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ELEMENTARY PARTICLES AND FIELDS
Theory

# On the Velocity Tensors*

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**Abstract**—A new object, called the velocity tensor, is introduced. It allows to formulate a generally covariant mechanics. Some properties of the velocity tensor are derived.

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## 1. INTRODUCTION

In classical mechanics [1] the velocity $\mathbf{v}(t)$ of a material point is defined as

$$\mathbf{v}(t) = \frac{d\mathbf{x}(t)}{dt}, \tag{1}$$

where $\mathbf{x}(t)$ is the trajectory function of the moving point and $t$ is the time coordinate in the chosen inertial reference frame. In special relativity [2] the notion of the three-dimensional velocity $\mathbf{v}(t)$ is generalized to the notion of the four-velocity defined as

$$u^{\mu}(\tau) = \frac{dx^{\mu}(\tau)}{d\tau}, \tag{2}$$

where the space–time position of the material point is given by four functions $x^{\mu}(\tau)$ ($\mu = 0, 1, 2, 3; x^0 = ct$) parametrized by the so-called proper time

$$d\tau = dt\sqrt{1 - \frac{\mathbf{v}^2(t)}{c^2}}. \tag{3}$$

Due to its dependence on velocity of the moving material point, the notion of the proper time $\tau$ is different for each material point and for nonuniform motions the proper time is not a uniformly changing function of the coordinate time $t$. Moreover, for nonuniform motions the proper time coincides with the coordinate time in continuously changing inertial reference frames (the momentarily rest frames). Only for uniformly moving material points the proper time coincides with the coordinate time in one reference frame (the rest frame of the moving material point). In addition, for many particle systems the trajectories of particles are parametrized by different proper times and it is almost impossible to describe the interaction between particles without the notion of propagating fields. Therefore relativistic mechanics cannot be so well developed as the nonrelativistic mechanics is.

Fortunately, there exists another way of passing from Galilean–Newton mechanics to the relativistic one [3] which is not based on Eq. (2). Indeed, it is easy to see that rewriting Eq. (1) in the form

$$d\mathbf{x}(t) - \mathbf{v}(t)\,dt = 0 \tag{4}$$

we can immediately generalize it to a relativistic (as a matter of fact, generally) covariant form

$$V^{\mu}_{\nu}(x)\,dx^{\nu} = 0, \tag{5}$$

where a new mixed tensor field $V^{\mu}_{\nu}(x)$ is introduced. We shall name this tensor as *the velocity tensor*.

It is clear that for nontrivial velocity tensors ($V^{\mu}_{\nu}(x) \neq \delta^{\mu}_{\nu}$) Eq. (5) define some submanifolds of the considered space–time. We shall require from the velocity tensors that these submanifolds should always be one-dimensional, what means that Eq. (5) must determine some curves interpreted as trajectories of the moving material points.

Form (5) has the obvious advantage over (1) and (2), that it does not use any evolution parameter and therefore it may be applied to systems with arbitrary number of material points by generalizing (5) to the set of relations

$$V^{\mu}_{a,\nu}(x_a)\,dx^{\nu}_a = 0, \tag{6}$$

where the index $a$ labels different material points.

At each space–time event the velocity tensors (different for different material points) fix the infinitesimal directions in which any material point located at that event may move. In addition, *forms (5) and (6) are invariant under arbitrary changes of space–time coordinates*. Therefore, they may be used to formulate a generally covariant scheme for classical mechanics.

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The aim of the present paper is to describe some interesting properties of the velocity tensors. We shall also provide the explicit construction of the general form of such tensors.

It is clear that velocity tensors are related to the kinematical part of mechanics. We shall also touch the dynamical aspect of mechanics.

## 2. GENERAL PROPERTIES OF THE VELOCITY TENSORS

Equation (5), in $n$-dimensional space—time, is an eigen equation for the $n \times n$-dimensional matrix $V$ (defined by the velocity tensor) for the eigenvalue 0, while the infinitesimal displacements $dx^\mu$ in any motion are the eigenvectors of the velocity tensors belonging to this eigenvalue.

Writing the characteristic equation for the general eigenvalue problem

$$V^\mu_\nu (x)\, dx^\nu = \lambda dx^\mu \tag{7}$$

we get the equation for the possible eigenvalues $\lambda$

$$\sum_{j=0}^{n} (-\lambda)^{n-j}\, \mathrm{Tr}_j V (x) = 0, \tag{8}$$

where $\mathrm{Tr}_j V (x)$ denotes the sums of diagonal minors of order $j$ of the matrix $V (x)$. Obviously, $\mathrm{Tr}_1 V (x)$ coincides with the ordinary trace of $V (x)$ and $\mathrm{Tr}_n V (x)$ is the determinant of $V (x)$. For shortness, we also use the convention

$$\mathrm{Tr}_0 V (x) = 1 \tag{9}$$

for any matrix $V (x)$.

Due to physical reason we must require that there should be only one eigenvalue equal to 0. This means that there should be a unique eigenvector for any velocity tensor which fixes the infinitesimal displacements in any motion. The characteristic equation (8) must be therefore of the form

$$\lambda^n = 0, \tag{10}$$

from which we get the following conditions for any velocity tensor:

$$\mathrm{Tr}_j V(x) = 0 \tag{11}$$

for all $j > 0$.

Conditions (11) are generally covariant requirements because all the $\mathrm{Tr}_j V$, being the coefficients in characteristic equation (8), are invariant under arbitrary similarity matrix transformations, and it is well known that for mixed tensors, treated as matrices, the general coordinate transformations locally become the similarity transformations

$$V (x) \to V' \left(x'\right) = S (x)\, V (x)\, S^{-1} (x), \tag{12}$$

where the matrix elements of $S (x)$ are given by

$$S^\mu_\nu (x) = \frac{\partial x'^\mu (x)}{\partial x^\nu} \tag{13}$$

for arbitrary changes of space—time coordinates $x^\mu \to x'^\mu (x)$.

Conditions (11) impose $n$ restrictions for the $n^2$ matrix elements of the velocity tensors. Further restrictions come from the requirement that, in each reference frame, from (5) it should follow that

$$dx^k = v^k (t)\, dt, \tag{14}$$

where $k = 1, \ldots, (n-1)$ and $v^k (t)$ are the components of the standard velocity. This gives us additional $n-1$ restrictions for the matrix elements of the velocity tensor. Finally, we shall require that in $n$-dimensional space—times the motions in all $n-k$ subspaces should be described exactly as they were described in the $n-k$-dimensional space—times. This means that restricting the motions to subspaces the form of the velocity tensor should reduce to the already established forms of the velocity tensors in the corresponding lower-dimensional subspaces. We shall refer to this requirement as to the reduction principle. It is easy to count that such a requirement gives additional $2^{n-1} - 2$ conditions for the matrix elements of any velocity tensor. Altogether we are left with $n^2 - n - (n-1) - \left(2^{n-1} - 2\right) = (n-1)^2 - \left(2^{n-1} - 2\right)$ free parameters of any velocity tensor. These free parameters should represent components of some $(n-1)$-dimensional vector which will guarantee the covariance of the velocity tensor under space rotations because this is the only simple geometrical interpretation of the remaining constants in the velocity tensors. In this way, we arrive at the equation

$$(n-1)^2 - \left(2^{n-1} - 2\right) = n - 1. \tag{15}$$

It is surprising that this equation has solution only for $n = 2, 3$, and 4. This means that our construction can be performed only in two, three and four-dimensional space—times, correspondingly.

## 3. GENERAL CONSTRUCTION OF THE VELOCITY TENSORS

We shall now present a simple method of the construction of all possible velocity tensors.

Let us consider space—times for which the passage between inertial reference frames is described by the linear change of coordinates

$$x^\mu \to x'^\mu = L^\mu_\nu (\mathbf{u})\, x^\nu, \tag{16}$$

where $\mu, \nu = 0, 1, 2, 3$ and $\mathbf{u}$ denotes the relative velocity of the two inertial reference frames. From (16) and the tensor character of the velocity tensor we get the transformation law for it (written in the matrix form)

$$V \to V' = L(\mathbf{u})\, V L^{-1}(\mathbf{u}) = L(\mathbf{u})\, V L(-\mathbf{u})\,. \tag{17}$$

It is clear that we should look for velocity tensors which are functions of the ordinary velocity of motion. Our basic assumption consists in the requirement that the functional forms of the velocity tensor are the same in each reference frame. This means that

$$V'\left(\mathbf{v}'\right) = V\left(\mathbf{v}'\right) \tag{18}$$

because only under such condition in each reference frame we can fulfill conditions (14). In this way, the transformation law (17) becomes to be a system of functional equations for the matrix elements of the matrix $V$ of the following form

$$V\left(\mathbf{v}'\right) = L(\mathbf{u})\, V(\mathbf{v})\, L(-\mathbf{u})\,, \tag{19}$$

where

$$v'^k = \frac{L_0^k(\mathbf{u}) + \sum_j L_j^k(\mathbf{u})\, v^j}{L_0^0(\mathbf{u}) + \sum_j L_j^0(\mathbf{u})\, v^j}. \tag{20}$$

Taking into account that the particle at rest in the unprimed reference frame moves with the velocity $-\mathbf{u}$ in the primed frame we can rewrite these functional equations in the explicit form

$$V\left(\frac{-u^k L_0^0(\mathbf{u}) + \sum_j L_j^k(\mathbf{u})\, v^j}{L_0^0(\mathbf{u}) + \sum_j L_j^0(\mathbf{u})\, v^j}\right) = L(\mathbf{u})\, V(\mathbf{v})\, L(-\mathbf{u})\,. \tag{21}$$

The solutions of these equations are obtained by the standard method. We first put $v^k = 0$, then change the signs of $u^k$ and, finally, rename $\mathbf{u}$ into $\mathbf{v}$. As a result, we get

$$V(\mathbf{v}) = L(-\mathbf{v})\, V L(\mathbf{v})\,, \tag{22}$$

where on the right-hand side the matrix $V$ has constant matrix elements equal to the elements of $V(0)$. The constant matrix elements of $V$ should be determined by the additional requirements the velocity tensors have to satisfy.

For all dimensions the first column of the velocity tensor $V$ consists of null elements. This follows from the fact that for particles at rest the eigenvector in (5) is of the form

$$\begin{pmatrix} dt \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{23}$$

Such eigenvector will satisfy Eq. (5) only if $V_0^\mu = 0$.

## 4. EXAMPLES

### 4.1. Two-Dimensional Space–Time

For $n = 2$ from conditions (11) it follows that

$$V = \begin{pmatrix} 0 & V_1^0 \\ 0 & 0 \end{pmatrix}, \tag{24}$$

where $V_1^0$ is an arbitrary nonzero number. Since Eq. (5) is homogeneous, this constant can be taken as 1.

For Galilean space–time

$$L(u) = \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix} \tag{25}$$

and from (22) we get

$$V(v) = \begin{pmatrix} -v & 1 \\ -v^2 & v \end{pmatrix}. \tag{26}$$

For Lorentz space–time

$$L(u) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \begin{pmatrix} 1 & -\frac{u}{c^2} \\ -u & 1 \end{pmatrix} \tag{27}$$

and from (22) we get

$$V(v) = \frac{1}{1 - \frac{v^2}{c^2}} \begin{pmatrix} -v & 1 \\ -v^2 & v \end{pmatrix}. \tag{28}$$

### 4.2. Higher Dimensional Space–Times

From the reduction principle and from the form of the velocity tensor in the two-dimensional space–time we immediately get that in all higher dimensional space–times the only nonzero components are the $V_k^0$. Therefore the final form of the velocity tensors is

$$V_\nu^\mu(\mathbf{v}(t)) = L_0^\mu(-\mathbf{v}(t)) \sum_k V_k^0 L_\nu^k(\mathbf{v}(t))\,, \tag{29}$$

where $\left(V_1^0, V_2^0, \ldots, V_n^0\right)$ are components of a $(n-1)$-dimensional vector under rotations in the subspace $\left(x^1, x^2, \ldots, x^{n-1}\right)$. Using this form of $V$ and the explicit forms of the Galilean and Lorentz transformations we easily can get the velocity tensor both for the Galilean and Lorentz space–times of any dimension.

## 5. DYNAMICS

Since our kinematical part of classical mechanics is generally covariant, it is necessary to determine such form of dynamical equations which also will be generally covariant. For this purpose we shall remind that the only generally covariant differential relation which may be reduced to the famous Newton relation

$$\frac{d\mathbf{p}\left(t\right)}{dt} = \mathbf{F}\left(t\right) \tag{30}$$

is of the form

$$\nabla_{\mu}\pi^{\mu\nu}\left(x\right) = F^{\nu}\left(x\right), \tag{31}$$

where $\pi^{\mu\nu}\left(x\right)$ is some tensorial density, and $F^{\nu}\left(x\right)$ is a vector density while $\nabla_{\mu}$ denotes a corresponding covariant derivative.

Assuming that $\pi^{\mu\nu}\left(x\right)$, like the velocity tensors, is a function of the ordinary velocity we easily can construct the explicit form of this quantity. This leads, exactly as for the velocity tensors, to the following form of the dynamical tensor

$$\pi^{\mu\nu}\left(\mathbf{v}\left(t\right)\right) = L^{\mu}_{\alpha}\left(-\mathbf{v}\left(t\right)\right)\pi^{\alpha\beta}L^{\nu}_{\beta}\left(-\mathbf{v}\left(t\right)\right), \tag{32}$$

where all $\pi^{\alpha\beta}$ are constants. Since, in contradiction to the velocity tensor, the dynamical tensor $\pi^{\mu\nu}\left(\mathbf{v}\right)$ need not to satisfy any additional conditions, we have here to do with $n^2$ arbitrary constants which describe the inertial properties of the considered particles. We may, however, diminish the number of arbitrary constants by requiring the symmetry of $\pi^{\mu\nu}\left(\mathbf{v}\right)$ and then only one parameter, the mass of the particle, describes its inertial property. In this case $\pi^{\mu\nu}$ simply is the energy–momentum tensor of the material point. Since the $\pi^{\mu\nu}\left(\mathbf{v}\left(t\right)\right)$ depends only on the time coordinate, it is clear that Eq. (32) reduces to Eq. (31).

## 6. CONCLUSIONS

We have introduced a new mechanical object called the velocity tensor and explicitly constructed the velocity tensors in space–times of any dimension. We hope that the notion of the velocity tensor will shed more light on the possible dynamics in general relativity. It also may be useful for relativistic many-body systems.

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