^2/R_0^2 = 3(\upsilon_s\kappa_g/R_0)^2 \tag{76}$$

and choose the scaled distance $\tau = r/R_0$ as an independent variable. In these terms the vacuum scalar-field equation (75) at $V_s = 0$ becomes

$$e^{-\zeta}\frac{d}{d\tau}\left(e^{\alpha+\zeta}\tau^2\frac{dX}{d\tau}\right) = 6\tau^2 e^{-X}, \tag{77}$$

while the tensor gravity ones remain unchanged modulo substitution $r \to \tau$.

Approximate equations Of particular phenomenological interest is the case $\upsilon_s \ll 1$ (see Sec. 4.2). Assuming in this case $X = \mathcal{O}(1)$ and $|\alpha|, |\zeta| \ll 1$ (to be confirmed) present the scalar-field and tensor gravity equations in the linear in α and ζ approximation, respectively, as follows:

$$\begin{aligned}
\frac{d}{d\tau}\left(\tau^2\frac{dX}{d\tau}\right) &= 6\tau^2 e^{-X}, \\
\frac{d}{d\tau}\left(\tau^2\frac{d}{d\tau}(\alpha - \upsilon_s^2 X)\right) &= 0, \\
\frac{d}{d\tau}\left(\tau^2\frac{d\zeta}{d\tau}\right) &= -\frac{\upsilon_s^2}{2}\left(\tau\frac{dX}{d\tau}\right)^2,
\end{aligned} \tag{78}$$

with the last tensor gravity equation

$$\frac{d}{d\tau}\left(\tau^2\frac{d}{d\tau}(\alpha - \zeta) - 2\tau(\alpha + \zeta)\right) = 0 \tag{79}$$

serving as a consistency condition. Clearly, the coupling of the tensor and scalar gravity modes is weak, whereas the self-coupling of the scalar mode proves to be strong. The second equation (78) has the general solution $\alpha - \upsilon_s^2 X = c_1/\tau + c_0$, with c_1 and c_0 being some constants.

The driving equation in the system above is that for X. Having solved the equation, one can easily find α and ζ. It follows from (78) that if $X(\tau)$ is a particular solution then an equivalence class of solutions may be obtained by the inhomogeneous scaling transformations:

$$X(\tau) \to \tilde{X}(\tau) = X(k_0\tau) - 2\ln k_0, \tag{80}$$

with $k_0 > 0$ being a constant. This is a reminiscence of the dynamical global symmetry (12) in the global non-symmetric gauge $ab = 1$. At face value, (80) reduces to reparametrization $R_0 \to R_0/k_0$ supplemented by a shift in X. Because R_0 is arbitrary and a shift in X does not matter at $V_s = 0$, we restrict ourselves by $k_0 = 1$.

To study the scalar-field equation, introduce the new variables

$$t = \ln 3\tau^2, \quad Z = X - t \tag{81}$$

and present the equation equivalently as follows:

$$\frac{d^2Z}{dt^2} + \frac{1}{2}\frac{dZ}{dt} = \frac{1}{2}\left(e^{-Z} - 1\right). \tag{82}$$
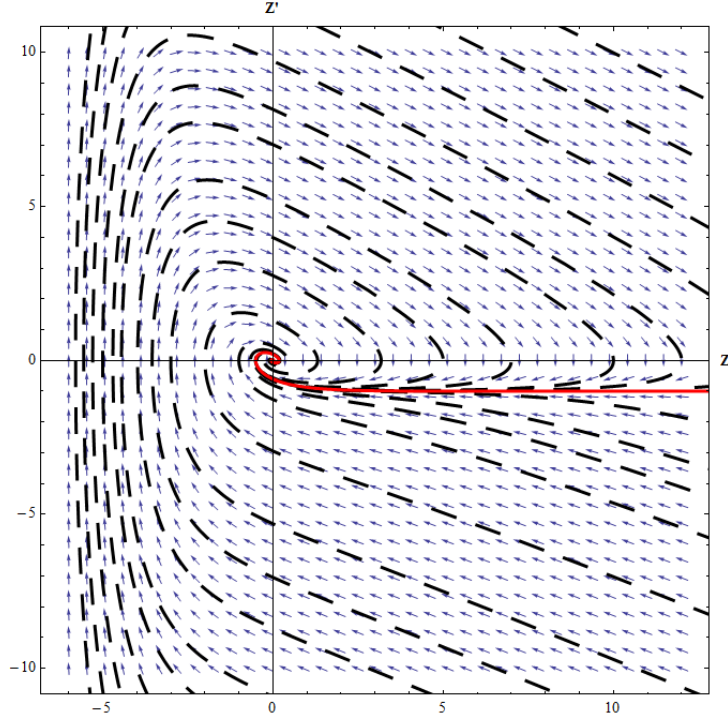
Figure 1: Phase plane $(Z, \dot{Z})$: arrows designate the normalized direction field determined by $d\dot{Z}/dZ$. The trajectories $(Z(t), \dot{Z}(t))$ are tangential to the direction field everywhere. Solid line – the regular trajectory ($\dot{Z} \to -1$ at $t \to -\infty$); dashed lines – irregular trajectories ($\dot{Z} \to -\infty$ at $t \to -\infty$). The origin $Z = \dot{Z} = 0$ is attractor at $t \to \infty$.

Putting further $\dot{Z} \equiv dZ/dt$ one can bring the second-order ordinary differential equation above to the autonomous first-order differential system:

$$\frac{dZ}{dt} = \dot{Z}$$
$$\frac{d\dot{Z}}{dt} = -\frac{1}{2}\dot{Z} + \frac{1}{2}\left(e^{-Z} - 1\right). \tag{83}$$

Such systems are known to be basically characterized by the types of their exceptional points given by the requirement $dZ/dt = d\dot{Z}/dt = 0$. The system above has the single such point, $Z = \dot{Z} = 0$, and the latter proves to belong to the stable focus type. The respective phase plane $(Z, \dot{Z})$ is presented in Figure 1. It clearly shows the attraction point and a distinguished trajectory (the solid line) to which all other trajectories (the dashed lines) tend to accumulate departing it nevertheless sooner or later.

Accordingly, there are three classes of solutions to (78):

(i) an *exceptional* solution reflected by the attraction point at the origin of phase plane;

(ii) a *regular* at the origin ($\tau = 0$) solution corresponding to the regular trajectory ($\dot{Z} \to -1$ at $t \to -\infty$);

(iii) the *irregular* at $\tau = 0$ solutions corresponding to the irregular trajectories ($\dot{Z} \to -\infty$ at $t \to -\infty$). Consider them in turn.

Exceptional solution The exact exceptional solution corresponding to $Z = \dot{Z} = 0$ looks like

$$\begin{aligned}
\bar{X} = \bar{\alpha}/v_s^2 &= \ln 3\tau^2, \\
\bar{\zeta}/v_s^2 &= -\ln 3\tau^2 + 2,
\end{aligned} \tag{84}$$

with the integration constants properly chosen. This solution may serve as a reference one, with all other solutions approaching it at $\tau \to \infty$. Note that $\bar{X}$ is invariant under the rescaling (80). Inverting (35) one gets in the leading v_s-order the hard-core dark modulus corresponding to $\bar{X}$:

$$\sqrt{-\bar{g}_*} \simeq e^{-\bar{X}} = 1/(3\tau^2). \tag{85}$$

Implementing (43) to $\bar{g}_*$ gives for the transverse coordinates the cuspy relation $\hat{\tau} = \tau^{1/3}$.

Regular solution Decomposing a regular solution in the powers of τ (only even powers prove to enter) we get the looked-for solution (endow it with the subscript h) as follows:

$$X_h = \alpha_h/v_s^2 = \begin{cases} \tau^2 - \frac{3}{10}\tau^4 + \frac{4}{30}\tau^6 + \mathcal{O}(\tau^8), & \tau \le 1, \\ \ln 3\tau^2, & \tau \gg 1 \end{cases} \tag{86}$$

and

$$\zeta_h/v_s^2 = \begin{cases} -\frac{1}{10}\tau^4 + \frac{2}{35}\tau^6 + \mathcal{O}(\tau^8), & \tau \le 1, \\ -\ln 3\tau^2 + 2, & \tau \gg 1, \end{cases} \tag{87}$$

with $X_h(0) = 0$ being imposed. The restriction (79) fulfils identically with the same accuracy both at $\tau \le 1$ and $\tau \gg 1$. Note that though X_h changes under the global transformations (80) its asymptotic remains invariant. The solution satisfies the matterless condition, $\alpha_h = v_s^2 X_h$ (see Sec. 4.1). The respective static space structure is nothing but the dark *halo*.[21] The parameter R_0 plays the role of the soft-core radius (see Sec. 4.2). The equations above can be extended analytically to $\tau^2 < 0$ (corresponding to $R_0^2 < 0$, $\Lambda_0 > 0$), though with loosing the halo-type solution. This explains the earlier made choice $\Lambda_0 \le 0$.

Inverting (35) one gets in the wide region of τ, where $|\zeta_h| = \mathcal{O}(v_s^2) \ll 1$, the soft-core dark modulus as follows:

$$\sqrt{-g_{*h}} \simeq e^{-X_h} = \begin{cases} 1 - \tau^2 + \mathcal{O}(\tau^4), & \tau \le 1, \\ 1/(3\tau^2), & \tau \gg 1. \end{cases} \tag{88}$$

Implementing (43) to g_{*h} gives then for the transverse coordinates the relation

$$\hat{\tau} \simeq \begin{cases} \tau, & \tau \le 1, \\ \tau^{1/3}, & \tau \gg 1. \end{cases} \tag{89}$$

To study the asymptotic behaviour of X_h in more detail put $X_h \equiv \bar{X} + \Delta\bar{X}_h$. In these terms the approximate scalar-field equation looks equivalently like

$$\frac{d}{d\tau}\left(\tau^2 \frac{d\Delta\bar{X}_h}{d\tau}\right) = 2\left(e^{-\Delta\bar{X}_h} - 1\right). \tag{90}$$

---

[21]Indeed, equating the Newtonian gravitational attraction force $F_g = m\alpha'_h/2$, acting in the given metric on a test particle with the mass m, to the centripetal force $F_c = mv_h^2/r$, corresponding to the circular rotation velocity v_h, we could already anticipate the asymptotic constant $v_h = v_s$, characteristic of the galaxy dark halos (see Sec. 4.2).
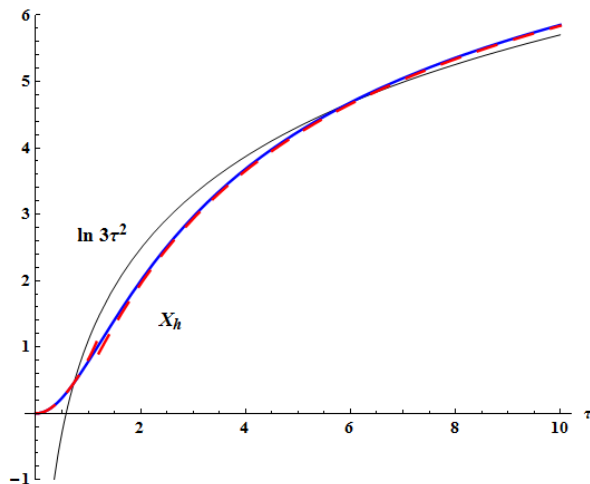
Figure 2: Regular solution X_h: bold solid line – numerical result; dashed line – piecewise analytical approximation with $\bar{\bar{\delta}}_0$ and $\bar{\tau}_0$ as in Figure 3. Thin line – exceptional solution.

Assuming $|\Delta\bar{X}_h| < 1$ (to be confirmed) and retaining the linear in $\Delta\bar{X}_h$ part we get the solution at $\tau > 1$ as follows:

$$\Delta\bar{X}_h = (\bar{\bar{\delta}}_0/\sqrt{\tau})\cos\left((\sqrt{7}/2)\ln\tau/\bar{\tau}_0\right). \tag{91}$$

Here $\bar{\bar{\delta}}_0$ and $\bar{\tau}_0$ are some integration constants to be inferred from matching with solution at $\tau \leq 1$ or from comparison with the numerical solution. Clearly, X_h oscillates with attenuation around $\bar{X}$ approaching the latter at $\tau \to \infty$. The regular solution is unique and may be prolonged in the arbitrary order in v_s^2 to the solution of the exact equations (75). The behaviour of X_h is shown in Figures 2 and 3. It is seen, in particular, that X_h gets strong already at the moderate τ, so that the account for potential $V_s(X)$ may become important in this region.[22]

3.4   Irregular non-harmonic solution

At last, consider the interpolating ansatz for λ given by (47). To satisfy this ansatz, drop off the requirement of regularity at the origin and look for a solution interpolating between the two previous solutions. Put without loss of generality $X = X_h + \Delta X_h$. In these terms the approximate scalar-field equation corresponding to $\lambda = \Lambda_0$ at $\tau > r_0/R_0$ looks equivalently like

$$\frac{d}{d\tau}\left(\tau^2\frac{d\Delta X_h}{d\tau}\right) = 6\tau^2 e^{-X_h}\left(e^{-\Delta X_h} - 1\right). \tag{92}$$

Similarly, putting $\alpha = \alpha_h + \Delta\alpha_h$ and $\zeta = \zeta_h + \Delta\zeta_h$ present the approximate gravity equations at $\tau > r_0/R_0$ as follows:

$$\frac{d}{d\tau}\left(\tau^2\frac{d}{d\tau}(\Delta\alpha_h - v_s^2\Delta X_h)\right) = 0,$$
$$\frac{d}{d\tau}\left(\tau^2\frac{d\Delta\zeta_h}{d\tau}\right) =$$

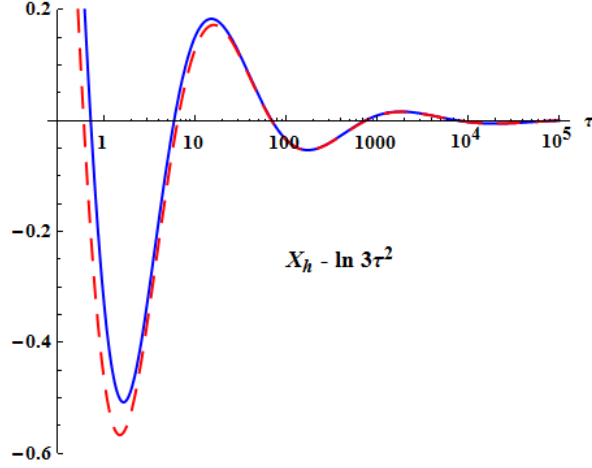[22]For additional details on the regular solution see [25, 26].

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Figure 3: The asymptotic of the regular solution X_h: solid line – numerical result for $\Delta \bar{X}_h = X_h - \bar{X}$; dashed line – analytical approximation with $\bar{\delta}_0 = -0.75$ and $\bar{\tau}_0 = 2.0$

$$= -v_s^2 \tau^2 \left(\frac{dX_h}{d\tau} + \frac{1}{2} \frac{d\Delta X_h}{d\tau} \right) \frac{d\Delta X_h}{d\tau}, \tag{93}$$

with the consistency condition remaining as before:

$$\frac{d}{d\tau} \left(\tau^2 \frac{d}{d\tau} (\Delta \alpha_h - \Delta \zeta_h) - 2\tau (\Delta \alpha_h + \Delta \zeta_h) \right) = 0. \tag{94}$$

Assuming $|\Delta X_h| < 1$ and $|d\Delta X_h/d\tau| < |dX_h/d\tau|$ (to be confirmed) consider the linear in ΔX_h approximation. Decomposing ΔX_h in the series in τ starting (by assumption to be verified) from $1/\tau$ and accounting for (86) we get the solution as follows:

$$\Delta X_h = \delta_i \begin{cases} -1/\tau + 3\tau - 2\tau^3 + \mathcal{O}(\tau^5), & r_0/R_0 < \tau \leq 1, \\ (\delta_0/\sqrt{\tau}) \cos \left((\sqrt{7}/2) \ln \tau/\tau_0 \right), & \tau > 1. \end{cases} \tag{95}$$

In the above, δ_i is a small normalization parameter to be fixed by subsequent matching with the dark fracture solution. Likewise, δ_0 and τ_0 are some integration constants which can, in principle, be fixed by further applying the perturbation procedure in ΔX_h. These constants can be estimated by matching the two branches of (95) or from comparison with the numerical results. Solving (93) we then get

$$\Delta \alpha_h = -\frac{\delta_\alpha}{\tau} + v_s^2 \Delta X_h, \tag{96}$$

$$\Delta \zeta_h = -\frac{\delta_\zeta}{\tau} - v_s^2 \delta_i \begin{cases} \tau - \frac{2}{5}\tau^3 + \mathcal{O}(\tau^5), & r_0/R_0 < \tau \leq 1, \\ \mathcal{O}(1/\sqrt{\tau}), & \tau > 1, \end{cases}$$

with δ_α and δ_ζ being some small parameters to be fixed by further approximation. The consistency condition (94) is fulfilled with the same accuracy. Several representative solutions corresponding to the irregular trajectories somewhat close to the regular one are shown in Figure 4. The less $|\delta_i|$, the better is the approximation.[23]

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[23] The decaying behaviour of the irregular solution with $\delta_i > 0$ is superseded eventually by the growing one at the tiny τ (not shown), what lies though beyond the region of approximation.
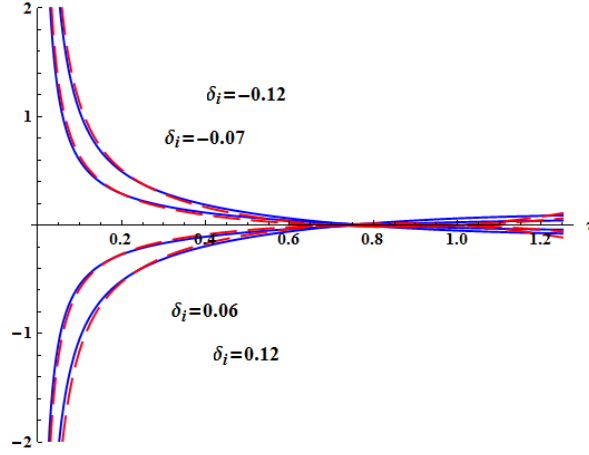
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Figure 4: Representative irregular solutions X: solid lines – numerical results for $\Delta X_h = X - X_h$; dashed lines – analytical approximation with the respective δ_i for the pole singularity.

To arrive at a solution valid approximately in the whole interval of τ, match the $1/\tau$-pole term in ΔX_h with the respective term in X_f, the latter corresponding to $\lambda = \Delta_0$ at $\tau < r_0/R_0$ (and similarly for $\Delta\alpha_h$ and $\Delta\alpha_h$). In view of (60) this implies

$$
\begin{aligned}
\delta_i &= r_s/(v_s R_0), \\
\delta_\alpha &= (r_g - v_s r_s)/R_0, \\
\delta_\zeta &= \left(\sqrt{r_g^2 + r_s^2} - r_g\right)\big/R_0.
\end{aligned}
\tag{97}
$$

In particular, the case $\delta_\alpha = 0$ ($r_g = v_s r_s$) proves to correspond to a matterless fracture, with $\ln a_f = v_s^2 X_f$ (see Sec. 4.1). Partite further ΔX_h as $\Delta X_h \equiv X_f + X_i$, where $X_f = -\delta_i/\tau = -r_s/(v_s r)$ is the scalar tail of fracture and the rest, X_i, is attributed to fracture-halo interference. Altogether, the total solution (designate it by the subscript l) can be presented as a coherent sum of three contributions, $X_l = X_h + \Delta X_h = X_f + X_h + X_i \equiv X_f + X_{heff}$ (and similarly for the metric components α_l and ζ_l). At $\tau \to \infty$ the interference disappears due to disappearance of X_f. At $\delta_i = 0$ ($r_s = 0$ or $R_0 \to \infty$) it disappears identically. The behaviour of X_i/δ_i is show in Figure 5. Finally note that in the Laurent decomposition of the exact X_l, the part comprising powers of $1/r$ may be associated with X_f, the even powers of r with X_h, and the odd powers of r with X_i.

The gravitational potential α_l reveals the property of confinement, with the reciprocal behaviour, $-r_g/r$, at the tail of fracture superseded eventually at the periphery of halo by the logarithmic potential of gravitational attraction, $v_s^2 \ln 3r^2/R_0^2$. The confinement scale R_0 is otherwise the halo soft-core radius (see Sec 4.1). Call the respective static space structure the dark *lacuna*.[24] Due to the coherent scalar field getting strong towards the periphery of halo, the gravitational confinement within a lacuna should in reality be only partial, being terminated ultimately by the potential V_s and/or the influence of the near-by lacunas. Generally, the lacunas depend on three distance scales which may be chosen as r_g, r_s and R_0. With the inner and outer scalar distribution radii, r_s and R_0, getting

---

[24]The gravitational attraction force $mr_g/2r^2$, acting on a test particle with mass m inside a lacuna due to a central fracture, is superseded eventually by the universal attraction force mv_s^2/r due to halo, which should dominate at $r > \max(R_0, r_g/v_s^2)$.
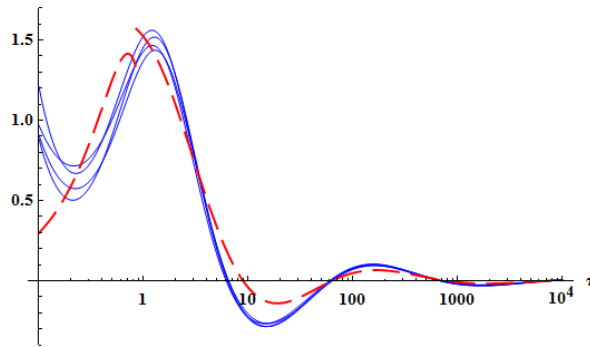
Figure 5: Representative irregular solutions X: solid lines – numerical results for interference term X_i/δ_i, with δ_i as in Figure 4; dashed line – piece-wise analytical approximation, with $\delta_0 = 0.85$ and $\tau_0 = 2.0$.

closer, the three-component structure of a lacuna becomes less prominent. Nevertheless, the asymptotic behaviour and gravitational confinement in a lacuna should still survive (at least in a parameter region). This is because the asymptotic properties of the solution are determined exclusively by the exponential term in the r.h.s. of the vacuum scalar-field equation (77) or (78).

4   Interpretation and applications

4.1   Energy content

Static space structures   The preceding results were obtained exclusively in the geometry framework without any recourse to DM. To the latter end let us reveal the energy content of the static space structures found previously. Consider an isolated gravitationally bound system. Let $T_{\mu\nu}$ be its total bare energy-momentum tensor, with gravity included only in the minimal fashion through metric. To incorporate the proper gravity contribution define the net energy-momentum pseudo-tensor $\check{T}_{\mu\nu} = T_{\mu\nu} + \check{\tau}_{\mu\nu}$, with $\check{\tau}_{\mu\nu}$ being the pseudo-tensor of the gravity itself. At that, $\check{\tau}_{\mu\nu}$ is well-known to be dependent both on definition and, possibly, the class of coordinates. The same concerns thus $\check{T}_{\mu\nu}$. According to Tolman [46], with $\check{\tau}_{\mu\nu}$ taken as the Einstein pseudo-tensor in the quasi-Galilean coordinates, the net gravitating energy/mass of a static isolated system may be expressed entirely through its bare energy-momentum tensor as follows:

$$M_g \;\; = \;\; \int \check{T}_0^0 \sqrt{-g} d^3 x = \int (T_0^0 - T_l^l) \sqrt{-g} d^3 x. \tag{98}$$

In other terms, the net energy density, with gravity properly incorporated, is $\check{\rho} \equiv \check{T}_0^0 = \rho + \Sigma p_l$, where ρ and p_l $(l = 1, 2, 3)$ are the bare energy density and pressure, respectively. Physically this means that gravity "pumps-in" energy equal to the proper work performed against pressure. Having been established in the particular spatial coordinates, M_g is to be prolonged to the arbitrary ones as a scalar. For validity of the isolation assumption, the ever-present cosmological constant Λ, which is paramount for the Universe as a whole, is to be excluded from $\check{\rho}$ for an isolated system. (Otherwise this would result in a double-counting when considering the evolution of the Universe.) In view of (26) we get most

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generally

$$M_g = 2\kappa_g^2 \int R_0^0 \sqrt{-g} \, d^3x, \tag{99}$$

with the net gravitating energy determined thus entirely by the R_0^0-component of the Ricci tensor.

In the reciprocal gauge, by means of (44) and the first equation (41) (with $1/b = a \equiv A$ and σ substituted by the spatial scalar $\ln A$) we get

$$2R_0^0 = (AC(\ln A)')'/C = -g^{kl}\nabla_k\nabla_l \ln a. \tag{100}$$

Here ∇_k is a spatial component of the four-dimensional covariant derivative. This relation is now valid in the arbitrary spatial gauge/coordinates. This gives

$$M_g = -\kappa_g^2 \int \sqrt{-g} g^{kl}\nabla_k\nabla_l \ln a \, d^3x. \tag{101}$$

In view of (15) (with X substituted by $\ln a$) one has

$$\sqrt{-g} g^{kl}\nabla_k\nabla_l \ln a = \partial_k(\sqrt{-g} g^{kl}\partial_l \ln a), \tag{102}$$

and the three-dimensional Gauss theorem reduces (101) to the integral over the remote two-dimensional surface S as follows:

$$M_g = -\kappa_g^2 \int \sqrt{-g} g^{kl}\partial_l \ln a \, dS_k, \tag{103}$$

with $\sqrt{-g} = \sqrt{abc}$.

Partite M_g onto the contributions of the non-gravitational matter and the systolons, respectively, M_m and M_s, including the proper gravity energy:

$$\begin{aligned} M_g &= M_m + M_s = \\ &= \int \left((T_{m0}^{\ 0} - T_{mn}^{\ n}) + (T_{s0}^{\ 0} - T_{sn}^{\ n}) \right) \sqrt{-g} \, d^3x. \end{aligned} \tag{104}$$

Put by default for the net partial energy densities:

$$\check{\rho}_m \equiv T_{m0}^{\ 0} - T_{mn}^{\ n}, \quad \check{\rho}_s \equiv T_{s0}^{\ 0} - T_{sn}^{\ n}. \tag{105}$$

In view of (38) the systolon net partial energy (with V_s neglected) is

$$M_s = -v_s^2\kappa_g^2 \int \sqrt{-g} g^{kl}\nabla_k\nabla_l X d^3x, \tag{106}$$

and the three-dimensional Gauss theorem gives then

$$M_s = -v_s^2\kappa_g^2 \int \sqrt{-g} g^{kl}\partial_l X dS_k. \tag{107}$$

Treated as general covariant scalars the expressions above determine in UBG the energy content of a static space structure through $\nabla \cdot \nabla \ln a$ and $\nabla \cdot \nabla X$. While the first term is the same as in GR with a (free massless) scalar field, the second term is peculiar to UBG. It is produced due to the unimodularity constraint missing in GR.[25] In particular, in the case $\ln a = v_s^2 X$ (modulo an additive constant) there follows $M_g = M_s$, $M_m = 0$ signifying a matterless, pure systolon static space structure.

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[25] Physically, the scalar-graviton field may be attributed to (singularity of) modulus as a scalar source. This may take place even in vacuum. In GR, to reproduce M_s one should couple the scalar field directly to a matter scalar charge. Thus, though the solution in the two theories is formally the same, its physics content differs [24].

Dark fractures   In what follows we adopt for a dark fracture the truncated real form, with a singularity only at $r = 0$ (see Sec 3.2). Everywhere, but for the origin, there fulfills $R_0^0 = 0$ and $\nabla \cdot \nabla X = 0$. Thus, the spatial integrals are saturated in this case exclusively at $r = 0$ by the point-like singularity. The latter being in fact known only implicitly, a spatial integral may be substituted via the three-dimensional Gauss theorem by the respective integral over the remote sphere. Having consistently defined the energy in the quasi-Galilean coordinates, one can use, just for calculations, the polar coordinates, with $\partial_l = \delta_l^r \partial_r$, $\sqrt{-g} dS_k = \delta_k^r c r^2 d\Omega^2$ and $g^{rr} = -A$. The net gravitating energy of a fracture is then as follows:

$$M_g = 4\pi \kappa_g^2 \nu_f r_f = r_g/(2G) \geq 0, \tag{108}$$

with $r_g \geq 0$ ($\nu_f \geq 0$) imposed. Likewise, the systolon net contribution is

$$M_s = 4\pi \kappa_g^2 \upsilon_s \eta_f \sqrt{1 - \nu_f^2} r_f = \upsilon_s r_s/(2G), \tag{109}$$

or otherwise, $M_s = \upsilon_s \eta_f \sqrt{1/\nu_f^2 - 1} M_g$. The net contribution of the non-gravitational matter is then

$$M_m = M_g - M_s = \left(1 - \upsilon_s \eta_f \sqrt{1/\nu_f^2 - 1}\right) M_g. \tag{110}$$

Imposing by default the requirements $M_m \geq 0$ and $M_s \geq 0$ ($\eta_f = 1$) one can envisage two following extreme cases.

   (*i*) *BHs*: $\nu_f = \nu_{f\max} = 1$, with $M_s = 0$ and $M_g = M_m$. Here r_g is determined conventionally by the matter net energy, $r_g = 2G M_m$.

   (*ii*) *Vacuum dark fractures*: $\nu_f = \nu_{f\min} = \upsilon_s/\sqrt{1 + \upsilon_s^2}$, with $M_m = 0$ and $M_g = M_s$. Here $r_g = \upsilon_s r_s$, with $r_g \ll r_s$ at $\upsilon_s \ll 1$. This accords with $\ln a_f = \upsilon_s^2 X_f$ (modulo a constant) as a generic condition of the absence of matter.[26]

Dark halos   For a dark halo the solution is regular, and according to (38) the systolon net energy density is well-defined as

$$\check{\rho}_s = -2\Lambda_s = -2\Lambda_0 e^{-X}. \tag{111}$$

Under the attractive effective potential Λ_s ($\Lambda_0 < 0$) this energy density is positive-definite, with $\check{\rho}_s = 0$ only at $\Lambda_0 = 0$. In view of (76) and the first equation (78) one has

$$\check{\rho}_s = \upsilon_s^2 \frac{\kappa_g^2}{R_0^2} \frac{1}{\tau^2} \frac{d}{d\tau}\left(\tau^2 \frac{dX}{d\tau}\right). \tag{112}$$

Substituting $X = X_h$ one then gets in particular the halo energy density profile as follows:

$$\rho_h = \rho_0 \begin{cases} 1 - \tau^2 + \frac{4}{5}\tau^4 + \mathcal{O}(\tau^6), & \tau \leq 1, \\ 1/(3\tau^2) + \mathcal{O}(1/\tau^{5/2}), & \tau \gg 1, \end{cases} \tag{113}$$

with ρ_0 standing for the central energy density:

$$\rho_0 = -2\Lambda_0 = 6\upsilon_s^2 \frac{\kappa_g^2}{R_0^2}. \tag{114}$$

---

[26] The gravitational attraction of the "normal" fractures ($\eta_f = 1$, $r_s \geq 0$) is enhanced, $M_g \geq M_m$. Admitting $\eta_f = -1$, relaxing thus requirement $M_s \geq 0$, one would get the "anomalous" fractures screening the matter, $M_g < M_m$.
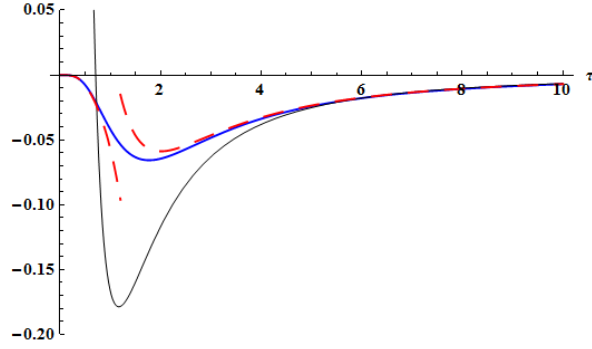
Figure 6: Normalized energy density profiles $\rho(\tau)/\rho_0$: the difference between a profile and the reference one $\rho_{ref}/\rho_0 = 1/(1 + \tau^2)$. Bold solid line – numerical result for the soft-core profile ρ_h/ρ_0; dashed line – piece-wise analytical approximation; thin line – the hard-core profile $\bar{\rho}/\rho_0 = 1/3\tau^2$ for exceptional solution.

Thus, a dark halo naturally enjoys the soft-core energy density profile, with R_0 being the halo core radius. At that, ρ_h/ρ_0 closely reproduces the dark modulus (88). For comparison, the exceptional solution $\bar{X}(\tau) = \ln 3\tau^2$ results in the hard-core profile

$$\bar{\rho} = \frac{1}{3}\frac{\rho_0}{\tau^2} = 2v_s^2\frac{\kappa_g^2}{r^2} \tag{115}$$

reproducing, accordingly, hard-core modulus (85). Asymptotically, $\rho_h(\tau)$ oscillates around $\bar{\rho}$ with attenuation, approaching $\bar{\rho}$ at $\tau \gg 1$. The normalized difference, $(\rho - \rho_{ref})/\rho_0$, between a profile and the reference one, $\rho_{ref} = \rho_0/(1 + \tau^2)$, is shown at the moderate τ in Figure 6.

It is seen, in particular, that the soft-core profile, ρ_h, decays at $\tau \leq 2$ faster than the reference one, both having the same behaviour near the origin.[27]

The halo net gravitating energy interior to τ is

$$M_{h<}(\tau) = 8\pi v_s^2\kappa_g^2 R_0 \begin{cases} \tau^3 - \frac{3}{5}\tau^5 + \mathcal{O}(\tau^7), & \tau \leq 1, \\ \tau + \mathcal{O}(\sqrt{\tau}), & \tau \gg 1. \end{cases} \tag{116}$$

At $r \gg R_0$, one has $M_{h<}(r) = 8\pi v_s^2\kappa_g^2 r = v_s^2 r/G$, with the respective energy for $\bar{X}$ rising linearly identically.

To clarity the physics content of a dark halo note that according to (37), (76) and (111) the exceptional solution in the leading v_s-approximation results in the bare energy-momentum tensor with

$$\bar{p}_0 = \bar{p}_\theta = \bar{p}_\varphi = 0, \quad \bar{p}_r = \frac{1}{3}\frac{\rho_0}{\tau^2}. \tag{117}$$

In turn, this results in the positive-definite net energy density $\bar{\rho} = \bar{p}_0 + \Sigma\bar{p}_n = \bar{p}_r = \rho_0/(3\tau^2) > 0$, being attributed entirely to tensor gravity. As for the regular solution X_h ($X_h'|_{r=0} = 0$), its bare energy density (37) near the origin is negative, while the net one (111) is nevertheless positive-definite everywhere ($\Lambda_0 < 0$). The same concerns the bare radial pressure (37). At $\tau \gg 1$ the regular solution behaves as $\bar{X}$, with the net energy density decaying like $1/\tau^2$, too. Thus, the dark halo is a gravitationally tightly bound structure, with the bulk of its net energy provided by tensor gravity to form the halo.

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[27]In the conventional DM approach ρ_{ref} is due to the so-called pseudo-isothermal sphere, while $\bar{\rho}$ corresponds to the true isothermal one (see, e.g., [47]).
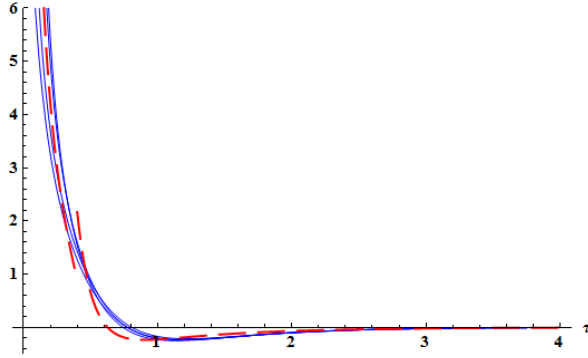
Figure 7: Normalized energy density profiles $\rho(\tau)/\rho_0$: solid lines – numerical results for interference term $\rho_i/(\rho_0\delta_i)$, with δ_i as in Figure 4; dashed line – piece-wise analytical approximation, with δ_0 and τ_0 as in Figure 5.

Dark lacunas For a dark lacuna one has $X_l = X_f + X_h + X_i \equiv X_f + X_{heff}$. The contribution to the net gravitating energy due to X_f is accounted for by the surface integral. For energy density of the effective dark halo, $\rho_{heff} = \rho_h + \rho_i$, Eq. (112) gives ρ_h as before and the interference contribution as follows:

$$\rho_i/\rho_0 = \delta_i \begin{cases} 1/\tau - 4\tau + \mathcal{O}(\tau^3), & \tau \leq 1, \\ \mathcal{O}(1/\tau^{5/2}), & \tau \gg 1. \end{cases} \tag{118}$$

Thus, ρ_{heff} comprises some cuspy $1/\tau$-correction compared to the soft-core profile ρ_h. For the normal fractures ($\eta_f = 1$) the interference is constructive, $\delta_i > 0$, with the effective halo net energy near the origin increasing. The numerical results for the normalized interference term, $\rho_i/(\rho_0\delta_i)$, are shown in Figure 7. The interference contribution to the lacuna net energy interior to τ is

$$M_{i<}(\tau) = 12\pi v_s \kappa_g^2 r_s \begin{cases} \tau^2 - 2\tau^4 + \mathcal{O}(\tau^6), & \tau \leq 1, \\ \mathcal{O}(\sqrt{\tau}), & \tau \gg 1. \end{cases} \tag{119}$$

The above results are obtained in the perturbative fashion. For the realistic lacunas the picture may become more complicated in detail though its salient features should, conceivably, survive.

4.2   Rotation curves and equivalent DM

Rotation velocity   Let $U^\mu = dx^\mu/d\tau$, $U \cdot U = g_{\mu\nu}U^\mu U^\nu = 1$, be the four-velocity of a test particle with respect to its proper time τ. If the particle interacts gravitationally only in a minimal fashion through metric, its U^μ for the free motion satisfies the geodesic equation, $dU^\lambda/d\tau + \Gamma^\lambda_{\mu\nu}U^\mu U^\nu = 0$, with $\Gamma^\lambda_{\mu\nu}$ standing for the Christoffel connection. For the circular rotation in a static spherically symmetric metric one has in the polar coordinates $U^r = 0$. Besides, one can put $\theta = \pi/2$, with $U^\theta = dU^\theta/d\tau = 0$. The remaining non-zero components are $U^\varphi = d\varphi/d\tau$ and $U^0 = dt/d\tau$. The ratio $\omega = U^\varphi/U^0 = d\varphi/dt$ is thus nothing but the angular velocity with respect to the observer's time. The visible circular rotation velocity is as follows:

$$u = \sqrt{cr^2}d\varphi/dt = \sqrt{cr^2}U^\varphi/U^0. \tag{120}$$

For the spherically symmetric metric (36) one has $\Gamma^r_{00} = a'/(2b)$, $\Gamma^r_{\varphi\varphi}|_{\theta=\pi/2} = -(cr^2)'/(2b)$ and $\Gamma^r_{r\varphi} \equiv 0$, so that the equation of motion gives $dU^r/d\tau = -(\Gamma^r_{00}(U^0)^2 + \Gamma^r_{\varphi\varphi}(U^\varphi)^2) = 0$, and thus

$$(U^\varphi)^2/(U^0)^2 = a'/(cr^2)'. \tag{121}$$

Altogether, the visible rotation velocity squared is

$$u^2 = a'/\ln(cr^2)'. \tag{122}$$

Otherwise, the rotation velocity with respect to the particle proper time looks like

$$V = u\,dt/d\tau = uU^0 = \sqrt{cr^2}\,U^\varphi. \tag{123}$$

Accounting for $U \cdot U = a(U^0)^2 - cr^2(U^\varphi)^2 = 1$ and (121), one then gets $V = v/\sqrt{1-v^2}$, where

$$v^2 = (\ln a)'/(\ln cr^2)'. \tag{124}$$

In the above, $v \equiv u/\sqrt{a} = V/\sqrt{1+V^2}$ is the visible rotation velocity accounting for the gravitational deceleration of time. For consistency, $v < 1$. Being gauge invariant, all these expressions are valid in the arbitrary radial coordinates. The choice of the expressions for the rotation velocity remains though convention dependent. By default, we choose v. Particularly, in the astronomic coordinates ($c = 1$) one gets conventionally $v^2 = r(\ln a)'/2$. In the non-relativistic weak-field limit, we are interested in, the difference between the definitions becomes irrelevant.[28]

Rotation curves In the leading v_s-approximation one can put $c = 1$, so that

$$v^2 = r\alpha'/2 = (\tau/2)d\alpha/d\tau. \tag{125}$$

The total velocity squared in a dark lacuna is thus the sum of three components, $v_l^2 = v_f^2 + v_h^2 + v_i^2 \equiv v_f^2 + v_{heff}^2$, where the fracture, halo and interference contributions, respectively, are as follows:

$$
\begin{aligned}
v_f^2 &= \frac{1}{2}\frac{r_g}{r}, & r \gg r_g, \\
v_h^2 &= v_s^2 \begin{cases} \tau^2 - \frac{3}{5}\tau^4 + \frac{12}{35}\tau^6 + \mathcal{O}(\tau^8), & \tau \le 1, \\ 1 + \mathcal{O}(1/\sqrt{\tau}), & \tau \gg 1, \end{cases} \\
v_i^2 &= \frac{3}{2}v_s^2\delta_i \begin{cases} \tau - 2\tau^3 + \mathcal{O}(\tau^5), & \tau \le 1, \\ \mathcal{O}(1/\sqrt{\tau}), & \tau \gg 1, \end{cases}
\end{aligned}
\tag{126}
$$

with v_i^2 being generally sign-indefinite. It is seen that the RC profile $v_{heff}(\tau)$ for the effective halo gets flat only asymptotically. At that, the exceptional solution $\bar{X}$ would result in the precisely flat profile $\bar{v} = v_{h\infty} = v_s$, around which all the profiles $v_{heff}(\tau)$ oscillates with attenuation, approaching $\bar{v}$ at $\tau \gg 1$. The respective results are shown in Figures 8–10. Evidently, the net velocity profile is in reality far from being exactly flat.

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[28]In passing, choosing u one would get that for BHs ($a = 1 - r_g/r$, $c = 1$) in all the aforementioned radial coordinates (see Sec. 3.2), but for isotropic ones, there exactly fulfills the third Kepler's law: $T^2 \sim r^3$, with T being the rotation period.
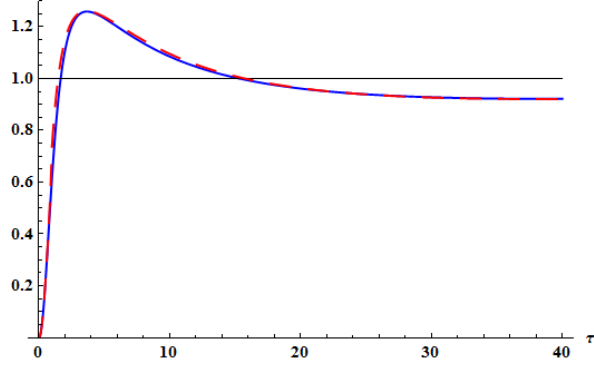
Figure 8: Normalized RC profile $v(\tau)$: bold solid line – numerical result for v_h^2/v_s^2 practically coinciding with analytical approximation (dashed line); thin line – $(\bar{v}^2/v_s^2 = 1)$ for exceptional solution.
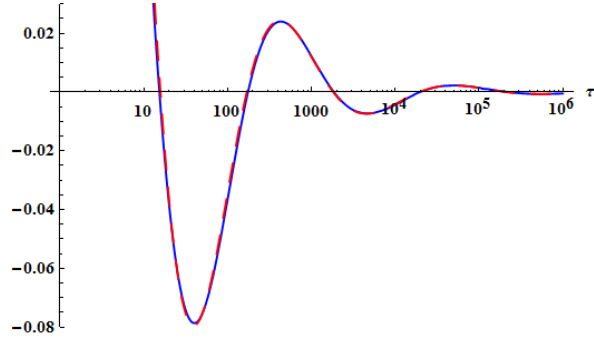

Figure 9: The asymptotic of the normalized RC profile $v(\tau)$: solid line – numerical result for $(v_h^2/v_s^2 - 1)$ practically coinciding with analytical approximation (dashed line).
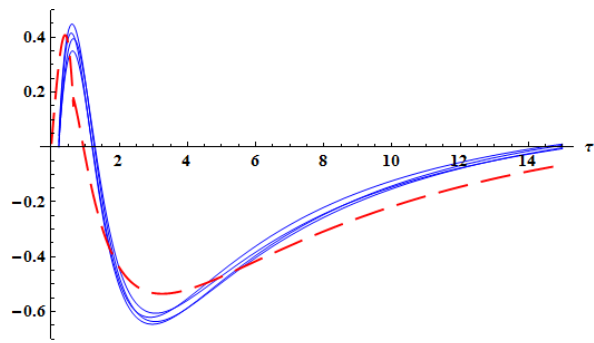

Figure 10: Normalized RC profile $v(\tau)$: solid lines – numerical results for interference term $v_i^2/(v_s^2\delta_i)$, with δ_i as in Figure 4; dashed line – piece-wise analytical approximation, with δ_0 and τ_0 as in Figure 5.

Equivalent non-relativistic DM Let us interpret the previously found energy density and rotation velocity profiles in terms of the non-relativistic DM, the latter being conventionally used in confronting with astronomic observations. For the non-relativistic DM halo in the flat space-time ($a = c = 1$) the Newton's dynamics results in the rotation velocity v_d determined through

$$v_d^2/r = GM_{d<}(r)/r^2, \tag{127}$$

where $M_{d<}(r) = 4\pi \int_0^r \rho_d(r) r^2 dr$ is the DM energy interior to r, with ρ_d being the DM energy density. The latter should thus satisfy

$$\rho_d = \frac{1}{4\pi G} \frac{(rv_d^2)'}{r^2}. \tag{128}$$

Imposing $v_d^2 \equiv v_{heff}^2 = v_h^2 + v_i^2$ and accounting for (125) we get the energy density for the equivalent DM as follows:

$$\rho_d = \frac{\kappa_g^2}{R_0^2} \frac{1}{\tau^2} \frac{d}{d\tau} \left(\tau^2 \frac{d\alpha_{heff}}{d\tau} \right), \tag{129}$$

where $\alpha_{heff} = \alpha_h + \alpha_i$. Further, because in the second line of (78) the term $1/\tau$ due to fracture drops off, one gets hereof $\alpha_{heff} = v_s^2 X_{heff}$, and (112) for X_{heff} gives $\rho_d = \rho_{heff} = \rho_h + \rho_i$. The profiles $\rho_d(\tau)$ oscillate with attenuation around $\bar{\rho}$ approaching the latter at $\tau \gg 1$ (cf., e.g., Figure 9).

Thus in the context of RCs, the previously found energy content of the dark halo coincides in a consistent manner with the non-relativistic DM interpretation. At that, RCs admit a complementary approach: either directly in terms of the static space structures in the geometry framework, without any recourse to DM, or in terms of the equivalent non-relativistic DM in the framework of the Newtonian dynamics. This justifies treating the coherent systolon field as DM.

For more clarity let us present ρ_d as follows:

$$\rho_d = (\rho_0/6)\chi^4 d^2 X_{heff}/d\chi^2, \tag{130}$$

where $\chi = 1/\tau$, ρ_0 is given by (114) and there roughly fulfils $X_{heff} \simeq X_h$, with

$$X_h = \begin{cases} 1/\chi^2, & \chi \geq 1, \\ \ln(3/\chi^2), & \chi \ll 1, \end{cases} \tag{131}$$

so that

$$\rho_d \simeq \rho_0 \begin{cases} 1, & \chi \geq 1, \\ \chi^2/3, & \chi \ll 1. \end{cases} \tag{132}$$

(Note in parentheses that χ is nothing but the radial harmonic coordinate for the dark halo, with the approximate scalar-field equation (78) being $d^2 X_h/d\chi^2 = (6/\chi^4)e^{-X_h}$. Also, under the choice of the unit of length $l_0 = R_0$ the fracture harmonic coordinate (53) smoothly matches at $r_f < r < R_0$ with χ for halo.) Evidently, it is only the relatively rapidly varying with χ part of the scalar-graviton field in halo which matters for the equivalent DM. At that, varying slower than logarithm (though, possibly, larger) part of the field may result in the equivalent DE through the potential $V_s(X)$ (omitted here). The spherical symmetry being not crucial, we expect that this is a generic property of the systolon DM and DE.

We believe that such a coherent scalar-graviton field is of principle importance only within the gravitationally bound systems, such as galaxies and clusters of galaxies. Beyond them the specific nature of the interior DM may become less important, with the Universe as a whole being described effectively by the conventional ΛCDM model (or a variation of it). This would result in the apparent two-component DM and could, perhaps, help unravelling the so-called core/cusp DM problem [47].[29]

Galaxy DM: coherent scalar field vs. continuous medium   To refine the nature of the equivalent DM let us consider an isotropic continuous medium with the conventional energy-momentum tensor

$$T_d^{\mu\nu} = (\rho_d + p_d)U_d^\mu U_d^\nu - p_d g^{\mu\nu}, \tag{133}$$

where ρ_d and $p_d = p_d(\rho_d)$ are, respectively, the proper energy density and pressure of the putative medium, and U_d^μ ($U_d \cdot U_d = 1$) is its local four-velocity. According to (26), in the comoving frame, $U_d^\mu = (1, 0, 0, 0)$, (assumed to coincide with that of galaxy) in the metric (36) there fulfills:

$$R_0^0 = \frac{1}{2\kappa_g^2}\Big(a\rho_d + (a+2)p_d\Big). \tag{134}$$

Imposing in the weak-field ($|a-1| \ll 1$) approximation

$$\rho_d + 3p_d = \rho_{heff}, \tag{135}$$

i.e., equating the net energy densities one reproduces the same R_0^0 for the medium and the effective halo. With R_0^0 largely determining $\ln a$ through (100), and $\ln a$, in turn, largely describing RCs through (124), this ensures the equivalence between the two descriptions, the coherent-field and continuous-medium ones, in the non-relativistic RC context. Under (135) such an equivalence fulfills irrespective of the equation of state $p_d = p_d(\rho_d)$. In particular, at $p_d \ll \rho_d$ one recovers the condition of the preceding paragraph, $\rho_d = \rho_{heff}$, for the cold/warm DM. As far as being determined largely by the Newtonian potential, $\ln a$, the same statement concerns other gravitational manifestations in the non-relativistic weak-field approximation.

Soft-core DM halos   The observational data on the DM-dominated (late-type LSB disk and gas-rich dwarf) galaxies are consistent with the cored energy density profile for the DM halos as follows:

$$\rho_d = \frac{\rho_C}{1 + (r/R_C)^2}, \tag{136}$$

where ρ_C and R_C are two free core parameters (see, e.g., [47]). The presence of a mild cusp is still admitted by the data. Such a profile leads to the asymptotic constant rotation velocity $v_\infty = (4\pi G\rho_C R_C^2)^{1/2}$. This behaviour is in qualitative agreement with that for the previously found one-parameter dark halo profile $\rho_h(r/R_0)$, which results in the asymptotic constant velocity $v_{h\infty} = (4\pi G\rho_0 R_0^2/3)^{1/2}$. But now the parameters ρ_0 and R_0 are, in fact, not independent ensuring the fixed universal $v_{h\infty} = v_s$. To disentangle the parameters one should use a three-parameter lacuna profile accounting for the fracture-halo interference. The latter gets significant with the interference parameter $\delta_i = \mathcal{O}(1)$,

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[29]Besides, as an incoherent component of DM built of systolons there might serve the (near) vacuum mini-fractures (see Sec. 4.1) distributed all over the Universe.

or with the scalar radii for fracture and halo related as $r_s \sim \upsilon_s R_0$, $\upsilon_s \ll 1$, what is not quite unrealistic. Other factors such as lacuna asphericity or rotation, influence of the scalar-field potential, influence of the distributed non-gravitational matter, etc, may also be of importance.[30,31]

With these caveats, the dark lacunas consisting of a stabilizing super-massive dark fracture at the origin surrounded by a dark halo might serve as a prototype model for the galaxy DM frames, to be supplemented ultimately by the distributed non-gravitational matter. With $\kappa_g = M_P/\sqrt{8\pi} = 2.4 \times 10^{18}$ GeV, $M_P = 1/\sqrt{G}$ standing for the Planck mass, the asymptotic rotation velocity in galaxies $v_\infty \sim 10^{-3}$ (in units of the speed of light) at its face value results in $\upsilon_s = \sqrt{2}\kappa_s/\kappa_g \sim 10^{-3}$. This implies that the mass scale appropriate to the scalar mode, $\kappa_s \sim 10^{15}$ GeV, is to be of the order of GUT scale. Could it be more than just a coincidence, with a common origin of the two scales (if any)?

Long-distance gravity modification: vacuum vs. Lagrangian   In the end let us present several comments concerning conceivable long-distance gravity modification in the context of galaxy DM.

(*i*) *Spontaneous vacuum modification*   The present approach to galaxy DM comprises an explicit modification of the gravity Lagrangian at a fixed ultraviolate mass scale $\kappa_s \sim 10^{-3}\kappa_g$. The effective theory is to be valid up to the high scales $\mu \leq \kappa_s$. An infrared parameter Λ_0 (or, equivalently, a long-distance scale R_0) appears at random due to spontaneous breaking of a global symmetry within a lacuna. The randomness of the infrared parameter insures more flexibility of the theory in the context of galaxy DM.

(*ii*) *Explicit kinetic Lagrangian modification*   The attempts at the explicit infrared Lagrangian modifications in the context of galaxy DM are numerous. First of all, one may mention the so-called $f(R)$-gravity, without or with a proper modification of matter Lagrangian (cf., e.g., [49, 50]). In this case, having no specific degree(s) of freedom, DM is just mimicked by the modification of the long-distance tensor gravity.[32] A related approach is given by the scalar-field theories with a non-canonical kinetic term, supplemented ultimately by a repulsive potential in Lagrangian (cf., e.g., [51, 52]), etc. Moreover, one may envisage the case wth a ghost quadratic kinetic term.[33] By construction, so modified Lagrangians are given by some functions $f(R/\mu_I^2)$ or $f(\nabla\sigma \cdot \nabla\sigma/\mu_I^4)$, etc., depending explicitly on a fixed infrared mass scale $\mu_I \ll \kappa_g$. Ultimately, this would imply that a more fundamental theory should settle down already on the relatively low scales $\mu \geq \mu_I$.[34]

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[30]In particular, despite the fact that LSB galaxies are DM-dominated, the account for their star disks may be important for RCs near the origin.

[31]From the DM point of view the dark lacunas may rather be considered as scalar "lumps" in asymptotically non-flat metric. This is to be contrasted with alternative attempts at treating galaxies in the framework of GR with a scalar field by means of scalar lumps in asymptotically flat metrics (cf., e.g., [48]).

[32] At that, a looked-for putative matter is rather "missing" than the dark one, being beyond the reach of direct searches.

[33]For the scalar-field ghost condensation as an alternative to DM in the context of the Universe cf, e.g., [53].

[34]It goes without saying that UBG could a priori admit, if desired, an arbitrary infrared modification of the kinetic Lagrangians both for the tensor and scalar modes. Presently such modifications are left aside.

5 Conclusion and prospects

The Unimodular Bimodal Gravity (UBG) is a theoretically viable development of UR and GR with a scalar field. It retains the principle ingredients of GR – the general covariance and masslessness of the transverse-tensor graviton. Beyond GR, the theory comprises in metric a propagating compression mode – the scalar graviton/systolon. With the latter treated as DM, UBG presents a unified description of the (tensor) gravity and DM. The appearance of a physically well-motivated scalar field is the crucial point of the theory. In its reduced version, with the spontaneously broken dynamical global symmetry, the theory, being rather restrictive, is apt to result in a number of definite predictions. In particular, in the static spherically symmetric case it predicts peculiar space structures – the dark lacunas – consisting of a compact singular dark fracture (a scalar-dressed BH) at the origin surrounded by an extended soft-core dark halo. Enjoying the property of gravitational confinement, with the logarithmic potential of gravitational attraction at the periphery, the dark lacunas ensure asymptotic flattening of RCs and may serve in cosmology as the DM frames for galaxies.

The scalar graviton/systolon physics presents conceivably a perspective field of the future investigations, both theoretical and phenomenological. In particular, the influence of the (near) massless scalar field may drastically change the structure of the GR BHs both at their event horizon and at asymptotic due to appearance of dark lacunas. Further studying the latter ones, as well as dark fractures and halos, including their more subtle aspects such as a putative asphericity or rotation, influence of the scalar-graviton potential and the distributed non-gravitational matter, application to various types of galaxies as well as to galaxy clusters is in order. The application of the theory to evolution of the Universe is likewise urgent to verify the theory (if any) and unravel ultimately the mystery of DM and DE.

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