Alternating subalgebras of Hecke algebras and alternating subgroups of braid groups

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Abstract

For a Coxeter system (G, S) the multi-parametric alternating subalgebra $H^+(G)$ of the Hecke algebra and the alternating subgroup $\mathcal{B}^+(G)$ of the braid group are defined. Two presentations for $H^+(G)$ and $\mathcal{B}^+(G)$ are given; one generalizes the Bourbaki presentation for the alternating subgroups of Coxeter groups, another one uses generators related to edges of the Coxeter graph.

1. Introduction

Let (G, S) be an arbitrary Coxeter system. The alternating subgroup G^+ is an index 2 subgroup of the Coxeter group G (we will omit the reference to S, as in G^+ or, see below, in $H(G)$, etc). In [1], a presentation of the alternating group G^+ is suggested. In this "Bourbaki" presentation, one vertex of the Coxeter graph plays a particular role. In [10], a different presentation has been given for alternating subgroups of Coxeter groups. In this presentation, the generators are related to the oriented edges of the Coxeter graph; for an irreducible Coxeter system (G, S), no particular vertex is distinguished.

The Hecke algebra $H(G)$ associated to the Coxeter group G is a flat deformation of the group ring of G. In [11], an analogue of the Hecke algebra is defined, in the one-parameter setting, for the alternating subgroups of the Coxeter groups. Here we extend this definition to the general multi-parameter situation and call the resulting algebra, denoted by $H^+(G)$, "the alternating subalgebra of the Hecke algebra". The algebra $H^+(G)$ is an index 2 subalgebra (see Section 2 for precise definitions) of the Hecke algebra $H(G)$ and is a flat deformation of the group ring of G^+. The algebra $H^+(G)$ is among the deformations of the group ring of G^+ studied in [5].

In addition, associated to a Coxeter system (G, S), there is a braid group $\mathcal{B}(G)$. Similarly to the alternating subgroup G^+ of G, we define an "alternating subgroup" $\mathcal{B}^+(G)$ of the braid group $\mathcal{B}(G)$.

We give a presentation à la Bourbaki for the alternating subalgebra $H^+(G)$ of the Hecke algebra and for the alternating subgroup $\mathcal{B}^+(G)$ of the braid group $\mathcal{B}(G)$. The Bourbaki presentation of $\mathcal{B}^+(G)$ (as well as the Bourbaki presentation of G^+) can be obtained by the Reidemeister–Schreier rewriting process [12, 13]; we present however a different proof. Then we prove a presentation of $H^+(G)$ and $\mathcal{B}^+(G)$ with generators related to the oriented edges of the Coxeter graph.

One advantage of the presentation of $H^+(G)$ with generators related to edges of the Coxeter graph is that, passing from the defining relations for G^+ to the defining relations for $H^+(G)$, only

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the characteristic equations for the generators are deformed. This is similar to the situation of the Coxeter group G and its Hecke algebra $H(G)$.

For type A, the presentations for the chain of algebras $H^+(A_n)$ and the chain of groups $\mathcal{B}^+(A_n)$ with generators related to edges of the Coxeter graph are local and stationary, in the sense of [14]; for types B and D these presentations of the chains of algebras $H^+(B_n)$ and $H^+(D_n)$, and the chains of groups $\mathcal{B}^+(B_n)$ and $\mathcal{B}^+(D_n)$ are local and eventually stationary. This extends results obtained in [10] for the alternating subgroups of Coxeter groups.

The paper is organized as follows. In Section 2 we give the definition of the alternating subalgebra $H^+(G)$ of the Hecke algebra, as the even, for a certain grading, subalgebra of the Hecke algebra $H(G)$. The Bourbaki presentation of $H^+(G)$ and the presentation of $H^+(G)$ with generators related to edges of the Coxeter graph are proved in Sections 3 and 4. In Section 5 we give analogues of these two presentations for the alternating subgroups of the braid groups. In Appendix, we obtain a recurrence relation and the generating function for the coefficients in the defining relations of the alternating subalgebras of the Hecke algebra.

Notation. Certain defining relations in this text involve a parameter $m \in \mathbb{Z}_{>0} \cup \infty$. It is understood that if $m = \infty$, the relation is absent.

For any non-negative integer k, $\{a, b\}_k$ denotes the product $abab\dots$ with k factors (by convention $\{a, b\}_0 := 1$); for example $\{a, b\}_1 := a$, $\{a, b\}_2 := ab$ and $\{a, b\}_3 := aba$. We also set, for any non-negative integer k, $\{a, b\}_{-k} := \{b, a\}_k$.

2. Definition of the alternating subalgebra of the Hecke algebra

Let (G, S) be a Coxeter system: S is the set of generators, $S = \{s_0, \dots, s_{n-1}\}$; the defining relations of the Coxeter group G are encoded by a symmetric matrix $\mathfrak{m}$ with $m_{ii} = 1$ and $2 \le m_{ij} \in \mathbb{Z}_{>0} \cup \infty$ for $0 \le i < j \le n-1$:

$$G = \langle\, s_0, \dots, s_{n-1} \mid \quad (s_i s_j)^{m_{ij}} = 1 \;\text{ for } i, j = 0, \dots, n-1 \text{ such that } i \le j \,\rangle. \tag{1}$$

The sign character is the unique homomorphism $\epsilon : G \to \{-1, 1\}$ such that $\epsilon(s_i) = -1$ for $i = 0, \dots, n-1$. Its kernel $G^+ := \ker(\epsilon)$ is called the alternating subgroup of G.

Recall that s_i and s_j are conjugate in the group G iff there are some $i_1, \dots, i_r \in \{0, \dots, n-1\}$ such that $m_{ii_1}, m_{i_1 i_2}, \dots, m_{i_r j}$ are odd, see, *e.g.*, [6]. Let $(q_0, \dots, q_{n-1})$ be a set of indeterminates such that $q_i = q_j$ if m_{ij} is odd. Let $\mathcal{A}$ be the ring of Laurent polynomials in q_i, $i = 0, \dots, n-1$, over $\mathbb{C}$. The *Hecke algebra* $H(G)$ is the algebra over $\mathcal{A}$ generated by $g_0, \dots, g_{n-1}$ with the defining relations:

$$g_i^2 = (q_i - q_i^{-1}) g_i + 1 \;\text{ for } i = 0, \dots, n-1, \tag{2}$$

$$\{g_i, g_j\}_{m_{ij}} = \{g_j, g_i\}_{m_{ij}} \quad \text{for } i, j = 0, \dots, n-1 \text{ such that } i < j. \tag{3}$$

The algebra $H(G)$ is a flat deformation of the group ring $\mathcal{A}G$: $H(G)$ has a basis whose elements are in one-to-one correspondence with the elements of G (see chap. IV, sec. 1 exercise 23 in [1], [3] and [4] for different proofs).

The assignment $g_i \mapsto -g_i^{-1}$ extends to an involutive homomorphism $\phi \colon H(G) \to H(G)$ of algebras, $\phi^2 = \mathrm{id}$. Thus, $H(G) = H^+(G) \oplus H^-(G)$, where $H^+(G)$ and $H^-(G)$ are eigenspaces of ϕ corresponding to eigenvalues $+1$ and -1; the involution ϕ defines a $\mathbb{Z}_2$-grading on $H(G)$. The subalgebra $H^+(G)$ of even elements is called the *alternating subalgebra* of the Hecke algebra $H(G)$.

Let $\mathcal{B} = \mathcal{B}^+ \oplus \mathcal{B}^-$ be a $\mathbb{Z}_2$-graded associative algebra. Assume that $\mathcal{B}^-$ contains an invertible element f. Then the left multiplication by f gives an isomorphism of vector spaces $\mathcal{B}^+$ and $\mathcal{B}^-$. We then say that $\mathcal{B}^+$ is a *subalgebra of index 2* of $\mathcal{B}$. Define the following elements of $H(G)$:

$$f_i := \frac{1}{q_i + q_i^{-1}}(g_i + g_i^{-1}) = \frac{2g_i - (q_i - q_i^{-1})}{q_i + q_i^{-1}}, \qquad i = 0, \dots, n-1.$$

Since $f_i^2 = 1$ and $\phi(f_i) = -f_i$, $i = 0, \dots, n-1$, $H^+(G)$ is a subalgebra of index 2 of $H(G)$; therefore, $H^+(G)$ is a flat deformation of the group ring $\mathcal{A}G^+$.

The elements f_i, $i = 0, \dots, n-1$, form a generating set of $H(G)$. The algebra $H^+(G)$ is generated by the elements $f_i f_j$, $i, j = 0, \dots, n-1$ and $i \neq j$.

Let $\beta_i := (q_i - q_i^{-1})/(q_i + q_i^{-1})$, $i = 0, \dots, n-1$. Since the expression $\{g_i, g_j\}_{m_{ij}} - \{g_j, g_i\}_{m_{ij}}$ is antisymmetric with respect to $i \leftrightarrow j$, the defining relations (3) of $H(G)$, in terms of the elements f_i, $i = 0, \dots, n-1$, can be rewritten in the form

$$\sum_{k=1}^{m_{ij}} a_k^{(m_{ij})} \big(\{f_i, f_j\}_k - \{f_j, f_i\}_k\big) + \sum_{k=0}^{m_{ij}-1} b_k^{(m_{ij})} \big(\{f_i, f_j\}_k + \{f_j, f_i\}_k\big) = 0 \ . \tag{4}$$

The leading coefficient $a_{m_{ij}}^{(m_{ij})}$ is non-zero and we normalize it to be $a_{m_{ij}}^{(m_{ij})} = 1$. With this choice, $a_k^{(m_{ij})}, b_k^{(m_{ij})} \in \mathbb{Z}[\beta_i, \beta_j]$ are polynomials in β_i, β_j with integer coefficients; $a_k^{(m_{ij})}$ is symmetric while $b_k^{(m_{ij})}$ is antisymmetric with respect to $\beta_i \leftrightarrow \beta_j$.

Lemma 1. *We have*

$$a_k^{(m_{ij})} = 0 \quad \textit{if} \quad k \not\equiv m_{ij} \,(\mathrm{mod}\, 2), \tag{5}$$

$$b_k^{(m_{ij})} = 0 \quad \text{for any } k. \tag{6}$$

Proof. The algebra $\mathcal{D}_{ij}$ with the generators g_i and g_j and the defining relations (2)–(3) is a flat deformation of the group ring of the dihedral group. The elements $1, \{g_i, g_j\}_k, \{g_j, g_i\}_k$, $k = 1, \dots, m_{ij} - 1$, and $\{g_i, g_j\}_{m_{ij}}$ form a basis $\mathfrak{B}$ of $\mathcal{D}_{ij}$. Denote by ρ_{ij} the expression in the left hand side of (4). As ϕ is an automorphism of $\mathcal{D}_{ij}$, the relation $\phi(\rho_{ij}) = 0$ holds in the algebra $\mathcal{D}_{ij}$. Assume that there exists $k \not\equiv m_{ij} (\mathrm{mod}\, 2)$ such that $a_k^{(m_{ij})} \neq 0$ or $b_k^{(m_{ij})} \neq 0$. Then $\rho_{ij} - (-1)^{m_{ij}} \phi(\rho_{ij}) = 0$ can be rewritten as a relation between the elements of $\mathfrak{B}$, a contradiction. Thus,

$$a_k^{(m_{ij})} = 0 \quad \text{and} \quad b_k^{(m_{ij})} = 0 \qquad \text{if} \quad k \not\equiv m_{ij} \,(\mathrm{mod}\, 2). \tag{7}$$

If m_{ij} is odd then $\beta_i = \beta_j$, so the expression $\{g_i, g_j\}_{m_{ij}} - \{g_j, g_i\}_{m_{ij}}$, rewritten in terms of the elements f_i, f_j, is antisymmetric with respect to $f_i \leftrightarrow f_j$; thus $b_k^{(m_{ij})}$ vanish. Assume, for m_{ij} even, that there exists an even k such that $b_k^{(m_{ij})} \neq 0$. Then, taking into account (7), we can rewrite $\rho_{ij} + f_i \rho_{ij} f_i$ as a relation between the elements of $\mathfrak{B}$, a contradiction. $\square$

To conclude, we obtain the following set of defining relations of $H(G)$:

$$f_i^2 = 1 \quad \text{for } i = 0, \dots, n-1, \tag{8}$$

and, multiplying the relation (4) by $(-1)^{m_{ij}} \{f_i, f_j\}_{m_{ij}}$,

$$\sum_{k=1}^{m_{ij}} a_k^{(m_{ij})} \Big((f_i f_j)^{\frac{m_{ij}+k}{2}} - (f_i f_j)^{\frac{m_{ij}-k}{2}} \Big) = 0 \quad \text{for } i, j = 0, \dots, n-1 \text{ such that } i < j \tag{9}$$

with the restriction (5).

For the types A and B, the algebra $H^+(G)$ was introduced in [8, 9]. For equal parameters, $q_i = q$, the coefficients $a_k^{(m_{ij})}$ have been calculated in [11]. We study the coefficients $a_k^{(m_{ij})}$ with arbitrary q_i and q_j in the Appendix.

3. Bourbaki presentation

Let (G, S) be a Coxeter system with the Coxeter matrix $\mathfrak{m}$. We first recall the Bourbaki presentation of G^+ suggested in [1], chap. IV, sec. 1, exercise 9 (see [2] for a proof). The alternating group G^+ is isomorphic to the group generated by $R_1, \dots, R_{n-1}$ with the defining relations:

$$\begin{cases} R_i^{m_{0i}} = 1 & \text{for } i = 1, \dots, n-1, \\ (R_i^{-1} R_j)^{m_{ij}} = 1 & \text{for } i, j = 1, \dots, n-1 \text{ such that } i < j. \end{cases} \tag{10}$$

The isomorphism with G^+ is given by $R_i \mapsto s_0 s_i$ for $i = 1, \dots, n-1$. The Bourbaki presentation of the alternating group G^+ depends on the choice of a generator carrying the subscript 0.

We prove in this Section a presentation of $H^+(G)$ similar to the Bourbaki presentation (10) of G^+.

Proposition 2. *For a Coxeter system (G, S) with the Coxeter matrix $\mathfrak{m}$, the alternating subalgebra $H^+(G)$ of the Hecke algebra is isomorphic to the algebra $\mathfrak{A}$ with the generators $Y_1^{\pm 1}, \dots, Y_{n-1}^{\pm 1}$ and the defining relations*

$$\begin{cases} \sum_{k=1}^{m_{0i}} a_k^{(m_{0i})} \Big(Y_i^{\frac{m_{0i}+k}{2}} - Y_i^{\frac{m_{0i}-k}{2}} \Big) = 0 & \textit{for } i = 1, \dots, n-1, \\ \sum_{k=1}^{m_{ij}} a_k^{(m_{ij})} \Big((Y_i^{-1} Y_j)^{\frac{m_{ij}+k}{2}} - (Y_i^{-1} Y_j)^{\frac{m_{ij}-k}{2}} \Big) = 0 & \textit{for } i, j = 1, \dots, n-1 \textit{ such that } i < j. \end{cases} \tag{11}$$

Proof. Define a map ψ from the set of generators $\{Y_1, \dots, Y_{n-1}\}$ to the algebra $H^+(G)$ by

$$Y_i \mapsto f_0 f_i \text{ for } i = 1, \dots, n-1.$$

Due to the relations (8)–(9), this map extends to a (surjective) homomorphism, which we denote again by ψ, from the algebra $\mathfrak{A}$ to $H^+(G)$. We shall prove that ψ is an isomorphism.

The left hand side of the first relation in (11) is invariant under the following sequence of operations: replace Y_i by Y_i^{-1} and then multiply by $-Y_i^{m_{0i}}$. One can verify directly that the left hand side of the second relation in (11) is invariant under the following sequence of operations: replace Y_i by Y_i^{-1}, Y_j

by Y_j^{-1}, then multiply from the left by $-Y_j^{-1}(Y_jY_i^{-1})^{m_{ij}}$ and from the right by Y_j. Therefore the map defined by $\omega : Y_i \mapsto Y_i^{-1}$ extends to an involutive automorphism of the algebra $\mathfrak{A}$.

With the help of ω, we define the cross-product $\tilde{\mathfrak{A}}$ of the algebra $\mathfrak{A}$ with the cyclic group C_2 with two elements. Let f be the generator of the group C_2. As a vector space, $\tilde{\mathfrak{A}}$ is isomorphic to $\mathfrak{A} \otimes \mathcal{A}C_2$. The generators of $\tilde{\mathfrak{A}}$ are the elements $Y_1^{\pm 1}, \dots, Y_{n-1}^{\pm 1}$ with the defining relations (11), and in addition the generator f with the defining relations $f^2 = 1$ and $fY_i = Y_i^{-1}f$, $i = 1, \dots, n-1$. The map

$$f \mapsto f_0 \quad \text{and} \quad Y_i \mapsto f_0 f_i \ \text{ for } i = 1, \dots, n-1,$$

extends to a morphism of algebras $\psi_1 : \tilde{\mathfrak{A}} \to H(G)$. The verification is straightforward (use (8)-(9)). On the other hand, one directly verifies that the map

$$f_0 \mapsto f \quad \text{and} \quad f_i \mapsto fY_i \ \text{ for } i = 1, \dots, n-1,$$

extends to a morphism of algebras $\psi_2 : H(G) \to \tilde{\mathfrak{A}}$. Moreover, the morphisms ψ_1 and ψ_2 are mutually inverse. The restriction of ψ_2 to $H^+(G)$ is the morphism inverse to ψ. □

4. Presentation using edges of the Coxeter graph

Let (G, S) be a Coxeter system with the Coxeter matrix $\mathfrak{m}$. We first recall the presentation given in [10] of G^+; it uses edges of the Coxeter graph $\mathcal{G}$ of (G, S).

Recall that vertices, indexed by the subscripts $0, 1, \dots, n-1$, of the Coxeter graph $\mathcal{G}$ are in one-to-one correspondence with the generators $s_0, \dots, s_{n-1}$ of G; vertices i and j are connected if and only if $m_{ij} \geq 3$ and then the edge between i and j is labeled by the number m_{ij}. In the sequel the edge between vertices i and j is denoted by (ij).

If $\mathcal{G}$ is not connected, let $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_m$ be its connected components. We choose an arbitrary vertex i_a of $\mathcal{G}_a$ for $a = 1, \dots, m$; we add an edge between i_l and i_{l+1} for $l = 1, \dots, m-1$ and label it by the number 2. The obtained connected graph $\mathcal{G}^c$ we call a *connected extension* of the Coxeter graph $\mathcal{G}$.

The presentation in [10] uses an orientation - chosen arbitrarily - of edges of the connected extension $\mathcal{G}^c$ of the Coxeter graph. For concreteness, if there is an edge between i and j with $i < j$, we orient it from i to j. We associate a generator r_{ij} to each oriented edge of $\mathcal{G}^c$. For a generator r_{ij} we denote by r_{ji} the inverse, $r_{ji} := r_{ij}^{-1}$.

Definition 3. *Two edges (ij) and (kl) of $\mathcal{G}^c$ are said to be not connected if $\{i, j\} \cap \{l, m\} = \varnothing$ and there is no edge connecting any of the vertices $\{i, j\}$ with any of the vertices $\{l, m\}$.*

The alternating group G^+ is isomorphic [10] to the group with the generators r_{ij} and the defining relations

$$\begin{cases} (r_{ij})^{m_{ij}} = 1 & \text{for all generators } r_{ij}, \\ r_{ii_1}r_{i_1i_2}\dots r_{i_ai} = 1 & \text{for cycles with edges } (ii_1), (i_1i_2)\dots, (i_ai), \\ (r_{ij}r_{jk})^2 = 1 & \text{for } r_{ij}, r_{jk} \text{ such that } i < k \text{ and } m_{ik} = 2, \\ (r_{ij}r_{jk}r_{kl})^2 = 1 & \text{for } r_{ij}, r_{jk}, r_{kl} \text{ such that } i < l \text{ and } m_{il} = 2, \\ r_{ij}r_{lm} = r_{lm}r_{ij} & \text{if } (ij) \text{ and } (lm) \text{ are not connected.} \end{cases} \tag{12}$$

The isomorphism with G^+ is given by $r_{ij} \mapsto s_i s_j$ for all generators r_{ij}.

We generalize this presentation to a presentation of $H^+(G)$. Associate an element y_{ij} to each generator r_{ij} of G^+, and set, for all y_{ij}, $y_{ji} := y_{ij}^{-1}$.

Proposition 4. *The alternating subalgebra $H^+(G)$ of the Hecke algebra is isomorphic to the algebra $\mathfrak{Y}$ with the generators y_{ij} and the defining relations*

$$\begin{cases} \sum_{k=1}^{m_{ij}} a_k^{(m_{ij})} \big(y_{ij}^{\frac{m_{ij}+k}{2}} - y_{ij}^{\frac{m_{ij}-k}{2}}\big) = 0 & \text{for all generators } y_{ij}, \quad (13) \\ y_{ii_1} y_{i_1 i_2} \dots y_{i_a i} = 1 & \text{for cycles with edges } (ii_1), (i_1 i_2) \dots, (i_a i), \quad (14) \\ (y_{ij} y_{jk})^2 = 1 & \text{for } y_{ij}, y_{jk} \text{ such that } i < k \text{ and } m_{ik} = 2, \quad (15) \\ (y_{ij} y_{jk} y_{kl})^2 = 1 & \text{for } y_{ij}, y_{jk}, y_{kl} \text{ such that } i < l \text{ and } m_{il} = 2, \quad (16) \\ y_{ij} y_{lm} = y_{lm} y_{ij} & \text{if } (ij) \text{ and } (lm) \text{ are not connected.} \quad (17) \end{cases}$$

Let $\mathfrak{c}_1, \dots, \mathfrak{c}_{\mathfrak{l}}$ be a set of generators of the fundamental group of $\mathcal{G}^c$. In the set of the defining relations it is sufficient to impose the relation (14) for the cycles $\mathfrak{c}_{\mathfrak{a}}$, $\mathfrak{a} = 1, \dots, \mathfrak{l}$.

Proof of the Proposition. Notice that if $m_{0i} = 2$ then the first relation in (11) reduces to $Y_i^2 = 1$ and also that, if $m_{ij} = 2$ then the second relation in (11) reduces to $(Y_i^{-1} Y_j)^2 = 1$. Due to this fact, the proof is very similar to the proof in [10] in the classical situation (that is, for the presentation (12) of G^+). So we only sketch it. The following map

$$y_{ij} \mapsto \begin{cases} Y_i^{-1} Y_j & \text{if } i \neq 0, \\ Y_j & \text{if } i = 0, \end{cases} \tag{18}$$

extends to an algebra homomorphism $\Phi : \mathfrak{Y} \to H^+(G)$.

Define now the map Ψ from the set of generators $\{Y_1, \dots, Y_{n-1}\}$ of $H^+(G)$ to $\mathfrak{Y}$ by

$$\Psi \ : \ Y_i \mapsto \grave{Y}_i := y_{0i_1} y_{i_1 i_2} \dots y_{i_k i} \ \text{ for all } i = 1, \dots, n-1, \tag{19}$$

where $(0, i_1, i_2, \dots, i_k, i)$ is an arbitrary path from the vertex 0 to the vertex i in $\mathcal{G}^c$. The map Ψ is well-defined since the element $\grave{Y}_i$ does not depend on the chosen path, due to the relation (14). The map Ψ extends to an algebra homomorphism from $H^+(G)$ to $\mathfrak{Y}$, which we still denote by Ψ. Moreover Ψ and Φ are mutually inverse. □

Remark. The defining relations (13)–(17) of the algebra $H^+(G)$ are deformations of the defining relations (12) of the group G^+. Only the characteristic equation for the generators is deformed. This is similar to the Hecke algebra situation (passing from the relations (1) to the relations (2)–(3), only the characteristic equation for the generators is deformed). This phenomenon does not appear in the deformation of the Bourbaki presentation (10) of G^+ to the Bourbaki presentation (11) of $H^+(G)$.

5. Alternating subgroups of braid groups

5.1 Definition

Let (G, S) be a Coxeter system with the Coxeter matrix $\mathfrak{m}$. The braid group $\mathcal{B}(G)$ is the group generated by $g_0, \dots, g_{n-1}$ with the defining relations:

$$\{g_i, g_j\}_{m_{ij}} = \{g_j, g_i\}_{m_{ij}} \quad \text{for } i, j = 0, \dots, n-1 \text{ such that } i < j. \tag{20}$$

Extend the sign character to the group $\mathcal{B}(G)$, that is, define the homomorphism $\epsilon : \mathcal{B}(G) \to \{-1, 1\}$ by $\epsilon(g_i) = -1$ for $i = 0, \dots, n-1$. The kernel, $\mathcal{B}^+(G) := \ker(\epsilon)$, we call the alternating subgroup of the braid group $\mathcal{B}(G)$. The group $\mathcal{B}^+(G)$ is generated by the elements $g_i g_j$ (and their inverses), $i, j = 0, \dots, n-1$.

Remark. There is a natural $\mathbb{Z}_2$-grading of the group ring of $\mathcal{B}(G)$ defined by ϵ. Let π be the natural surjection of the group ring of $\mathcal{B}(G)$ to the Hecke algebra $H(G)$ (the quotient by the relation (2)). Recall the grading on $H(G)$ defined by the involution ϕ. It should be noted that $\phi\pi \neq \pi\epsilon$, the image of $\mathcal{B}^+(G)$ under π does not belong to $H^+(G)$; in other words, the grading on $H(G)$ is not induced by π from the grading on the group ring of $\mathcal{B}(G)$.

5.2 Bourbaki presentation for alternating subgroups of braid groups

We extend the Bourbaki presentations (10) and (11) of the group G^+ and the algebra $H^+(G)$ to the group $\mathcal{B}^+(G)$. The presentation depends on a choice of a generator g_0, carrying the subscript 0, among the generators of $\mathcal{B}(G)$.

Proposition 5. *For a Coxeter system (G, S) with the Coxeter matrix $\mathfrak{m}$, the alternating subgroup $\mathcal{B}^+(G)$ of the braid group is isomorphic to the group B with the generators $\mathsf{R}_0, \dots, \mathsf{R}_{n-1}$ and $\mathsf{R}'_0, \dots, \mathsf{R}'_{n-1}$ and the defining relations*

$$\begin{cases} \mathsf{R}'_0 = 1\,, \\ \{\mathsf{R}'_i, \mathsf{R}_j\}_{m_{ij}} = \{\mathsf{R}'_j, \mathsf{R}_i\}_{m_{ij}} & \textit{for } i, j = 0, \dots, n-1 \textit{ such that } i < j, \\ \{\mathsf{R}_i, \mathsf{R}'_j\}_{m_{ij}} = \{\mathsf{R}_j, \mathsf{R}'_i\}_{m_{ij}} & \textit{for } i, j = 0, \dots, n-1 \textit{ such that } i < j. \end{cases} \tag{21}$$

Proof. Define a map ψ from the set of generators of B to $\mathcal{B}^+(G)$ by

$$\mathsf{R}_i \mapsto g_0 g_i,\ i = 0, \dots n-1, \quad \text{and} \quad \mathsf{R}'_i \mapsto g_i g_0^{-1},\ i = 0, \dots n-1. \tag{22}$$

One directly verifies that ψ extends to a homomorphism, which we still denote by ψ, from B to $\mathcal{B}^+(G)$. Moreover, the homomorphism ψ is surjective. Indeed, for $i, j = 0, \dots, n-1$, we have $g_i g_j = \psi(\mathsf{R}'_i \mathsf{R}_j)$. We shall prove that ψ is actually an isomorphism.

Define a map ω from the the set of generators of B to B by

$$\mathsf{R}_i \mapsto \mathsf{R}'_i \mathsf{R}_0,\ i = 0, \dots n-1, \quad \text{and} \quad \mathsf{R}'_i \mapsto \mathsf{R}_0^{-1} \mathsf{R}_i,\ i = 0, \dots n-1.$$

It is straightforward to verify that ω defines an automorphism of B. Moreover, the automorphism ω^2 is inner: for any $x \in B$, we have $\omega^2(x) = \mathsf{R}_0^{-1} x \mathsf{R}_0$.

The automorphism ω generates the action of the infinite cyclic group C on B. Let $B \rtimes C$ be the corresponding semidirect product; the group $B \rtimes C$ is generated by the generators of B and an element g, and we add to the defining relations of B the relation $gxg^{-1} = \omega(x)$ for each generator x of B.

Now let Q be the quotient of the group $B \rtimes C$ by the relation $g^2 = \mathsf{R}_0^{-1}$. The following map

$$g \mapsto g_0^{-1}, \quad \mathsf{R}_i \mapsto g_0 g_i, \ i = 0, \dots, n-1, \quad \text{and} \quad \mathsf{R}'_i \mapsto g_i g_0^{-1}, \ i = 0, \dots, n-1,$$

extends to a homomorphism ψ_1 from Q to $\mathcal{B}(G)$. The verification is the same as for the map ψ, given by (22), with, in addition, the verification of the relations in Q concerning the generator g; these are satisfied by construction.

The following map

$$g_i \mapsto g^{-1}\mathsf{R}_i, \ i = 0, \dots, n-1,$$

extends to a homomorphism ψ_2 from $\mathcal{B}(G)$ to Q. We omit the straightforward calculations here.

Moreover, the morphisms ψ_1 and ψ_2 are mutually inverse. The restriction of ψ_2 to $\mathcal{B}^+(G)$ is the inverse of ψ; thus, the homomorphism ψ, given by (22), is the required isomorphism between B and $\mathcal{B}^+(G)$. □

Remarks. **(i)** Denote by τ the standard anti-automorphism of the group $\mathcal{B}(G)$, sending g_i to g_i, $i = 0, \dots, n-1$. The action of τ on the generators of the Bourbaki presentation is given by

$$\mathsf{R}_i \to \mathsf{R}'_i \mathsf{R}_0 \ \text{ and } \ \mathsf{R}'_i \to \mathsf{R}_0^{-1}\mathsf{R}_i, \quad i = 0, \dots, n-1.$$

(ii) The Coxeter group G is the quotient of the braid group $\mathcal{B}(G)$ by the relations $g_i^2 = 1$, $i = 0, \dots, n-1$. In the alternating setting, we have a similar result; the alternating subgroup G^+ of the Coxeter group G is the quotient of the group $\mathcal{B}^+(G)$ by the relations $\mathsf{R}_0 = 1$ and $\mathsf{R}'_i = \mathsf{R}_i^{-1}$, $i = 1, \dots, n-1$. Indeed, in this quotient, the relations (21) reduce to

$$\begin{cases} \mathsf{R}_j^{m_{0j}} = 1 & \text{for } j = 1, \dots, n-1, \\ (\mathsf{R}_i^{-1}\mathsf{R}_j)^{m_{ij}} = 1 & \text{for } i, j = 1, \dots, n-1 \text{ such that } i < j. \end{cases}$$

These are the defining relations of the Bourbaki presentation of G^+, see (10).

(iii) The Reidemeister–Schreier rewriting process [12, 13] (see *e.g.* [7] for a more recent exposition), allows to find a presentation of a subgroup H of a group G, given a presentation of G and a suitable information about H. We apply this process to the subgroup $\mathcal{B}^+(G)$ of $\mathcal{B}(G)$. Decompose $\mathcal{B}(G)$ into the disjoint union of its right cosets with respect to $\mathcal{B}^+(G)$, $\mathcal{B}(G) = \mathcal{B}^+(G) \cup \mathcal{B}^+(G)g_0$. For any $a \in \mathcal{B}(G)$, define $\overline{a} \in \{1, g_0\}$ by $\mathcal{B}^+(G)a = \mathcal{B}^+(G)\overline{a}$ and let $\mathsf{S} := \{g_0, g_1, \dots, g_{n-1}\}$. The Reidemeister–Schreier rewriting process asserts that $\mathcal{B}^+(G)$ is isomorphic to the group with a set of generators $\mathfrak{S}$ and a set of defining relations $\mathfrak{D}$ defined as follows.

• Elements of $\mathfrak{S}$ are in one-to-one correspondence with elements $\gamma(a, g) := ag(\overline{ag})^{-1}$, $g \in \mathsf{S}$ and $a \in \{1, g_0\}$, such that $ag(\overline{ag})^{-1} \neq 1$; we obtain generators $\mathfrak{R}_i \in \mathfrak{S}$, $i = 0, \dots, n-1$, corresponding to $g_0 g_i(\overline{g_0 g_i})^{-1} = g_0 g_i$, and generators $\mathfrak{R}'_i \in \mathfrak{S}$, $i = 1, \dots, n-1$, corresponding to $g_i(\overline{g_i})^{-1} = g_i g_0^{-1}$. Define for convenience $\mathfrak{R}'_0 := 1$.

• For a word $w = g_{i_1} \dots g_{i_k}$ in the the alphabet S let $\pi(w) := \hat{\gamma}(1, g_{i_1})\hat{\gamma}(\overline{g_{i_1}}, g_{i_2}) \dots \hat{\gamma}(\overline{g_{i_1} \dots g_{i_{k-1}}}, g_{i_k})$ where $\hat{\gamma}(1, g_0) := \mathfrak{R}'_0 = 1$ and, otherwise, $\hat{\gamma}(a, g)$ is the generator corresponding to $\gamma(a, g)$. The relations $\pi(a\{g_i, g_j\}_{m_{ij}}) = \pi(a\{g_j, g_i\}_{m_{ij}})$, $a \in \{1, g_0\}$ and $i, j = 0, \dots, n-1$, $i < j$, form the set $\mathfrak{D}$.

It is straightforward to see that the relations in $\mathfrak{D}$ are $\{\mathfrak{R}'_i, \mathfrak{R}_j\}_{m_{ij}} = \{\mathfrak{R}'_j, \mathfrak{R}_i\}_{m_{ij}}$ and $\{\mathfrak{R}_i, \mathfrak{R}'_j\}_{m_{ij}} = \{\mathfrak{R}_j, \mathfrak{R}'_i\}_{m_{ij}}$, $i, j = 0, \dots, n-1$, $i < j$.

Thus, the presentation of the Proposition 5 coincides with the one obtained by the Reidemeister–Schreier rewriting process (with $\{1, g_0\}$ as the “Schreier transversal”) for the subgroup $\mathcal{B}^+(G)$ of $\mathcal{B}(G)$. For the alternating subgroup G^+ of the Coxeter group G the Reidemeister–Schreier rewriting process leads to the presentation (10).

(iv) The set of generators in the presentation of $\mathcal{B}^+(G)$ given by the Proposition 5 is, in general, not minimal. For example, if there is some $j \in \{1, \dots, n-1\}$ such that $m_{0j} = 2$, then the second and third relations in (21) (for $i = 0$) imply that $\mathsf{R}'_j = \mathsf{R}_j \mathsf{R}_0^{-1} = \mathsf{R}_0^{-1} \mathsf{R}_j$. Furthermore, if there is some $j \in \{1, \dots, n-1\}$ such that m_{0j} is odd, then the second and third relations in (21) (for $i = 0$) imply that $\mathsf{R}_j^{\frac{m_{0j}-1}{2}} = (\mathsf{R}'_j \mathsf{R}_0)^{\frac{m_{0j}-1}{2}} \mathsf{R}'_j$ and $\mathsf{R}_0 (\mathsf{R}'_j \mathsf{R}_0)^{\frac{m_{0j}-1}{2}} = \mathsf{R}_j^{\frac{m_{0j}+1}{2}}$. Thus, we have that

$$\mathsf{R}'_j = \mathsf{R}_j^{-\frac{m_{0j}+1}{2}} \mathsf{R}_0 \mathsf{R}_j^{\frac{m_{0j}-1}{2}} \quad \text{for } j = 1, \dots, n-1 \text{ such that } m_{0j} \text{ is odd.}$$

Nevertheless, in general, both sets, R_i and R'_i, of generators are needed. Consider, for example, the braid group U generated by g_0 and g_1 and the defining relations $g_0 g_1 g_0 g_1 = g_1 g_0 g_1 g_0$ (that is, $\{g_0, g_1\}_4 = \{g_1, g_0\}_4$). In this case, the generators for the alternating subgroup suggested by the Proposition 5 are g_0^2, $g_0 g_1$ and $g_1 g_0^{-1}$. The element $g_1 g_0^{-1}$ does not belong to the subgroup generated by g_0^2 and $g_0 g_1$. Indeed, let $\bar{U}$ be the quotient of U by the normal subgroup generated by g_0^2 and let $\bar{g}_i$ be the images of g_i in $\bar{U}$. It is known that $\bar{U}$ is isomorphic to $S_2 \ltimes C^2$, where S_2 is the symmetric group on 2 elements and C is the infinite cyclic group; S_2 acts on C^2 by permuting the two copies of C. Suppose that there exists an integer x such that $\bar{g}_1 \bar{g}_0 = (\bar{g}_0 \bar{g}_1)^x$. If $x = 1$, that is $\bar{g}_1 \bar{g}_0 = \bar{g}_0 \bar{g}_1$, then the group $\bar{U}$ would be isomorphic to $S_2 \times C$, which is impossible. Assume that $x \neq 1$. We have $(\bar{g}_1 \bar{g}_0)^2 = (\bar{g}_0 \bar{g}_1)^{2x}$ which, together with the defining relation $(\bar{g}_1 \bar{g}_0)^2 = (\bar{g}_0 \bar{g}_1)^2$ leads to $(\bar{g}_0 \bar{g}_1)^{2(x-1)} = 1$, contradicting to the fact that $\bar{U}$ is infinite. A similar calculation shows that the element $g_0 g_1 \in U$ does not belong to the subgroup generated by g_0^2 and $g_1 g_0^{-1}$.

(v) The elements R_i, $i = 0, \dots, n-1$, generate the alternating subgroup of the braid group of any simply-laced type, see the remark **(v)**. We give the presentation for the alternating subgroup of the braid group of type A. Label the generators of the braid group $\mathcal{B}(A_n)$ in the standard way; that is, $\mathcal{B}(A_n)$ is generated by $g_0, g_1, \dots, g_{n-1}$ with the defining relations:

$$\begin{cases} g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} & \text{for } i = 0, \dots, n-2, \\ g_i g_j = g_j g_i & \text{for } i, j = 0, \dots, n-1 \text{ such that } |i-j| > 1. \end{cases} \tag{23}$$

The group $\mathcal{B}^+(A_n)$ is isomorphic to the group generated by $\mathsf{R}_0, \mathsf{R}_1, \dots, \mathsf{R}_{n-1}$ with the defining relations:

$$\begin{cases} \mathsf{R}_0 \mathsf{R}_1 \mathsf{R}_0 = \mathsf{R}_1^2 \mathsf{R}_0^{-1} \mathsf{R}_1^2, & \\ \mathsf{R}_0 \mathsf{R}_j = \mathsf{R}_j \mathsf{R}_0 & \text{for } j = 2, \dots, n-1, \\ \mathsf{R}_2 \mathsf{R}_1 \mathsf{R}_2 = \mathsf{R}_1^2 \mathsf{R}_0^{-1} \mathsf{R}_2 \mathsf{R}_0^{-1} \mathsf{R}_1^2, & \\ \mathsf{R}_2 \mathsf{R}_1^2 \mathsf{R}_2 = \mathsf{R}_0 \mathsf{R}_1 \mathsf{R}_2 \mathsf{R}_0^{-1} \mathsf{R}_1 \mathsf{R}_0, & \\ \mathsf{R}_1^2 \mathsf{R}_j = \mathsf{R}_j \mathsf{R}_1 \mathsf{R}_0, \quad \mathsf{R}_j \mathsf{R}_1^2 = \mathsf{R}_0 \mathsf{R}_1 \mathsf{R}_j & \text{for } j = 3, \dots, n-1, \\ \mathsf{R}_i \mathsf{R}_{i+1} \mathsf{R}_i = \mathsf{R}_{i+1} \mathsf{R}_i \mathsf{R}_{i+1} & \text{for } i = 2, \dots, n-2, \\ \mathsf{R}_i \mathsf{R}_j = \mathsf{R}_j \mathsf{R}_i & \text{for } i, j = 2, \dots, n-1 \text{ such that } |i-j| > 1. \end{cases} \tag{24}$$

The verification that this presentation is equivalent to (21) for the type A is straightforward once one notices that here we have $\mathsf{R}'_1 = \mathsf{R}_0^{-1}\mathsf{R}_1^2\mathsf{R}_0^{-1}$ and $\mathsf{R}'_j = \mathsf{R}_j\mathsf{R}_0^{-1} = \mathsf{R}_0^{-1}\mathsf{R}_j$ for $j = 2,\dots,n-1$.

It is interesting to note that in terms of generators $\bar{\mathsf{R}}_0 := \mathsf{R}_0^{-1}$ and R_i, $i = 1,\dots,n-1$, one can rewrite all relations (24) without inverses of generators and define thus a monoid of positive elements.

5.3 Presentation using edges of the Coxeter graph of alternating subgroups of braid groups

The group $\mathcal{B}^+(G)$ admits a presentation similar to the presentations of the group G^+ and the algebra $H^+(G)$, see (12) and the Proposition 4. Associate, as in Section 4, a generator r_{ij} to any oriented edge (the edges are oriented from i to j if $i < j$) of the graph $\mathcal{G}^c$. Set $\mathsf{r}_{ji} := \mathsf{r}_{ij}^{-1}$ for all generators r_{ij}.

Proposition 6. *For a Coxeter system (G, S) with the Coxeter matrix $\mathfrak{m}$, the alternating subgroup $\mathcal{B}^+(G)$ of the braid group is isomorphic to the group $\daleth$ generated by the elements r_{ij} and $\mathsf{t}_0,\dots,\mathsf{t}_{n-1}$ with the defining relations*

$$\begin{cases} \mathsf{r}_{ii_1}\mathsf{r}_{i_1i_2}\dots\mathsf{r}_{i_ai} = 1 \qquad \textit{for cycles with edges } (ii_1),(i_1i_2)\dots,(i_ai), \\ \mathsf{r}_{ij}\mathsf{r}_{jk}\mathsf{t}_k = \mathsf{r}_{kj}\mathsf{r}_{ji}\mathsf{t}_i, \quad \mathsf{t}_k\mathsf{r}_{ij}\mathsf{r}_{jk} = \mathsf{r}_{kj}\mathsf{r}_{ji}\mathsf{t}_i & \textit{if } i<k \textit{ and } m_{ik}=2, \\ \mathsf{r}_{ij}\mathsf{r}_{jk}\mathsf{r}_{kl}\mathsf{t}_l = \mathsf{r}_{lk}\mathsf{r}_{kj}\mathsf{r}_{ji}\mathsf{t}_i, \quad \mathsf{t}_l\mathsf{r}_{ij}\mathsf{r}_{jk}\mathsf{r}_{kl} = \mathsf{r}_{lk}\mathsf{r}_{kj}\mathsf{r}_{ji}\mathsf{t}_i & \textit{if } i<l \textit{ and } m_{il}=2, \\ (\mathsf{r}_{ij}\mathsf{t}_j)^{\frac{m_{ij}}{2}} = (\mathsf{r}_{ji}\mathsf{t}_i)^{\frac{m_{ij}}{2}}, \quad (\mathsf{t}_j\mathsf{r}_{ij})^{\frac{m_{ij}}{2}} = (\mathsf{r}_{ji}\mathsf{t}_i)^{\frac{m_{ij}}{2}} & \textit{if } m_{ij}>2 \textit{ and } m_{ij} \textit{ is even}, \\ (\mathsf{r}_{ij}\mathsf{t}_j)^{\frac{m_{ij}-1}{2}}\mathsf{r}_{ij} = (\mathsf{r}_{ji}\mathsf{t}_i)^{\frac{m_{ij}-1}{2}}, \quad (\mathsf{r}_{ij}\mathsf{t}_j)^{\frac{m_{ij}+1}{2}} = \mathsf{t}_i(\mathsf{r}_{ji}\mathsf{t}_i)^{\frac{m_{ij}-1}{2}} & \textit{if } m_{ij}>2 \textit{ and } m_{ij} \textit{ is odd}, \\ \mathsf{r}_{ij}\mathsf{r}_{lm} = \mathsf{r}_{lm}\mathsf{r}_{ij} \qquad \textit{if } (ij) \textit{ and } (lm) \textit{ are not connected.} \end{cases} \tag{25}$$

Proof. The proof is similar to the proof of the Proposition 4. We skip the calculations and indicate below only the mutually inverse isomorphisms between the group $\daleth$ and the group $\mathcal{B}^+(G)$ with the presentation of the Proposition 5:

$$\mathsf{r}_{ij} \mapsto \mathsf{R}'_i\mathsf{R}'^{-1}_j \quad \text{and} \quad \mathsf{t}_i \mapsto \mathsf{R}'_i\mathsf{R}_i, \ \ i = 0,\dots,n-1,$$

and

$$\mathsf{R}_0 \mapsto \mathsf{t}_0, \ \ \mathsf{R}'_0 \mapsto 1, \qquad \mathsf{R}_i \mapsto \mathsf{r}_{0i_1}\mathsf{r}_{i_1i_2}\dots\mathsf{r}_{i_ai}\mathsf{t}_i \ \text{ and } \ \mathsf{R}'_i \mapsto \mathsf{r}_{ii_a}\dots\mathsf{r}_{i_2i_1}\mathsf{r}_{i_10}, \ \ i = 1,\dots,n-1,$$

where, for $i = 1,\dots,n-1$, $(0, i_1, i_2,\dots, i_a, i)$ is a path in the graph $\mathcal{G}^c$ from the vertex 0 to the vertex i. The second map is well-defined since the image of R_i (respectively, of R'_i) does not depend on the chosen path, due to the first relation in (25). □

The isomorphism between the group generated by the elements r_{ij} and $\mathsf{t}_0,\dots,\mathsf{t}_{n-1}$ with the defining relations (25) and the subgroup $\mathcal{B}^+(G)$ of the braid group $\mathcal{B}(G)$ is given by:

$$\mathsf{t}_i \mapsto g_i^2, \ \ i = 0,\dots,n-1, \quad \text{and} \quad \mathsf{r}_{ij} \mapsto g_ig_j^{-1}, \ \ \text{for all generators } \mathsf{r}_{ij}. \tag{26}$$

Remarks. (i) The action of the standard anti-automorphism τ (see remark **(i)** after the proof of the Proposition 5) on the generators of the presentation given by the Proposition 6 is

$$\mathsf{t}_i \mapsto \mathsf{t}_i, \ \ i = 0,\dots,n-1, \quad \text{and} \quad \mathsf{r}_{ij} \mapsto \mathsf{t}_j^{-1}\mathsf{r}_{ji}\mathsf{t}_i, \ \ \text{for all generators } \mathsf{r}_{ij}.$$

(ii) This remark is the analogue, for this presentation, of the remark **(ii)** after the proof of the Proposition 5. The alternating subgroup G^+ of the Coxeter group G is the quotient of the group $\mathcal{B}^+(G)$, with the presentation (25), by the relations $\mathsf{t}_i = 1$, $i = 0, \dots, n-1$. Indeed, in this quotient, the relations (25) reduce immediately to the defining relations (12) of G^+.

(iii) In the type A situation, with the same labeling of the Coxeter graph as in remark **(v)** after the proof of the Proposition 5, the presentation using edges of the Coxeter graph is the following. Set $\mathsf{r}_i := \mathsf{r}_{i-1i}$, $i = 1, \dots, n-1$. The group $\mathcal{B}^+(A_n)$ is isomorphic to the group generated by $\mathsf{r}_1, \dots, \mathsf{r}_{n-1}$ and $\mathsf{t}_0, \dots, \mathsf{t}_{n-1}$ with the defining relations:

$$\left\{\begin{array}{lll} \mathsf{r}_i\mathsf{r}_{i+1}\mathsf{t}_{i+1} = \mathsf{r}_{i+1}^{-1}\mathsf{r}_i^{-1}\mathsf{t}_{i-1}, \quad \mathsf{t}_{i+1}\mathsf{r}_i\mathsf{r}_{i+1} = \mathsf{r}_{i+1}^{-1}\mathsf{r}_i^{-1}\mathsf{t}_{i-1} & & \text{for } i = 1, \dots, n-2, \\ \mathsf{r}_i\mathsf{r}_{i+1}\mathsf{r}_{i+2}\mathsf{t}_{i+2} = \mathsf{r}_{i+2}^{-1}\mathsf{r}_{i+1}^{-1}\mathsf{r}_i^{-1}\mathsf{t}_{i-1}, \quad \mathsf{t}_{i+2}\mathsf{r}_i\mathsf{r}_{i+1}\mathsf{r}_{i+2} = \mathsf{r}_{i+2}^{-1}\mathsf{r}_{i+1}^{-1}\mathsf{r}_i^{-1}\mathsf{t}_{i-1} & & \text{for } i = 1, \dots, n-3, \\ \mathsf{r}_i\mathsf{t}_i\mathsf{r}_i = \mathsf{r}_i^{-1}\mathsf{t}_i^{-1}, \quad (\mathsf{r}_i\mathsf{t}_i)^2 = \mathsf{t}_{i-1}\mathsf{r}_i^{-1}\mathsf{t}_{i-1} & & \text{for } i = 1, \dots, n-1, \\ \mathsf{r}_i\mathsf{r}_j = \mathsf{r}_j\mathsf{r}_i \qquad \text{for } i, j = 1, \dots, n-1 \text{ such that } |i-j| > 2. & & \end{array}\right. \tag{27}$$

It is immediate to check (with the help of the isomorphism (26)) that the following relations are satisfied

$$\mathsf{r}_i\mathsf{r}_j = \mathsf{r}_j\mathsf{r}_i, \quad \mathsf{r}_i\mathsf{t}_j = \mathsf{t}_j\mathsf{r}_i, \quad \mathsf{t}_i\mathsf{t}_j = \mathsf{t}_j\mathsf{t}_i \qquad \text{if } |i-j| > 2.$$

So the presentation (27) of the chain of the groups $\mathcal{B}^+(A_n)$ is local and stationary, in the sense of [14].

Appendix. Coefficients in the defining relations of the alternating subalgebras

The coefficients $a_k^{(m_{ij})}$ appearing in (9), (11) and (13) are easy to calculate for small m_{ij}. We define, in this Appendix, certain integers $\alpha_{k,l,l'}^{(m)}$ in terms of which the elements $a_k^{(m_{ij})}$, for any m_{ij}, can be expressed. We give the recurrent (in m) relations for $\alpha_{k,l,l'}^{(m)}$ and find the generating function for $\alpha_{k,l,l'}^{(m)}$. In the one-parameter situation, we recover a closed formula from [11] for $a_k^{(m_{ij})}$.

Recursion. Let f_1 and f_2 be the generators of an algebra with the defining relations $f_1^2 = f_2^2 = 1$. Define the elements $\alpha_{k,l,l'}^{(m)} \in \mathbb{Z}$, with $m, l, l' \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}$, by

$$\{(f_1 + x), (f_2 + y)\}_m = \sum_{l,l' \geq 0} x^l y^{l'} \sum_{k \in \mathbb{Z}} \alpha_{k,l,l'}^{(m)} \{f_1, f_2\}_k \ . \tag{28}$$

By construction, for a given m, only a finite number of elements $\alpha_{k,l,l'}^{(m)}$ are non-zero; we have:

$$\alpha_{k,l,l'}^{(m)} \neq 0 \quad \Rightarrow \quad |k| \leq m - l - l', \quad k \equiv m - l - l' \, (\mathrm{mod}\, 2), \quad l \leq \lfloor (m+1)/2 \rfloor \ \text{ and } \ l' \leq \lfloor m/2 \rfloor \ .$$

Here $\lfloor x \rfloor$ is the integer part of x. The coefficients $a_k^{(m_{ij})}$, in terms of $\alpha_{k,l,l'}^{(m)}$, read

$$a_k^{(m_{ij})} = \sum_{l,l' \geq 0} \beta_i^l \beta_j^{l'} \, (\alpha_{k,l,l'}^{(m_{ij})} - \alpha_{-k,l',l}^{(m_{ij})}) \ .$$

Lemma 7. *The elements* $\alpha^{(m)}_{k,l,l'}$ *satisfy the following initial condition and recursion:*

$$\alpha^{(0)}_{k,l,l'} = \delta^0_k \, \delta^0_l \, \delta^0_{l'} \ , \tag{29}$$

$$\alpha^{(m+1)}_{k,l,l'} = \alpha^{(m)}_{k-1,l',l} + \alpha^{(m)}_{-k,l',l-1} \ , \tag{30}$$

where δ^i_j *is the Kronecker delta.*

Proof. The initial condition (29) is obviously verified. For the recurrence relation (30), one only has to notice that $\{(f_1+x),(f_2+y)\}_{m+1} = (f_1+x)\,\{(f_2+y),(f_1+x)\}_m$ and then use the induction hypothesis and $f_1\,\{f_2,f_1\}_k = \{f_1,f_2\}_{k+1}$ for any $k \in \mathbb{Z}$. $\square$

Generating function. Let $C(t,u,v,s) := \sum_{m,l,l' \geq 0} \sum_{k \in \mathbb{Z}} \alpha^{(m)}_{k,l,l'}\, t^m u^l v^{l'} s^k$. The formulas (29)–(30) imply

$$C(t,u,v,s) - 1 = tsC(t,v,u,s) + tuC(t,v,u,s^{-1}) \ . \tag{31}$$

Exchanging, in (31), u and v, or replacing s by s^{-1}, or doing both simultaneously, we obtain the following system of equations:

$$\begin{pmatrix} 1 & -ts & 0 & -tu \\ -ts & 1 & -tv & 0 \\ 0 & -tu & 1 & -ts^{-1} \\ -tv & 0 & -ts^{-1} & 1 \end{pmatrix} \begin{pmatrix} C(t,u,v,s) \\ C(t,v,u,s) \\ C(t,u,v,s^{-1}) \\ C(t,v,u,s^{-1}) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} .$$

Inverting the matrix in the left hand side, we find the generating function

$$C(t,u,v,s) = \frac{1 + t(u+s) + t^2(vs + us^{-1} - s^{-2} - uv) + t^3(1-u^2)(v-s^{-1})}{1 - t^2(s^2 + s^{-2} + 2uv) + t^4(u^2-1)(v^2-1)} \ . \tag{32}$$

One-parameter situation, $q_i = q_j$**.** Now $\beta_i = \beta_j$ and the coefficients $a^{(m_{ij})}_k$ in terms of $\alpha^{(m)}_{k,L}$, read

$$a^{(m_{ij})}_k = \sum_{L \geq 0} \beta^L_i \,(\alpha^{(m_{ij})}_{k,L} - \alpha^{(m_{ij})}_{-k,L}) \ , \quad \text{where} \ \ \alpha^{(m)}_{k,L} := \sum_{l,l' \geq 0 \,:\, l+l'=L} \alpha^{(m)}_{k,l,l'} \ . \tag{33}$$

Define $D(t,u,s) := \sum_{m,L \geq 0} \sum_{k \in \mathbb{Z}} \alpha^{(m)}_{k,L}\, t^m u^L s^k$. By definition, $D(t,u,s) = C(t,u,u,s)$. By (32),

$$D(t,u,s) = \frac{1 + t(u - s^{-1})}{1 - t(s+s^{-1}) + t^2(1-u^2)} = \frac{(1-ts^{-1}) + tu}{(1-ts)(1-ts^{-1})}\, d(t,u,s) \ , \tag{34}$$

where

$$d(t,u,s) := \left(1 - \frac{t^2u^2}{(1-ts)(1-ts^{-1})}\right)^{-1} \ . \tag{35}$$

The even in u part of $D(t,u,s)$ is

$$\frac{1}{1-ts}\, d(t,u,s) = \sum_{a,b,c\geq 0} \frac{(a+b)!(a+c-1)!}{a!\,(a-1)!\,b!\,c!}\, t^{2a+b+c}\, u^{2a}\, s^{b-c}\ . \tag{36}$$

The odd in u part of $D(t,u,s)$ is

$$\frac{tu}{(1-ts)(1-ts^{-1})}\, d(t,u,s) = \sum_{a,b,c\geq 0} \frac{(a+b)!(a+c)!}{a!\,a!\,b!\,c!}\, t^{2a+1+b+c}\, u^{2a+1}\, s^{b-c}\ . \tag{37}$$

Thus $\alpha^{(m)}_{k,L}$ ($L,m\in\mathbb{Z}_{\geq 0}$; $k\in\mathbb{Z}$) vanish unless $L\leq m$, $|k|\leq m-L$ and $m+k\equiv L\,(\mathrm{mod}\,2)$, and then

$$\begin{cases} \alpha^{(m)}_{k,L} = \begin{pmatrix} (m+k)/2 \\ (m+k-L)/2 \end{pmatrix}\begin{pmatrix} (m-k-2)/2 \\ (m-k-L)/2 \end{pmatrix} & , \quad L\equiv 0\,(\mathrm{mod}\,2)\ , \\ \alpha^{(m)}_{k,L} = \begin{pmatrix} (m+k-1)/2 \\ (m+k-L)/2 \end{pmatrix}\begin{pmatrix} (m-k-1)/2 \\ (m-k-L)/2 \end{pmatrix} & , \quad L\equiv 1\,(\mathrm{mod}\,2)\ . \end{cases} \tag{38}$$

For $k\geq 0$, $\alpha^{(m)}_{-k,L} = \alpha^{(m)}_{k,L}$ if $L\equiv 1\,(\mathrm{mod}\,2)$, and $\alpha^{(m)}_{-k,L} = \dfrac{m-k}{m+k}\alpha^{(m)}_{k,L}$ if $L\equiv 0\,(\mathrm{mod}\,2)$. Substituting into (33), one finds

$$a^{(m_{ij})}_k = \sum_{p=0}^{\lfloor (m-1)/2\rfloor} \beta_i^{2p}\, \frac{2k}{m_{ij}+k}\, \alpha^{(m_{ij})}_{k,2p} \quad \text{for } k=1,\dots,m_{ij}. \tag{39}$$

The formulas (38) and (39) appear in [11].

Examples. We write explicitly the relation (13) for $m_{ij}\leq 6$ (this is all what is needed for the finite Coxeter groups other than the dihedral groups D_n with $n>6$) in the multiparameter setting:

- for $m_{ij}=2$, $y_{ij}^2 = 1$;
- for $m_{ij}=3$, $y_{ij}^3 = \beta_i^2(y_{ij}-y_{ij}^2)+1$;
- for $m_{ij}=4$, $y_{ij}^4 = 2\beta_i\beta_j(y_{ij}-y_{ij}^3)+1$;
- for $m_{ij}=5$, $y_{ij}^5 = 3\beta_i^2(y_{ij}-y_{ij}^4)+(\beta_i^4+\beta_i^2)(y_{ij}^2-y_{ij}^3)+1$;
- for $m_{ij}=6$, $y_{ij}^6 = 4\beta_i\beta_j(y_{ij}-y_{ij}^5)+(3\beta_i^2\beta_j^2+\beta_i^2+\beta_j^2)(y_{ij}^2-y_{ij}^4)+1$.

The one-parameter situation is obtained when one sets $\beta_i=\beta_j$.

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References

[1] Bourbaki N., *Groupes et algèbres de Lie*, Springer (2002).

[2] Brenti F., Reiner V. and Roichman Y., *Alternating subgroups of Coxeter groups*, J. Combin. Theo. A 115(5) (2008) 845–877. ArXiv: math/0702177

[3] Casselman B., *The construction of Hecke algebras associated to a Coxeter group*, in: Harmonic Analysis on Reductive, p-adic Groups, Contemporary Math. 543 (2011) 91–102.

[4] Eriksson K., *A combinatorial proof of the existence of the generic Hecke algebra and R-polynomials*, Math. Scand. 75 (1994) 169–177.

[5] Etingof P. and Rains E., *New deformations of Coxeter groups II*, Geom. Funct. Anal. 17 (2008) 1851–1871. ArXiv: math/0604519

[6] Humphreys J., *Reflection groups and Coxeter groups*, Cambridge University Press (1990).

[7] Lyndon R. and Schupp P., *Combinatorial Group Theory*, Springer (1977).

[8] Mitsuhashi H., *The q-analogue of the alternating group and its representations*, J. Algebra 240(2) (2001) 535–558. ArXiv: math/9912121

[9] Mitsuhashi H., *$\mathbb{Z}_2$-graded Clifford system in the Hecke algebra of type B_n*, J. Algebra 264(1) (2003) 231–250.

[10] Ogievetsky O. and Poulain d'Andecy L., *Alternating subgroups of Coxeter groups and their spinor extensions*, to appear. ArXiv: 1112.6347

[11] Ratliff L., *The alternating Hecke algebra and its representations*, PhD thesis, Sydney Univ. (2007).

[12] Reidemeister K., *Knoten und Gruppen*, Abh. Math. Sem. (Hamburg) 5 (1926) 7–23.

[13] Schreier O., *Die Untergruppen der freien Gruppen*, Abh. Math. Sem. (Hamburg) 5 (1927) 161–183.

[14] Vershik A., *Local stationary algebras*, Amer. Math. Soc. Trans. 148(2) (1991) 1–13.