Topics in Quantum Field Theory in Curved Space

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In these lectures we consider some topics of Quantum Field Theory in Curved Space. In the first one particle creation in curved space is studied from a mathematical point of view, especially, particle production at a given time using the so called "instantaneous diagonalization method". As a first application we study particle production in a no-oscillating model where re-heating may be explained from gravitational particle creation. In the second one we re-calculate, with all the mathematical details, particle production in the Starobinsky model. Particle production by strong electromagnetic fields (Schwinger's effect) and particle production by moving mirrors simulating black hole collapse are also studied. In the second lecture we calculate the re-normalized two-point function using the adiabatic regularization. The conformally and minimally coupled cases are considered for a scalar massive and massless field. We reproduce previous results in a rigorous mathematical form and clarify some empirical approximations and bounds. The re-normalized stress tensor is also calculated in several situations. Finally, in last lecture quantum correction due to a massless fields conformally coupled with gravity are considered in order to study the avoidance of singularities that appear in the flat Friedmann-Robertson-Walker (FRW) model. It is assumed that the universe contains a barotropic perfect fluid with state equation $p = \omega\rho$ (being ρ the energy density and p the pressure). The dynamics of the model is studied for all values of the parameter ω, and also for all values of the two parameters, that we will call α and β, provided by the quantum corrections. We will see that only the case $\alpha > 0$ could avoid the singularities. Then when $\omega > -1$, in order to obtain an expanding Friedmann universe at late times (only a one-parameter family of solutions, no a general solution, has this behavior at late times), the initial conditions of the no-singular solutions at early times must be very fine tuned. These no-singular solutions are: a general solution (a two-parameter family) leaving the contracting de Sitter phase, and a one-parameter family leaving the contracting Friedmann stage. On the other hand for $\omega < -1$ (phantom field), the problem of the avoidance of singularities is more involved because if one considers an expanding Friedmann stage at early times, then instead of fine tune the initial conditions one also has to fine tune the parameters α and β to obtain a behavior without future singularities, because only a one-parameter family of solutions follows a contracting Friedmann phase at late times, and only a particular solution behaves like a contracting de Sitter universe. The rest of solutions have future singularities.

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Contents

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I. INTRODUCTION

In these lectures we'll try to give a self-consistent presentation of the quantum field theory in curved space and also to show some of its applications. In the first lecture we will give a mathematical presentation of the subject and we will re-derive, with all the details, some of its applications to the theory of gravitational re-heating. Our presentation is no-standard, in the sense that we start explaining the "adiabatic vacuum prescription" introduced by Parker in his thesis [1–3], and the "instantaneous diagonalization method", introduced in Russian literature at the beginning of 70's, based in the idea that the number of created particles in a given mode at a given time is the energy of the mode at this time divided by the energy of a single particle in that mode [4–7]. As an instructive example we calculate particle creation in the flat Friedmann-Robertson-Walker (FRW) chart of the de Sitter space where one can sees the difference between both prescriptions. After that, we introduce the "in" and "out" states in asymptotically flat spaces where particle creation can be defined in the standard way [8–13]. As an application, we re-calculate the gravitational particle production in a transition from the de Sitter phase to the radiation-dominated one [14], and we also discuss the problem of a second inflationary stage related with the back-reaction [15–17]. This example is important because it describes approximately the inflationary phase followed by a transition to a radiation dominated universe, and the particle production process may be used to explain the pre-heating in inflationary non-oscillatory models [14, 18, 19]. Finally we study, in a great detail, particle production in the Starobinsky model [20–22] because we believe that there isn't a clear explanation of particle creation in this model. In this case, in order to obtain the well-known results, firstly one must disregards the power-law expansion of the universe and only retains the oscillating behavior of the scale factor at late times, secondly one has to assume that the energy density of the created particles is a well-defined quantity in this model and then one has to choose a particular form of it, and finally one has to assume that one kind of particles, named scalarons, are the responsible for the late time behavior of the universe, and also that is the decay of these scalarons what produce particles.

Particle production by strong electromagnetic fields [23] are also studied, and Schwinger's formula [24] that calculates the probability that the vacuum state remains unchanged in the presence of a constant electric field is deduced, in an elementary but not at all mathematically correct way, using standard methods of Quantum Field Theory. More precisely, it is deduced calculating the Bogoliubov coefficients for every mode via the W.K.B. method in the complex plane. At the end a rigorous demonstration is outlined.

The last part of this first lecture is devoted to the study of particle production by moving mirrors. Our interest is concentrated in trajectories that simulate the black body collapse, and to get the same kind of results obtained by Hawking in [25], i.e., to obtain the black body spectrum. This occurs for perfect reflecting mirrors, but we'll show that for semi-transparent moving mirrors the radiation spectrum is a bit different.

In the second lecture the vacuum quantum fluctuations are studied. We calculate the re-normalized two-point function subtracting adiabatic modes up to order two. We do the calculation for conformally coupled fields and also for minimally coupled ones re-obtaining, in a consistent way, all the early well-known results. Calculating the two-point function is very important in the context of inflation, for example, the eternal inflation phenomenon is manifested in some inflationary model (new inflationary universe [26–28], chaotic inflation [29]), i.e., the large-scale quantum fluctuations of the inflaton field termed by its two-point function lead to a process of infinitely self-reproducing inflationary mini-universes [30]. Studying the back-reaction of particles produced in the pre-heating phase also requires the two-point function of the inflaton field and the two-point function of the light particles involved in such a process [31–33].

The mean problem with the two-point function is that it is ultra-violet divergent and requires re-normalization. The simplest method for obtaining divergence-free expressions is the adiabatic regularization based on subtracting some generalized WKB modes [34] that have the same behavior at large frequencies as the exact modes; the divergent terms then cancel. But the procedures for calculating the re-normalized two-point function differed somewhat in early work. The authors assumed some not fully justified frequency cut-off or made unjustified approximations in order to obtain finite quantities [26, 35–37].

Our task in this lecture is to matematically clarify the features appearing in those works. Firstly, the massless case is

studied (conformally coupled and minimally coupled case), reproducing all the previous results in full detail. After this, we study a massive field in the de Sitter phase where it's assumed that its mass is smaller than the Hubble parameter (this is typical in the inflationary models [30]). Here we derive the two-point function at late time in full detail and accurately demonstrate the mean formula obtained in [26]. We finish this lecture reviewing some important results about the stress tensor re-nomalization which will be used in last lecture.

In last lecture we study the avoidance of cosmological singularities if one takes into account the vacuum corrections due to a massless conformally coupled field.

It's well-known that the classical solutions of the general relativity for a Friedmann-Robertson-Walker (FRW) model contain, in general, singularities (Big Bang, Big Rip, future sudden singularities), this means that near these singularities the space-time curvature is arbitrarily large. Then, for curvatures on the order of the Planck length, quantum effects have to be taken into account. These quantum effects, can violate the so-called energy conditions [38], and consequently they can modify drastically the classical solution. For this reason, it is possible that quantum effects avoid the classical singularities [39, 40].

We consider the quantum effects produced by massless fields conformally coupled with gravity. This is an special case where, for a flat FRW universe, the quantum vacuum stress tensor, that depends on two regularization parameters, that we call α and β, can be calculated explicitly. Then due to the trace anomaly and the equation of conservation, one easily calculates the vacuum energy density that contributes to the modified Friedmann equation. This equation cannot be analytically integrated, but a qualitative phase-space study can be performed. This is the main objective of this lecture.

First, we introduce the quantum effects and write the modified Friedmann equation that depends on the parameters α and β, which we'll assume that can take all possible values. After that, we study the simplest case, i.e. $\alpha = 0$, in this case the modified Friedmann equation becomes a first order differential equation and can be integrated. Our conclusion, in that case, is that the singularities are not avoided. Another simple case corresponds to the case of an empty universe (it doesn't contain any barotropic fluid, only quantum effects are taken into account). An special case ($\alpha < 0$, $\beta < 0$) is the Starobinsky model [20]. We'll see that, in that case, all solutions contain singularities, except when $\beta < 0$, where it appears an unstable de Sitter solution, and an unstable solution that connects the de Sitter solution with the point $H = \dot{H} = 0$ (being H the Hubble parameter). Finally, we study the general case, i.e., a universe filled by a barotropic perfect fluid with state equation $p = \omega\rho$. The only case where no-singular solutions may appear is when $\alpha > 0$ and $\beta < 0$. Then taking the same point of view as [41, 42], we show that when $\omega > -1$, the no-singular early time behaviors that can lead, at late times, to the Friedmann expanding stage, are a contracting de Sitter phase and a contracting Friedmann phase. However their initial conditions can be very fine tuned in order to match with the expanding Friedmann stage. On the other hand, for $\omega < -1$ the late time behavior of no-singular solutions that come from the expanding Friedmann stage at early time, are the contracting Friedmann phase and the contracting de Sitter one. In this case, instead of fine tune the initial conditions, one also has to fine tune the parameters α and β in order to obtain no-singular solutions. But in both cases, the no-singular solutions are unstable in the sense that an small perturbation leads them to a singular behavior.

The units used in these lectures are $c = \hbar = 1$.

II. PARTICLE CREATION BY CLASSICAL FIELDS

Particle creation by gravitational fields is studied in this section. The theory developed is applied to a non-oscillating inflationary model and to the Starobinsky one.

A. Graviational particle production

1. *Quantum fields in curved space-time: General Theory*

It's well known that the Lagrangian density of a scalar field is [11] $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2 - \xi R\phi^2)$, and its corresponding Klein-Gordon equation is given by

$$(-\nabla_\mu\nabla^\mu + m^2 + \xi R)\phi = 0, \tag{1}$$

where ξ is the coupling constant and R is the scalar curvature. If one considers the flat FRW metric $ds^2 = -dt^2 + C(t)d\mathbf{x^2} = C(\eta)(-d\eta^2 + d\mathbf{x^2})$ (being η the conformal time), the modes of the form $\phi_{\mathbf{k}}(\mathbf{x}, \eta) \equiv (2\pi)^{-3/2}C^{-1/2}(\eta)e^{i\mathbf{kx}}\chi_{\mathbf{k}}(\eta)$ will satisfy the equation

$$\chi''_{\mathbf{k}}(\eta) + \Omega^2_{\mathbf{k}}(\eta)\chi_{\mathbf{k}}(\eta) = 0, \tag{2}$$

where we have introduced the notation $\Omega^2_{\mathbf{k}}(\eta) \equiv \omega^2_{\mathbf{k}}(\eta) + (\xi - 1/6)C(\eta)R(\eta)$, with $\omega^2_{\mathbf{k}}(\eta) = m^2C(\eta) + |\mathbf{k}|^2$ and $R(\eta) = 3\left(\frac{C''}{C^2} - \frac{1}{2}\frac{C'^2}{C^3}\right) = 6\frac{a''}{a^3}$ (being $C \equiv a^2$).

Firstly, we are interested in the conformally coupled case, i.e., in the case $\xi = 1/6$, where equation (2) becomes

$$\chi''_{\mathbf{k}}(\eta) + \omega^2_{\mathbf{k}}(\eta)\chi_{\mathbf{k}}(\eta) = 0. \tag{3}$$

This is the equation of a set of no-interacting harmonic oscillators, so to built a quantum theory, we can consider the following Hamiltonian operator $\hat{\mathcal{H}}(\eta) \equiv \frac{1}{2}\left(\hat{\Pi}^2 + \omega^2(\eta)\hat{\phi}^2\right) - \frac{1}{2}\omega(\eta)$ corresponding to a single harmonic oscillator. In Heisenberg picture the operators $\hat{\Pi}$ and $\hat{\phi}$ satisfy the equations $\hat{\phi}' = \hat{\Pi}$ and $\hat{\Pi}' = -\omega^2\hat{\phi}$, that is, $\hat{\phi}$ satisfy the Klein-Gordon equation $\hat{\phi}'' + \omega^2\hat{\phi} = 0$, and thus, one can writes:

$$\begin{pmatrix} \hat{\phi}(\eta) \\ \hat{\Pi}(\eta) \end{pmatrix} = \begin{pmatrix} \chi(\eta) \\ \chi'(\eta) \end{pmatrix} \hat{A}_\chi + \begin{pmatrix} \chi^*(\eta) \\ \chi'^*(\eta), \end{pmatrix} \hat{A}^\dagger_\chi, \tag{4}$$

where the mode function χ is a solution of the Klein-Gordon equation and $\hat{A}_\chi$ is, in that picture, a constant operator that we will call the "annihilation operator relative to the mode χ".

From the commutation relation $[\hat{\Pi}, \hat{\phi}] = -i$, one can deduces that χ must satisfies the relation $\chi'\chi^* - \chi'^*\chi = -i$, which means that the annihilation operator is given by

$$\hat{A}_\chi = -i\left(\chi'^*(\eta)\hat{\phi}(\eta) - \chi^*(\eta)\hat{\Pi}(\eta)\right). \tag{5}$$

Once this operator has been introduced one can defines the "vacuum state relative to the mode χ", namely $|0;\chi\rangle$, as the quantum state that satisfies $\hat{A}_\chi|0;\chi\rangle = 0$. It's clear from this definition that there isn't particle production at any time, because $\langle\chi;0|\hat{A}^\dagger_\chi\hat{A}_\chi|0;\chi\rangle = 0$ all the time [43]. However, this definition depends on the choice of the mode χ. Effectively, if one chooses two different mode functions, namely χ_1 and χ_2, since $\hat{A}_{\chi_1} = \alpha_{1,2}\hat{A}_{\chi_2} + \beta^*_{1,2}\hat{A}^\dagger_{\chi_2}$ with $\alpha_{1,2} = -i\mathcal{W}[\chi_2;\chi_1^*]$ and $\beta^*_{1,2} = -i\mathcal{W}[\chi_2^*;\chi_1^*]$ (where $\mathcal{W}$ denotes the Wronskian), then an observer in the $|0;\chi_2\rangle$ vacuum state can observes χ_1-particles because one has $\mathcal{N}_{1,2} \equiv \langle\chi_2;0|\hat{A}^\dagger_{\chi_1}\hat{A}_{\chi_1}|0;\chi_2\rangle = |\beta_{1,2}|^2$.

In this way, if one considers the family of solutions to the Klein-Gordon equation, namely $\chi_{\eta'}(\eta)$, defined by the initial condition

$$\chi_{\eta'}(\eta') \equiv f(\eta'); \quad \chi'_{\eta'}(\eta') \equiv g(\eta'), \quad \text{where } f \text{ and } g \text{ are some arbitrary functions,} \tag{6}$$

one can calculates the number of χ_τ-particles detected by an observer in the $|0;\chi_{\tau'}\rangle$ vacuum state, that is, the number of produced particles at time τ from the vacuum state at time τ', with the formula

$$\mathcal{N}(\tau;\tau') \equiv \langle\chi_{\tau'};0_H|\hat{A}^\dagger_{\chi_\tau,H}\hat{A}_{\chi_\tau,H}|0_H;\chi_{\tau'}\rangle = |\beta(\tau;\tau')|^2, \quad \text{with} \quad \beta(\tau;\tau') = i\mathcal{W}[\chi_{\tau'};\chi_\tau]. \tag{7}$$

Note that, different families of solutions give rise to different definitions of the vacuum state. For example, to define the **adiabatic vacuum modes** first we consider the ϵ-Klein-Gordon equation $\epsilon\nu'' + \omega^2(\eta)\nu = 0$, where here, ϵ is a dimensionless parameter that one shall set $\epsilon = 1$ at the end of the calculations. At order "n", a WKB solution of the Klein-Gordon equation is (see for details [44]):

$$\chi_{n;WKB}(\tau;\epsilon) \equiv \sqrt{\frac{1}{2W_n(\tau;\epsilon)}}e^{-\frac{i}{\epsilon}\int^\tau W_n(\eta;\epsilon)d\eta}, \tag{8}$$

where $W_0 = \omega$ and

$$W_n = \text{terms until order } \epsilon^{2n} \text{ of } \left(\sqrt{\omega^2 - \epsilon^2\left[\frac{1}{2}\frac{W''_{n-1}}{W_{n-1}} - \frac{3}{4}\frac{(W'_{n-1})^2}{W^2_{n-1}}\right]}\right). \tag{9}$$

Once one has introduced the WKB solutions, the adiabatic vacuum at order n is defined through the family $\chi_{n;\eta'}(\eta)$ that satisfy the initial condition

$$\chi_{n;\eta'}(\eta') = \chi_{n;WKB}(\eta';\epsilon = 1); \quad \chi'_{n;\eta'}(\eta') = \chi'_{n;WKB}(\eta';\epsilon = 1). \tag{10}$$

From this definition, at the n order, the β-Bogoliubov coefficient is given by $\beta_n(\tau;\tau') = i\mathcal{W}[\chi_{n;\tau'};\chi_{n;\tau}]$, and the number of produced particles at time τ can be calculated from the formula

$$\begin{aligned}\mathcal{N}_n(\tau;\tau') \equiv |\beta_n(\tau;\tau')|^2 = \quad & \frac{1}{W_n(\tau)}\left(\frac{1}{2}[|\chi'_{n;\tau'}(\tau)|^2 + W_n^2(\tau)|\chi_{n;\tau'}(\tau)|^2] - \frac{1}{2}W_n(\tau)\right) \\ & + \frac{W_n'^2(\tau)}{8W_n^3(\tau)}|\chi_{n;\tau'}(\tau)|^2 + \frac{W_n'(\tau)}{4W_n^2(\tau)}(\chi'^*_{n;\tau'}(\tau)\chi_{n;\tau'}(\tau) + \chi'_{n;\tau'}(\tau)\chi^*_{n;\tau'}(\tau)), \end{aligned} \tag{11}$$

where $W_n(\tau) \equiv W_n(\tau;\epsilon=1)$.

Another important family of solutions to the Klein-Gordon equation, namely $\chi_{diag;\eta'}(\eta)$, is given by the initial condition

$$\chi_{diag;\eta'}(\eta') = \chi_{0;WKB}(\eta';\epsilon=1); \quad \chi'_{diag;\eta'}(\eta') = -i\omega(\eta')\chi_{0;WKB}(\eta';\epsilon=1). \tag{12}$$

This family defines the so-called **instantaneous Hamiltonian diagonalization method**, and their Bogoliubov coefficients

$$\alpha_{diag}(\tau;\tau') = -i\mathcal{W}[\chi_{diag;\tau'};\chi^*_{diag;\tau}], \quad \beta_{diag}(\tau;\tau') = i\mathcal{W}[\chi_{diag;\tau'};\chi_{diag;\tau}], \tag{13}$$

can be calculated as follows: Writing $\chi_{diag;\tau'}(\eta) = \alpha_{diag}(\tau;\tau')\chi_{diag;\tau}(\eta) + \beta_{diag}(\tau;\tau')\chi^*_{diag;\tau}(\eta)$, at $\eta = \tau$ one gets the system

$$\left\{\begin{array}{rcl}\chi_{diag;\tau'}(\tau) & = & \alpha_{diag}(\tau;\tau')\chi_{0;WKB}(\tau;\epsilon=1) + \beta_{diag}(\tau;\tau')\chi^*_{0;WKB}(\tau;\epsilon=1) \\ \chi'_{diag;\tau'}(\tau) & = & -i\omega(\tau)\left(\alpha_{diag}(\tau;\tau')\chi_{0;WKB}(\tau;\epsilon=1) - \beta_{diag}(\tau;\tau')\chi^*_{0;WKB}(\tau;\epsilon=1)\right),\end{array}\right. \tag{14}$$

which can be used to obtain the interesting formula

$$\mathcal{N}_{diag}(\tau;\tau') \equiv |\beta_{diag}(\tau;\tau')|^2 = \frac{1}{\omega(\tau)}\left(\frac{1}{2}[|\chi'_{diag;\tau'}(\tau)|^2 + \omega^2(\tau)|\chi_{diag;\tau'}(\tau)|^2] - \frac{1}{2}\omega(\tau)\right), \tag{15}$$

which shows that this quantity is the energy at time τ of the mode $\chi_{diag;\tau'}$ divided by the energy, at time τ, of a single particle.

Another way to obtain the Bogoliubov coefficients can be done from the system (14) if one takes into account that the mode function $\chi_{diag;\tau'}(\tau)$ satisfy the equation $\chi''_{diag;\tau'}(\tau) + \omega^2(\tau)\chi_{diag;\tau'}(\tau) = 0$. As a function of the variable τ, one has

$$\left\{\begin{array}{rcl}\alpha'_{diag}(\tau;\tau') & = & \omega'(\tau)\left(\nu^*_{0;WKB}(\tau;\epsilon=1)\right)^2\beta_{diag}(\tau;\tau') \\ \beta'_{diag}(\tau;\tau') & = & \omega'(\tau)\left(\nu_{0;WKB}(\tau;\epsilon=1)\right)^2\alpha_{diag}(\tau;\tau').\end{array}\right. \tag{16}$$

This system can be solved by iteration, for example, if in the first iteration one chooses $\alpha_{diag}(\tau;\tau') \cong 1$, one will arrive at formula

$$\beta_{diag}(\tau;\tau') \cong \int_{\tau'}^{\tau}\frac{\omega'(\eta)}{2\omega(\eta)}e^{-2i\int^\eta\omega(\eta')d\eta'}d\eta. \tag{17}$$

Example II.1. *(Particle creation in the flat FRW chart of the de Sitter space-time [45, 46])*

In the de Sitter phase the scalar factor is given by $a(\eta) = -1/(H\eta)$, with $-\infty < \eta < 0$ (being H the Hubble parameter), and the frequency has the form $\omega(\eta) = \sqrt{\omega_0^2 + \frac{m^2}{H^2\eta^2}}$, where ω_0 is a constant. We are interested in the case $m \gg H$ which correspond to the adiabatic approximation (see below). An easy calculation yields

$$\chi_{0;WKB}(\eta;\epsilon=1) = \sqrt{\frac{1}{2\omega(\eta)}}e^{i\omega_0\sqrt{\eta^2+m^2/(H^2\omega_0^2)}}\left(\frac{|\eta|}{\sqrt{\eta^2+m^2/(H^2\omega_0^2)}+m/(H\omega_0)}\right)^{im/H}, \tag{18}$$

which shows, when $\eta \to -\infty$, that

$$\chi_{0;WKB}(\eta;\epsilon=1) \to \sqrt{\frac{1}{2\omega_0}}e^{-i\omega_0\eta}; \quad \chi'_{0;WKB}(\eta;\epsilon=1) \to -i\omega_0\nu_{0;WKB}(\eta;\epsilon=1). \tag{19}$$

The mode solution that satisfy the initial condition (19) is given in terms of the Hankel functions [11]

$$\chi(\eta) = C\sqrt{\frac{\pi\eta}{4}}H_{\mu}^{(2)}(\omega_0\eta), \tag{20}$$

with $\mu \equiv \sqrt{\frac{1}{4} - \frac{m^2}{H^2}} \cong im/H, \quad C \equiv e^{-i(\frac{\pi\mu}{2}+\frac{\pi}{4})} \cong e^{\pi m/(2H)}e^{-i\frac{\pi}{4}}$ *and* $\eta = e^{-i\pi}|\eta|$.

Using the asymptotic form of the Hankel functions at late times, i.e., when $|\eta|\omega_0 \ll 1$ *[47]*

$$\chi(\eta) \cong -C\sqrt{\frac{\eta}{4\pi}}\frac{H}{m}\left[e^{-\pi m/H}\Gamma(1-im/H)\left(\frac{\omega_0\eta}{2}\right)^{im/H} - \Gamma(1+im/H)\left(\frac{\omega_0\eta}{2}\right)^{-im/H}\right], \tag{21}$$

an easy calculation provides that

$$|\beta_{diag}(0;-\infty)|^2 \cong \frac{H^3}{32\pi m^3}\left|\Gamma\left(1-im/H\right)\right|^2 e^{\pi m/H}. \tag{22}$$

Using at this point [47] $|\Gamma(1+iy)|^2 = \pi y/\sinh(\pi y)$*, we conclude that, when* $|\eta|\omega_0 \ll 1$*, the number of produced particles, using the diagonalization method, is given by [46]*

$$\mathcal{N}_{diag}(0;-\infty) \cong \frac{H^2}{16m^2}. \tag{23}$$

However if we use the zero order (the other orders give the same result) adiabatic vacuum modes, the square of the β*-Bogoliubov coefficient will be given by*

$$|\beta_0(0;-\infty)|^2 = \left|\chi(0)\chi'_{0;WKB}(0;\epsilon=1) - \chi'(0)\chi_{0;WKB}(0;\epsilon=1)\right|^2. \tag{24}$$

Inserting (21), in this last formula, we obtain

$$\mathcal{N}_0(0;-\infty) \cong \frac{H}{2\pi m}\left|\Gamma\left(1+im/H\right)\right|^2 e^{-\pi m/H} = \left(e^{2\pi m/H}-1\right)^{-1}. \tag{25}$$

This is the thermal spectrum obtained in the flat FRW chart of the de Sitter space-time [45, 48].

Remark II.1. *The two methods give a different result because when* $|\eta|\omega_0 \ll 1$*, one has*

$$\chi'_{0,WKB}(\eta;\epsilon=1) \cong -i\omega(\eta)\chi_{0,WKB}(\eta;\epsilon=1) + \frac{1}{2\eta}\chi_{0,WKB}(\eta;\epsilon=1) \neq -i\omega(\eta)\chi_{0,WKB}(\eta;\epsilon=1), \tag{26}$$

this is due to the fact that $\lim_{\eta\to 0^-}\omega(\eta) = \infty$*, that is, at late time there is not a well-defined "out" region (see below for a precise definition of the "out" region).*

In general (for arbitrary values of ξ), for a given set of modes satisfying $\mathcal{W}[\chi^*_{\mathbf{k}}, \chi_{\mathbf{k}}] = -i$ one can expands the quantum field, in the Heisenberg picture, as follows: $\hat{\phi}(\mathbf{x},\eta) = \sum_{\mathbf{k}}\hat{A}_{\chi_{\mathbf{k}}}\phi_{\mathbf{k}}(\mathbf{x},\eta) + \hat{A}^{\dagger}_{\chi_{\mathbf{k}}}\phi^*_{\mathbf{k}}(\mathbf{x},\eta)$, then one can defines the quantum vacuum state relative to the modes $\phi_{\mathbf{k}}(\mathbf{x},\eta) = (2\pi)^{-3/2}C^{-1/2}(\eta)e^{i\mathbf{kx}}\chi_{\mathbf{k}}(\eta)$, namely $|0;\chi\rangle$, which must satisfies $\hat{A}_{\chi_{\mathbf{k}}}|0;\chi\rangle = 0$ for all values of $\mathbf{k}$. However if one considers another set of modes, namely $\widetilde{\phi}_{\mathbf{k}}(\mathbf{x},\eta)$, one also may develops the quantum field as $\hat{\phi}(\mathbf{x},\eta) = \sum_{\mathbf{k}}\hat{A}_{\widetilde{\chi}_{\mathbf{k}}}\widetilde{\phi}_{\mathbf{k}}(\mathbf{x},\eta) + \hat{A}^{\dagger}_{\widetilde{\chi}_{\mathbf{k}}}\widetilde{\phi}^*_{\mathbf{k}}(\mathbf{x},\eta)$, where $\widetilde{\phi}_{\mathbf{k}} \equiv \alpha_{\mathbf{k}}\phi_{\mathbf{k}} + \beta_{\mathbf{k}}\phi^*_{\mathbf{k}}$ and thus, $\hat{A}_{\widetilde{\chi}_{\mathbf{k}}} \equiv \alpha^*_{\mathbf{k}}\hat{A}_{\chi_{\mathbf{k}}} - \beta^*_{\mathbf{k}}\hat{A}^{\dagger}_{\chi_{\mathbf{k}}}$ with $|\alpha_{\mathbf{k}}|^2 - |\beta_{\mathbf{k}}|^2 = 1$. The vacuum $|0;\widetilde{\chi}\rangle$ relative to the modes $\widetilde{\phi}_{\mathbf{k}}(\mathbf{x},\eta)$, is related with the other vacuum trough the relation

$$|0;\widetilde{\chi}\rangle = \prod_{\mathbf{k}}\exp\left\{\frac{1}{2}\frac{\beta^*_{\mathbf{k}}}{\alpha^*_{\mathbf{k}}}(\hat{A}^{\dagger}_{\chi_{\mathbf{k}};\tau})^2\right\}|0;\chi\rangle, \tag{27}$$

and the operator "number of particles in the mode $\mathbf{k}$" that depends on the choice of the set of modes, for example, for the set $\phi_{\mathbf{k}}(\mathbf{x},\eta)$ is $\hat{\mathcal{N}}_{\chi_{\mathbf{k}}} \equiv \hat{A}^{\dagger}_{\chi_{\mathbf{k}}}\hat{A}_{\chi_{\mathbf{k}}}$ satisfies $\langle\chi;0|\hat{\mathcal{N}}_{\chi_{\mathbf{k}}}|0;\chi\rangle = 0$, however one has $\langle\widetilde{\chi};0|\hat{\mathcal{N}}_{\chi_{\mathbf{k}}}|0;\widetilde{\chi}\rangle = |\beta_{\mathbf{k}}|^2$.

Once we have introduced these definitions, one can considers the family of solutions to the Klein-Gordon equation, namely $\chi_{\mathbf{k};\eta'}(\eta)$, defined by the initial condition

$$\chi_{\mathbf{k};\eta'}(\eta') \equiv f_{\mathbf{k}}(\eta'); \quad \chi'_{\mathbf{k};\eta'}(\eta') \equiv g_{\mathbf{k}}(\eta'), \quad \text{where } f_{\mathbf{k}} \text{ and } g_{\mathbf{k}} \text{ are some arbitrary functions,} \tag{28}$$

to calculate the number density of χ_τ-particles per unit volume detected by an observer in the $|0;\chi_{\tau'}\rangle$ vacuum state, that is, the number density of produced particles at time τ from the vacuum state at time τ'. The general formula is

$$N(\tau;\tau') \equiv \frac{1}{(2\pi a)^3}\int d^3\mathbf{k}\langle\chi_{\tau'};0|\hat{A}^\dagger_{\chi_{\mathbf{k};\tau}}\hat{A}_{\chi_{\mathbf{k};\tau}}|0;\chi_{\tau'}\rangle = \frac{1}{(2\pi a)^3}\int d^3\mathbf{k}|\beta_{\mathbf{k}}(\tau;\tau')|^2, \tag{29}$$

with $\beta_{\mathbf{k}}(\tau;\tau') = i\mathcal{W}[\chi_{\mathbf{k};\tau'};\chi_{\mathbf{k};\tau}]$.

Remark II.2. *In the conformally coupled case one also can defines the energy density of produced particles at time τ from the vacuum state at time τ' as follows $\rho(\tau;\tau') \equiv \frac{1}{(2\pi a)^3 a}\int d^3\mathbf{k}\omega_{\mathbf{k}}(\tau)|\beta_{\mathbf{k}}(\tau;\tau')|^2$.*

Example II.2. *(Particle production in the adiabatic approximation)*
This approximation is based in the assumption $\Omega'_{\mathbf{k}} \ll \Omega^2_{\mathbf{k}}$. In the conformally coupled case this assumption becomes $\omega'_{\mathbf{k}} \ll \omega^2_{\mathbf{k}}$ and it is always satisfied when $H \ll m$. In that case one has

$$\chi_{\mathbf{k},diag;\tau'}(\tau) \cong \chi_{\mathbf{k},0;WKB}(\tau;\epsilon=1) = \sqrt{\frac{1}{2\omega_{\mathbf{k}}(\tau)}}e^{-i\int_{\tau'}^{\tau}\omega_{\mathbf{k}}(\eta)d\eta}, \tag{30}$$

and when $\omega'_{\mathbf{k}}(\tau') = 0$ one can inserts this expression in formula (15) to obtain

$$|\beta_{\mathbf{k},diag}(\tau;\tau')|^2 \cong \frac{\omega'^2_{\mathbf{k}}(\tau)}{16\omega^4_{\mathbf{k}}(\tau)} = \frac{m^4C'^2(\tau)}{64\omega^6_{\mathbf{k}}(\tau)}, \tag{31}$$

which helps us to conclude that, using the instantaneous diagonalization method, the number density of created particles per unit volume is given by [5]

$$N_{diag}(\tau;\tau') \cong \frac{mH^2}{32\pi^2}\int_0^\infty \frac{x^2}{(x^2+1)^3}dx = \frac{mH^2}{512\pi}, \tag{32}$$

and their energy density is

$$\rho_{diag}(\tau;\tau') \cong \frac{m^2H^2}{32\pi^2}\int_0^\infty \frac{x^2}{(x^2+1)^{5/2}}dx = \frac{m^2H^2}{96\pi}. \tag{33}$$

Remark II.3. *Note that formula (23) can be easily obtained from formula (31) applied to the de Sitter phase.*

To finish we consider and asymptotically flat FRW space-time, that is, we assume that $\lim_{\eta\to\pm\infty}C(\eta) = C_\pm$, and we take the following set of modes $\phi_{in,\mathbf{k}}(\mathbf{x},\eta) \equiv (2\pi)^{-3/2}C^{-1/2}(\eta)e^{i\mathbf{kx}}\frac{e^{-i\omega_{-,\mathbf{k}}\eta}}{\sqrt{2\omega_{-,\mathbf{k}}}}$ when $\eta \to -\infty$, and the the set $\phi_{out,\mathbf{k}}(\mathbf{x},\eta) \equiv (2\pi)^{-3/2}C^{-1/2}(\eta)e^{i\mathbf{kx}}\frac{e^{-i\omega_{+,\mathbf{k}}\eta}}{\sqrt{2\omega_{+,\mathbf{k}}}}$ when $\eta \to \infty$ (being $\omega_{\pm,\mathbf{k}} = \sqrt{m^2C_\pm + |\mathbf{k}|^2}$). In this context we can define the "in" and "out" vacuum states, namely $|0_{in}\rangle$ and $|0_{out}\rangle$, and the "in" and "out" annihilation operators $\hat{A}_{in,\mathbf{k}}$ and $\hat{A}_{out,\mathbf{k}}$. Then, the average number of produced pairs, at late times, in the $\mathbf{k}$ mode, is given by $\langle 0_{in}|\hat{\mathcal{N}}_{out,\mathbf{k}}|0_{in}\rangle = |\beta_{\mathbf{k}}|^2$, where $\hat{\mathcal{N}}_{out,\mathbf{k}} = \hat{A}^\dagger_{out,\mathbf{k}}\hat{A}_{out,\mathbf{k}}$ is the operator number of "out" particles, and the beta Bogoliubov coefficient is obtained trough the relation $\phi_{in,\mathbf{k}} \equiv \alpha_{\mathbf{k}}\phi_{out,\mathbf{k}} + \beta_{\mathbf{k}}\phi^*_{out,\mathbf{k}}$.

In that context, the number density of created particle per unit volume, at late times, is given by

$$N = \frac{1}{(2\pi a)^3}\int d^3\mathbf{k}|\beta_{\mathbf{k}}|^2, \tag{34}$$

and their energy density by [49]

$$\rho = \frac{1}{(2\pi a)^3 a}\int d^3\mathbf{k}\omega_{+,\mathbf{k}}|\beta_{\mathbf{k}}|^2. \tag{35}$$

In general, it is impossible to solve the mode equation (2) but one can rewrite this differential equation in an integral one as follows [49, 50]:

$$\chi_{\mathbf{k}}(\eta) = \frac{e^{-i\omega_{-,\mathbf{k}}\eta}}{\sqrt{2\omega_{-,\mathbf{k}}}} + \frac{1}{\omega_{-,\mathbf{k}}}\int_{-\infty}^{\eta} V_{\mathbf{k}}(\eta')\sin(\omega_{-,\mathbf{k}}(\eta-\eta'))\chi_{\mathbf{k}}(\eta')d\eta', \tag{36}$$

where $V_{\mathbf{k}}(\eta) = \omega^2_{-,\mathbf{k}} - \Omega^2_{\mathbf{k}}(\eta)$.

Applying Picard's method to lowest order, i.e, replacing $\chi_{\mathbf{k}}(\eta')$ by $\frac{e^{-i\omega_{-,\mathbf{k}}\eta'}}{\sqrt{2\omega_{-,\mathbf{k}}}}$ one obtains the following approximation that works very well for massless nearly conformally coupled fields

$$\alpha_{\mathbf{k}} \cong 1 + \frac{i}{2\omega_{-,\mathbf{k}}}\int_{\mathbb{R}} V_{\mathbf{k}}(\eta)d\eta \quad \beta_{\mathbf{k}} \cong -\frac{i}{2\omega_{-,\mathbf{k}}}\int_{\mathbb{R}} e^{-2i\omega_{-,\mathbf{k}}\eta}V_{\mathbf{k}}(\eta)d\eta \tag{37}$$

Note that, one will have to assume $\lim_{\eta\to\pm\infty} V_{\mathbf{k}}(\eta) = 0$ if one wants well defined Bogoliubov coefficients. This always happens in the massless case, and using Plancherel's theorem is not difficult, in the massless case, to prove that

$$N = \frac{(\xi - 1/6)^2}{16\pi a^3}\int_{\mathbb{R}} a^4(\eta)R^2(\eta)d\eta. \tag{38}$$

2. Particle production in the transition from de Sitter phase to a radiation dominated universe

Consider the following scale factor [14]

$$a(\eta) = \begin{cases} -\frac{1}{H\eta} & \text{for } \eta \leq \eta_0 \\ H(\eta-\eta_0) - \frac{1}{H\eta_0} & \text{for } \eta_0 \leq \eta, \end{cases} \tag{39}$$

where $\eta_0 < 0$ is the time when the sudden transition occurs.

This example is interesting because in the radiation phase massless particles cannot be produced, so in that phase, the number density of created particles and their energy density are well defined quantities. Moreover, it describes approximately the inflationary phase followed by a transition to a radiation dominated universe, and the obtained result can be used to explain the reheating process of the universe after inflation in some no-oscillatory models [19].

First, we wil consider massless nearly conformally coupled particles. In the de Sitter phase, one has $\Omega^2_{\mathbf{k}}(\eta) = |\mathbf{k}|^2 + (12\xi - 2)/\eta^2$, and in the radiation one $\Omega^2_{\mathbf{k}}(\eta) = |\mathbf{k}|^2$. Using formula (37) one gets

$$\beta_{\mathbf{k}} \cong \frac{i}{|\mathbf{k}|}\int_{-\infty}^{\eta_0} e^{-2i|\mathbf{k}|\eta}\frac{6\xi - 1}{\eta^2}d\eta, \tag{40}$$

and form formula (38) one easily obtains

$$N = -\frac{(6\xi-1)^2}{12\pi a^3\eta_0^3} = \frac{(6\xi-1)^2H^3}{12\pi}\left(\frac{a_0}{a}\right)^3, \tag{41}$$

where $a_0 \equiv a(\eta_0)$.

It isn't difficult to show, from formula (37), that the energy density diverges. However if one assumes that the transition is not abrupt one obtains (see [14])

$$\rho \sim \frac{(6\xi-1)^2}{a^4\eta_0^4} = (6\xi-1)^2H^4\left(\frac{a_0}{a}\right)^4. \tag{42}$$

Now, we consider massless minimally coupled particles where the mode functions that describe the vacuum state in the de Sitter phase are given by

$$\chi_{\mathbf{k}}(\eta) = -\sqrt{\frac{\pi\eta}{4}}H^{(2)}_{3/2}(|\mathbf{k}|\eta) = \sqrt{\frac{1}{2|\mathbf{k}|}}e^{-i|\mathbf{k}|\eta}\left(1 + \frac{1}{i|\mathbf{k}|\eta}\right), \tag{43}$$

and in the radiation one, these modes have the form

$$\chi_{\mathbf{k}}(\eta) = \frac{1}{\sqrt{2|\mathbf{k}|}}\left(\alpha_{\mathbf{k}} e^{-i|\mathbf{k}|\eta} + \beta_{\mathbf{k}} e^{i|\mathbf{k}|\eta}\right). \tag{44}$$

Matching at time $\eta = \eta_0$ one obtains $\beta_{\mathbf{k}} = \frac{1}{2|\mathbf{k}|^2\eta_0^2} e^{-2i|\mathbf{k}|\eta_0}$ what implies that the number density of produced particles is infrared divergent, and their energy density is both, infrared and ultraviolet divergent.

To eliminate the infrared divergency one can imagines that at very early times the universe is in a radiation phase, then at a given time, for example $\eta = -H^{-1}$ ($t = 0$), there is an abrupt transition to the inflationary phase described approximately by a de Sitter one (see for details [36, 51]). On the other hand, to avoid the ultraviolet divergencies and can assumes that inflation finishes with an smooth transition to the radiation phase [14] because in that case modes with $|k| \gg |\eta|_0^{-1}$ have a very small contribution. Anyway if one is only interested in production of particles whose modes leave the Hubble horizon, that is, in modes that satisfy $H < |\mathbf{k}| < |\eta_0|^{-1}$, one can uses the formula $\beta_{\mathbf{k}} = \frac{1}{2|\mathbf{k}|^2\eta_0^2} e^{-2i|\mathbf{k}|\eta_0}$ to obtain

$$N = \frac{1}{8\pi^2 a^3 \eta_0^4}\int_H^{|\eta_0|^{-1}} |\mathbf{k}|^{-2} d|\mathbf{k}| = \frac{H^3}{8\pi^2}\left(\frac{a_0}{a}\right)^3\left(\frac{1}{H|\eta_0|} - 1\right), \tag{45}$$

and

$$\rho = \frac{1}{8\pi^2 a^4 \eta_0^4}\int_H^{|\eta_0|^{-1}} |\mathbf{k}|^{-1} d|\mathbf{k}| = \frac{H^4}{8\pi^2}\left(\frac{a_0}{a}\right)^4 \ln\left(\frac{1}{H|\eta_0|}\right). \tag{46}$$

Now we consider the general massless case where the modes that defines the vacuum state in the de Sitter phase are

$$\chi_{\mathbf{k}}(\eta) = C\sqrt{\frac{\pi\eta}{4}} H_\nu^{(2)}(|\mathbf{k}|\eta), \tag{47}$$

with $\nu \equiv \sqrt{\frac{9}{4} - 12\xi}$ and $C \equiv e^{-i(\frac{\pi\nu}{2} + \frac{\pi}{4})}$.

For modes that satisfy $|\mathbf{k}\eta_0| \ll 1$ one can uses the asymptotic formula

$$H_\nu^{(2)}(z) \cong \frac{i}{\pi}(z/2)^{-\nu}\Gamma(\nu) - \frac{ie^{i\pi\nu}}{\sin(\pi\nu)}(z/2)^{\nu}\frac{1}{\Gamma(\nu+1)}. \tag{48}$$

Then matching at point $\eta = \eta_0$ one obtains for $\nu > 0$ (i.e., for $\xi < 3/16$) [52]

$$|\beta_{\mathbf{k}}|^2 \cong \frac{1}{16\pi}\left(|\mathbf{k}||\eta_0|/2|\right)^{-2\nu-1}\Gamma^2(\nu)\left(1/2 - \nu\right)^2. \tag{49}$$

In the opposite case $|\mathbf{k}\eta_0| \gg 1$, from the asymptotic formula [47]

$$H_\nu^{(2)}(z) \cong C\sqrt{\frac{2}{\pi z}}\left(1 - i\frac{4\nu^2 - 1}{8z}\right)e^{-iz}, \tag{50}$$

after matching at point $\eta = \eta_0$ one obtains

$$|\beta_{\mathbf{k}}|^2 \cong \frac{1}{16}\frac{4\nu^2 - 1}{|\mathbf{k}|^2\eta_0^2} e^{-i|\mathbf{k}|\eta_0}, \tag{51}$$

what means that the energy density is always ultraviolet divergent.

From these results one concludes that for $5/48 < \xi < 3/16$ ($0 < \nu < 1$) there isn't infrared divergencies and thus the number density of particles converges. Moreover, if one assumes that the transition is smooth then their energy density is also finite. On the other hand, when $\xi \leq 5/48$ ($\nu \geq 1$) the number density is infrared divergent (and for $\xi \leq 0$ ($\nu \geq 3/2$) their energy is also infrared divergent). The solution to the avoidance of divergencies in this last case is the same as for the minimally coupled case. Then if one is only interested in modes that leave the Hubble Horizon, one can uses the approximation $|\mathbf{k}\eta_0| \ll 1$ to obtain the formulae

$$N \cong \frac{4^\nu}{32\pi^3(\nu-1)}\Gamma^2(\nu)\left(1/2 - \nu\right)^2 H^3\left(\frac{a_0}{a}\right)^3\left(\frac{1}{(H|\eta_0|)^{2(\nu-1)}} - 1\right), \tag{52}$$

and

$$\rho \cong \frac{4^{\nu}}{64\pi^3(2\nu-3)}\Gamma^2(\nu)\left(1/2-\nu\right)^2 H^4\left(\frac{a_0}{a}\right)^4\left(\frac{1}{(H|\eta_0|)^{(2\nu-3)}}-1\right). \tag{53}$$

Remark II.4. *Note that the case $\nu = 1/2$ is the conformally coupled case and thus there isn't particle production.*

To obtain results for massive particles we will apply the diagonalization method, more precisely, to calculate $|\beta_{\mathbf{k},diag}(\eta_0;-\infty)|^2$ we have to use formula (15) with, and that is very important, $\omega_{\mathbf{k}}(\eta_0) = \sqrt{|\mathbf{k}|^2+\frac{m^2}{H^2|\eta_0|^2}}$. Then for light particles (particles with small mass compared with the Hubble parameter, i.e., with $m \ll H$) no-conformally coupled one can uses the formulae (52) and (53). However for conformally coupled particles we've obtained, after a cumbersome calculation, in the range $m/H \ll |\mathbf{k}\eta_0| \ll 1$

$$|\beta_{\mathbf{k},diag}(\eta_0;-\infty)|^2 \sim \frac{m^2}{H^4\eta_0^4|\mathbf{k}|^4}, \tag{54}$$

then, integrating over the modes that leave the Hubble horizon one has

$$\rho_{diag}(\eta_0) \sim m^4\left(\frac{a_0}{a}\right)^4 \ln\left(\frac{1}{H|\eta_0|}\right). \tag{55}$$

In the opposite case, that is, for $m \gg H$, from the example 2.1 in the range $|\mathbf{k}\eta_0| \ll 1$, one gets

$$|\beta_{\mathbf{k},diag}(\eta_0;-\infty)|^2 \sim \frac{H^2}{m^2}, \tag{56}$$

and integrating over the modes that leave the Hubble horizon one has

$$\rho_{diag}(\eta_0) \sim \frac{H^5}{m}\left(\frac{a_0}{a}\right)^4. \tag{57}$$

Finally it's important to remark that for light particles it is well-known that its energy density during the de Sitter phase is of the order H^4 (see equation (194) in lecture II), and then if one takes a inflationary model where the energy density of the inflaton field, namely ρ_v, satisfies $\rho_v \ll m_p^4$ (being m_p the Planck mass), for example if it is at the grand unified theory scale [53], then from the Friedmann equation $H^2 = \frac{8\pi\rho_v}{3m_p^2}$ one will deduce that $H^4 \ll \rho_v$, that is, the back-reaction effect will be negligible. However, when $\rho_v \sim m_p$, in [15, 16] the authors showed, in the minimally coupled case, that there is a new inflationary state (which makes the results obtained above incorrect) driven by the field that produce light particles, namely ϕ, and not by the inflaton field, namely Φ, and thus instead of studying gravitational particle production, one should study the mechanism of production of particles of the field Φ by the oscillations of the field ϕ due to the potential energy density $\frac{1}{2}m^2\phi^2$.

3. Particle production in the Starobinsky model

In this model the universe emerges form the de Sitter phase and at late times the scale factor is approximately given by (see for example [20, 21]) $a(t) \cong t^{2/3}[1+\frac{2}{3Mt}\sin(Mt)]$ (in terms of the Hubble parameter $H(t) = \frac{4}{3t}\cos^2(Mt/2)\left[1-\frac{\sin(Mt)}{Mt}\right]$ where M is a constant that is related with the vacuum polarization effect produced by massless conformally coupled particles. More precisely, this model corresponds to an empty universe (it doesn't contain any barotropic fluid), and only quantum effects due to massless conformally coupled fields are taken into account [20–22, 41]. The energy density that contributes to the Friedmann equation depends on two parameters [39, 54, 55], one of them is M and the other is called H_0 in [21]. Then, when these two parameters are positive there is an unstable no-singular solution, that emerges from the de Sitter phase with $H(t) = H_0$ and at late times approaches to a matter dominated universe with $H(t) = \frac{4}{3t}\cos^2(Mt/2)\left[1-\frac{\sin(Mt)}{Mt}\right]$ (A more detailed description of this model is given in last lecture).

First at all we study the production of massless nearly conformally coupled particles. To do this we'll follow Vilenkin's viewpoint [21] (see also [22]). The idea is to calculate the rate of particle per unit volume and per unit time, this concept

was introduced in [50] but without any explanation. Here we'll try to justify the results obtained in [50]. Note that to calculate the particle rate per unit time one needs to calculate the particle production at a given time and then to calculate its derivative with respect to time, that is, one has to use formula (29) with $\tau' = -\infty$. To perform this calculation we've to choose a given family of mode solutions. From our point of view the most natural choice is

$$\chi_{\mathbf{k},\tau}(\eta) = \frac{e^{-i|\mathbf{k}|\eta}}{\sqrt{2|\mathbf{k}|}} + \frac{1}{|\mathbf{k}|}\int_\tau^\eta V_{\mathbf{k}}(\eta')\sin(|\mathbf{k}|(\eta-\eta'))\chi_{\mathbf{k},\tau}(\eta')d\eta', \tag{58}$$

with $V_{\mathbf{k}}(\eta) = \omega^2_{-,\mathbf{k}} - \Omega^2_{\mathbf{k}}(\eta) = -(\xi - 1/6)C(\eta)R(\eta)$.

Note that these modes satisfy

$$\chi_{\mathbf{k},\tau}(\tau) = \frac{e^{-i|\mathbf{k}|\tau}}{\sqrt{2|\mathbf{k}|}} \quad \text{and} \quad \chi'_{\mathbf{k},\tau}(\tau) = -i|\mathbf{k}|\frac{e^{-i|\mathbf{k}|\tau}}{\sqrt{2|\mathbf{k}|}}. \tag{59}$$

Now applying Picard's method to lowest order one obtains

$$\beta_{\mathbf{k}}(\tau;-\infty) \cong -\frac{i}{2|\mathbf{k}|}\int_{\mathbb{R}} \theta(\tau-\eta)e^{-2i|\mathbf{k}|\eta}V_{\mathbf{k}}(\eta)d\eta, \tag{60}$$

where we've introduced the Heaviside function θ. Then from Plancherel's theorem one obtains the following density of produced particles at time τ

$$N(\tau;-\infty) \cong \frac{(\xi-1/6)^2}{16\pi a^3}\int_{-\infty}^{\tau} a^4(\eta)R^2(\eta)d\eta. \tag{61}$$

To calculate the rate of particles per unit volume and unit time, one has to multiply this last quantity by a^3 and to take the derivative with respect the cosmic time and finally to divide by a^3, the final result is

$$\frac{1}{a^3}\frac{d}{dt}(a^3 N(\tau;-\infty)) \cong \frac{1}{16\pi}(\xi-1/6)^2R^2(\tau), \tag{62}$$

that coincides with formula (2.29) of [21].

Using the formula $R = 12H^2 + 6\frac{dH}{dt}$, one has approximately at late times $R \cong -\frac{4M}{t}\sin(Mt)$. Then averaging over a period of oscillation one finally obtains

$$\overline{\frac{1}{a^3}\frac{d}{dt}(a^3 N(\tau;-\infty))} \cong \frac{M^2}{2\pi t^2}(\xi-1/6)^2. \tag{63}$$

At this moment an important remark is in order. If one makes, at late times, the approximation $a(t) \cong t^{2/3}$ (that corresponds to the expansion law of a matter-dominated universe) one has $t \sim \eta^3$, then $R(\eta) \sim \eta^{-3}$ what means that, at late times, $V_{\mathbf{k}}(\eta) \sim \eta$ and thus one has an infinite production of particles. One can avoid this problem neglecting the effect of power-law expansion and taking as a scalar factor the function $a(t) \cong [1 + \frac{2}{3Mt}\sin(Mt)]$, in that case one has $t \sim \eta$ and thus, at late times, one has $V_{\mathbf{k}}(\eta) \sim \eta^{-1}$.

To calculate the energy density per unit time one has to take into account that the energy density at a given time, in general, diverges for a no-conformally coupled fields (see for example [56]), then one has to re-normalize this quantity. Moreover, the re-normalized energy density depends, in general, on the regularization method used, and in the final result it appears some vacuum polarization terms, that are very difficult to separate of those relative to the particle production (only in the conformally coupled case this is possible to do).

For these reasons in order to "calculate" the energy density per unit we use the fact that the main contribution to the particle production come from modes with $k \sim M/2$. This can be deduced taking into account the oscillating behavior of the curvature at late times, effectively, inserting $V_{\mathbf{k}} \cong (\xi - 1/6)\frac{4M}{\eta}\sin(M\eta)$ in the beta Bogoliubov coefficient, one deduces that the oscillating character of the integral disappear when $k = M/2$. Then one might concludes that an approximation for the energy density per unit time, averaged over an oscillation, is given by

$$\overline{\frac{d}{dt}\rho(\tau;-\infty)} \cong \frac{M}{2}\overline{\frac{1}{a^3}\frac{d}{dt}(a^3 N(\tau;-\infty))} \cong \frac{M^3}{4\pi t^2}(\xi-1/6)^2. \tag{64}$$

At this point is important to remember that Starobinsky proposed in [20] that the oscillating behavior of the scale factor can be though as a coherent oscillations of a massive field described by particles of mass M (scalarons), then with this point of view, gravitational particle production could be understood as a decay of scalarons (due to its rapid oscillations) into other particles like massless no-conformally coupled particles, massive conformally coupled ones, etc...

In this way, one can calculates the rate at which the energy of scalarons is dissipated

$$\Gamma \equiv \frac{\overline{\frac{d}{dt}\rho(\tau;-\infty)}}{\overline{\rho(\tau)}}, \tag{65}$$

where to obtain the energy density of the scalarons $\rho(\tau)$ one must uses the "effective" Friedmann equation $H^2 = \frac{8\pi\rho(\tau)}{3m_p^2}$, being m_p the Planck mass (we say "effective" because in the Starobinsky model the energy density that appears in the equations is only due to the vacuum polarization, but from Starobinsky's viewpoint, at late time, one could thinks that scalarons drive the expansion of the universe).

Now since $\overline{H} = \frac{2}{3t}$ one has

$$\Gamma \cong \frac{3}{2}\frac{M^3}{m_p^2}(\xi - 1/6)^2. \tag{66}$$

Finally at time $t \sim \Gamma^{-1}$ the oscillations of the scalarons field are damped, the created particles thermalize, and the universe becomes radiation dominated. To calculate its temperature, one has to use the thermodynamical relation $\rho = \frac{\pi^2}{30}N(T)T^4$, where $N(T)$ is the effective number of relativistic degrees of freedom at temperature T, the effective Friedmann equation, and the fact that reheating ends when $\Gamma \sim H$ (see for example [57]), then one obtains

$$T_{th} \sim \sqrt{\Gamma m_p} \sim |\xi - 1/6| M^{3/2} m_p^{-1/2}. \tag{67}$$

To end the section, we study massive conformally coupled particle production in the Starobinsky model. First at all note that, to apply formula (37) one must assumes that $\lim_{t\to+\infty} a(t) = \lim_{t\to-\infty} a(t)$ [49], then one doesn't have to work with the Starobinsky model because it doesn't satisfy this assumption. We will work with the an scale factor that at early times is $a(t) \cong 1$ and at late time is $a(t) \cong [1 + \frac{2}{3Mt}\sin(Mt)]$. We also assume that $m \ll M$, then since $\omega_{-,\mathbf{k}}^2 = |\mathbf{k}|^2 + m^2$, at late times, one has $V_{\mathbf{k}}(\eta) = \omega_{-,\mathbf{k}}^2 - \omega_{\mathbf{k}}^2(\eta) = m^2(1 - a^2(\eta)) \cong \frac{4m^2}{3M\eta}\sin(M\eta)$.

Now if one chooses the family of solutions

$$\chi_{\mathbf{k},\tau}(\eta) = \frac{e^{-i\omega_{-,\mathbf{k}}\eta}}{\sqrt{2\omega_{-,\mathbf{k}}}} + \frac{1}{\omega_{-,\mathbf{k}}}\int_\tau^\eta V_{\mathbf{k}}(\eta')\sin(\omega_{-,\mathbf{k}}(\eta-\eta'))\chi_{\mathbf{k},\tau}(\eta')d\eta', \tag{68}$$

one obtains

$$\beta_{\mathbf{k}}(\tau;-\infty) \cong -\frac{i}{2\omega_{-,\mathbf{k}}}\int_{\mathbb{R}} \theta(\tau-\eta)e^{-2i\omega_{-,\mathbf{k}}\eta}V_{\mathbf{k}}(\eta)d\eta. \tag{69}$$

Thus, since now we cannot apply Plancherel's theorem, to calculate the density of produced particles at time τ, we make the change of variable $|\mathbf{k}|^2 + m^2 = x^2M^2$ and make the approximation $xM\sqrt{x^2M^2 - m^2} \cong x^2M^2 - \frac{m^2}{2}$, then one gets

$$\begin{aligned} N(\tau;-\infty) &\cong \frac{M^3}{2\pi^2 a^3(\tau)}\left[\int_{m/M}^\infty x^2|\beta_{\mathbf{k}}(\tau;-\infty)|^2dx - \frac{m^2}{M^2}\int_{m/M}^\infty |\beta_{\mathbf{k}}(\tau;-\infty)|^2dx\right] \\ &\cong \frac{M^3}{2\pi^2 a^3(\tau)}\int_0^\infty x^2|\beta_{\mathbf{k}}(\tau;-\infty)|^2dx. \end{aligned} \tag{70}$$

Now we can apply Plancherel's theorem to obtain

$$N(\tau;-\infty) \cong \frac{1}{16\pi a^3(\tau)}\int_0^\tau V_{\mathbf{k}}^2(\eta)d\eta, \tag{71}$$

and then, at late times, one has

$$\frac{1}{a^3}\frac{d}{dt}(a^3 N(\tau;-\infty)) \cong \frac{m^4}{9\pi M^2\tau^2}\sin^2(M\tau), \tag{72}$$

where we have used the approximation $a(\tau)\cong 1$. Averaging over an oscillation one has

$$\overline{\frac{1}{a^3}\frac{d}{dt}(a^3 N(\tau;-\infty))} \cong \frac{m^4}{18\pi M^2\tau^2}. \tag{73}$$

To calculate the energy density per unit time, we use once again that the main contribution to the particle creation comes from modes with $k\sim M/2$, then

$$\overline{\frac{d}{dt}\rho(\tau;-\infty)} \cong \frac{M}{2}\overline{\frac{1}{a^3}\frac{d}{dt}(a^3 N(\tau;-\infty))} \cong \frac{m^4}{36\pi M^2\tau^2}. \tag{74}$$

And finally, the rate at which the energy of scalarons is dissipated is

$$\Gamma \equiv \frac{\overline{\frac{d}{dt}\rho(\tau;-\infty)}}{\overline{\rho(\tau)}} \cong \frac{m^4}{6Mm_p^2}, \tag{75}$$

this is the result obtained by Starobinsky [20].

B. Particle production by strong electromagnetic fields: Schwinger's formula

Here we deduce the well-known Schwinger formula [24] for spin and spinless particles using the W.K.B. approximation, that is, we compute the probability that the vacuum state remains unchanged in the presence of a constant electric field using the semi-classical approach.

First, we consider, in the Minkowski space-time, the Klein-Gordon field in a box of volume L^3, coupled with an external uniform vector potential $\mathbf{f}(t)$. The Klein-Gordon equation is equivalent to a Hamiltonian system, composed by an infinite number of harmonic oscillators with frequencies which depend on time. The mode equations are:

$$\chi''_{\mathbf{k}} + \omega^2_{\mathbf{k}}(t)\chi_{\mathbf{k}} = 0 \quad \text{with} \quad \mathbf{k}\in\mathbb{Z}^3, \tag{76}$$

where now $\omega^2_{\mathbf{k}}(t) = \left|\frac{2\pi\mathbf{k}}{L} + e\mathbf{f}(t)\right|^2 + m^2$ (being e the electric charge).

We assume that $\lim_{t\to\pm\infty}\omega_{\mathbf{k}}(t) = \omega_{\mathbf{k},\pm}$, and we write the "in"-states as linear combinations of the "out"-states as follows

$$\chi_{in,\mathbf{k}}(t) = \alpha_{\mathbf{k}}\chi_{out,\mathbf{k}}(t) + \beta_{\mathbf{k}}\chi^*_{out,\mathbf{k}}(t), \tag{77}$$

then one has, $\hat{A}_{out,\mathbf{k}} = \alpha_{\mathbf{k}}\hat{A}_{in,\mathbf{k}} + \beta^*_{\mathbf{k}}\hat{A}^\dagger_{in,\mathbf{k}}$.

Let $|n_{\mathbf{k}}\rangle$ be the "in"-state that contains n particles in the $\mathbf{k}$ mode, and let $|n_{\mathbf{k}})$ be the "out"-state that contains n particles in the $\mathbf{k}$ mode. Then, it is easy to obtain the following relations [6]:

$$|0_{\mathbf{k}}\rangle = \widetilde{C}_{\mathbf{k}}\sum_{n=0}^{\infty}\left(\frac{\beta^*_{\mathbf{k}}}{\alpha^*_{\mathbf{k}}}\right)^n |n_{\mathbf{k}}), \quad |0_{\mathbf{k}}) = C_{\vec{k}}\sum_{n=0}^{\infty}\left(-\frac{\beta^*_{\mathbf{k}}}{\alpha_{\mathbf{k}}}\right)^n |n_{\mathbf{k}}\rangle, \tag{78}$$

with $|\widetilde{C}_{\mathbf{k}}|^2 = |C_{\mathbf{k}}|^2 = |\alpha_{\mathbf{k}}|^{-2}$.

From these relations, we deduce that the probability that a particle in the $\mathbf{k}$ mode is produced [2, 6, 58], is

$$|(n_{\mathbf{k}}|0_{\mathbf{k}}\rangle|^2 = \frac{|\beta_{\mathbf{k}}|^{2n}}{(1+|\beta_{\mathbf{k}}|^2)^{n+1}}, \tag{79}$$

and the average number of produced particle in the $\mathbf{k}$ mode, is

$$\langle 0_{\mathbf{k}}|\hat{a}^+_{out,\mathbf{k}}\hat{a}_{out,\mathbf{k}}|0_{\mathbf{k}}\rangle = |\beta_{\mathbf{k}}|^2. \tag{80}$$

Example II.3. *(Adiabatic approximation) From formulae (29) and (31) one obtains the following formula for the number density of produced particles*

$$N_{diag}(t;-\infty)=\frac{\alpha}{512\pi m}|\mathbf{E}(t)|, \tag{81}$$

where $\alpha=e^2$ is the fine structure constant and $\mathbf{E}(t)=-\dot{\mathbf{f}}(t)$ is the external electric field. However, their energy density diverges. To obtain a well-known defined quantity one has to re-normalize the electric charge, after this one has (see details in [6, 69–71])

$$\rho_{diag}(t;-\infty)=\frac{7\alpha^2}{1920\pi^2m^4}|\mathbf{E}(t)|^4+\frac{\alpha}{960\pi^2m^2}\left(|\dot{\mathbf{E}}(t)|^2-2\mathbf{E}(t)\cdot\ddot{\mathbf{E}}(t)\right). \tag{82}$$

Moreover it is also possible to calculate, after charge re-normalization, the induced electric field, that is, the corrections to the external electric field due to vacuum fluctuations [6, 69]

$$\mathbf{E}_{vac}(t)=\frac{\alpha}{120\pi m^2}\ddot{\mathbf{E}}(t)-\frac{7\alpha^2}{360\pi m^5}\mathbf{E}(t)|\mathbf{E}(t)|^2, \tag{83}$$

and the effective Lagrangian density for the electric field [69]

$$\mathcal{L}_{eff}(t)=\frac{1}{8\pi}|\mathbf{E}(t)|^2+\frac{7\alpha^2}{5760\pi^2m^4}|\mathbf{E}(t)|^4+\frac{\alpha}{960\pi^2m^2}|\dot{\mathbf{E}}(t)|^2, \tag{84}$$

which generalizes the Euler-Heisenberg formula [24], for a time-dependent field.

From these results obtained above one can deduce that the probability that the vacuum state remains unchanged, namely P is

$$P=\prod_{\mathbf{k}\in\mathbb{Z}^3}\frac{1}{1+|\beta_{\mathbf{k}}|^2}=\exp\left(-\sum_{\mathbf{k}\in\mathbb{Z}^3}\log\left(1+|\beta_{\mathbf{k}}|^2\right)\right)=\exp\left(-\sum_{\mathbf{k}\in\mathbb{Z}^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}|\beta_{\mathbf{k}}|^{2n}\right) \tag{85}$$

As an application of this result, we'll find Schwinger's formula for scalar particles. Consider the case $\mathbf{f}(t)=(0,0,f(t))$, where

$$f(t)=\begin{cases}-ET & \text{if} \quad t<-T\\ Et & \text{if} \quad -T<t<T\\ ET & \text{if} \quad t>T,\end{cases} \tag{86}$$

being E the electric field and with $T\gg 1$. We suppose for example $eE>0$ (The case $eE<0$ is analogous).

The Schwinger formula gives the probability that the vacuum state remains unchanged. Then, using the notation

$$N\equiv\frac{2TL^3E^2\alpha}{8\pi^3},\quad S\equiv\frac{\pi m^2}{eE}, \tag{87}$$

Schwinger's formula for spinless particles is [24]

$$P=\exp\left(-N\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\exp\left(-nS\right)\right). \tag{88}$$

In order to deduce this formula, we compute the β Bogoliubov coefficient using the relativistic tunneling effect [59–61], i.e., using the formulae given by the W.K.B. method in the complex plane [62–64]in a formal way. (we say in a formal way because our field is not analytic, and the formula (89) that we'll use is only mathematically justified for analytic fields). The result is [65]

$$|\beta_{\mathbf{k}}|^2=\begin{cases}\exp\left(-Im\int_\gamma\sqrt{\left(k_\perp^2+m^2+\left(\frac{2\pi k_3}{L}+eEz\right)^2\right)}dz\right) & \text{if} \quad \left|\frac{2\pi k_3}{L}\right|<eET\\ 0 & \text{if} \quad \left|\frac{2\pi k_3}{L}\right|>eET,\end{cases} \tag{89}$$

where $k_\perp \equiv \frac{2\pi}{L}(k_1,k_2)$, and γ is a simple curve in the complex plane, containing the complex turning points $\frac{-\frac{2\pi k_3}{L}\pm\sqrt{k_\perp^2+m^2}}{eE}$ as interior points (Note that for $\left|\frac{2\pi k_3}{L}\right| > eET$ there isn't turning points [66], that's the reason why for that modes its beta-Bogoliubov coefficient vanishes). Now, it's easy to verify that

$$|\beta_{\mathbf{k}}|^2 = \begin{cases} \exp\left(-\frac{\pi(k_\perp^2+m^2)}{eE}\right) & \text{if } \left|\frac{2\pi k_3}{L}\right| < eET \\ 0 & \text{if } \left|\frac{2\pi k_3}{L}\right| > eET, \end{cases} \tag{90}$$

and therefore, the probability that the vacuum state remains unchanged is

$$P = \exp\left(-\sum_{\mathbf{k}\in\mathbb{Z}^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}|\beta_{\mathbf{k}}|^{2n}\right) = \exp\left(-N\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\exp\left(-nS\right)\right), \tag{91}$$

in agreement with Schwinger's result.

In the same way one obtains that the average number of produced pairs per unit volume and per unit time, is

$$\frac{E^2\alpha}{8\pi^3}\exp\left(-\frac{\pi m^2}{eE}\right). \tag{92}$$

To obtain Schwinger's formula for fermions, one has to use the Pauli Exclusion Principle to get the following relation between Bogoliubov's coefficients [65, 67, 68]

$$|\alpha_{\mathbf{k}}|^2 + |\beta_{\mathbf{k}}|^2 = 1. \tag{93}$$

Now $|\beta_{\mathbf{k}}|^2$ is the probability that a particle is created in the $\mathbf{k}$ mode, because it's the same that the average number of produced particles in that mode, and consequently $|\alpha_{\mathbf{k}}|^2$ is the probability that no particles are produced in that mode. Therefore, due to the existence of two states with spin $1/2$, the probability that the vacuum state remains unchanged is

$$P = \prod_{\mathbf{k}\in\mathbb{Z}^3}(1-|\beta_{\mathbf{k}}|^2)^2 = \exp\left(-2\sum_{\mathbf{k}\in\mathbb{Z}^3}\sum_{n=1}^{\infty}\frac{1}{n}|\beta_{\mathbf{k}}|^{2n}\right) = \exp\left(-2N\sum_{n=1}^{\infty}\frac{1}{n^2}\exp\left(-nS\right)\right). \tag{94}$$

Remark II.5. *In that Section we've obtained Schwinger's formula in an easy but formal way. However if one wants a complete demonstration of the formula with all the details, one can look up at [72–75]. Essentially, the idea is to make the change of variable* $y = \sqrt{\frac{2}{eE}}(\frac{2\pi k_3}{L} + eEt)$*, then the Klein-Gordon equation behaves*

$$\chi_{\mathbf{k}}'' + \widetilde{\omega}_{\mathbf{k}}^2(y)\chi_{\mathbf{k}} = 0, \tag{95}$$

where, $\widetilde{\omega}_{\mathbf{k}}(y) \equiv \sqrt{\frac{1}{4}y^2 - A_{\mathbf{k}}}$ *and* $A_{\mathbf{k}} \equiv \frac{-1}{2eE}(k_\perp^2+m^2)$*.*

In this case, the function $\chi_{\mathbf{k},0;WKB}(y;\epsilon=1) \equiv \sqrt{\frac{1}{2\widetilde{\omega}_{\mathbf{k}}(y)}}e^{-i\int^y\widetilde{\omega}_{\mathbf{k}}(\tau)d\tau}$*, has the asymptotic behavior*

$$\chi_{\mathbf{k},0;WKB}(y;\epsilon=1) = \begin{cases} e^{iy^2/4}(y^2)^{-1/4-iA_{\mathbf{k}}/2} & y\to-\infty \\ e^{-iy^2/4}(y^2)^{-1/4+iA_{\mathbf{k}}/2} & y\to\infty. \end{cases} \tag{96}$$

Note that $\chi'_{\mathbf{k},0;WKB}(y;\epsilon=1) = -i\widetilde{\omega}_{\mathbf{k}}(y)\chi_{\mathbf{k},0;WKB}(y;\epsilon=1)$ *when* $y\to\pm\infty$*, this means that the diagonalization metod and zero order adiabatic vacuum modes give the same value for* $\beta_{\mathbf{k}}$*.*

On the other hand, a independent set of solutions of (95) is given by the two following parabolic cylinder functions [47]

$$u_{\mathbf{k},1}(y) = \exp\left(-\frac{i}{4}y^2\right)M\left(-\frac{i}{2}A_{\mathbf{k}}+\frac{1}{4},\frac{1}{2},\frac{i}{2}y^2\right) \tag{97}$$

$$u_{\mathbf{k},2}(y) = \frac{1}{\sqrt{2}}\exp\left(-\frac{i}{4}y^2\right)y\exp\left(-\frac{i\pi}{4}\right)M\left(-\frac{i}{2}A_{\mathbf{k}}+\frac{3}{4},\frac{3}{2},\frac{i}{2}y^2\right), \tag{98}$$

where M *is the Kummer's function.*

We now define the mode solution

$$\chi_{\mathbf{k}}(y) \equiv \hbar^{1/2} B_{\mathbf{k}}^{-1} e^{i\pi/8} e^{-\pi A_{\mathbf{k}}/4} 2^{-1/4 - iA_{\mathbf{k}}/2} \varphi_{\mathbf{k}}(y), \tag{99}$$

with

$$\varphi_{\mathbf{k}}(y) \equiv \frac{\Gamma\left(\frac{1}{4} + \frac{i}{2}A_{\mathbf{k}}\right)}{\Gamma\left(\frac{1}{2}\right)} u_{\mathbf{k},1}(y) + \frac{\Gamma\left(\frac{3}{4} + \frac{i}{2}A_{\mathbf{k}}\right)}{\Gamma\left(\frac{3}{2}\right)} u_{\mathbf{k},2}(y), \tag{100}$$

and

$$B_{\mathbf{k}} \equiv \frac{\Gamma\left(\frac{1}{4} + \frac{i}{2}A_{\mathbf{k}}\right)}{\Gamma\left(\frac{1}{4} - \frac{i}{2}A_{\mathbf{k}}\right)} + i\frac{\Gamma\left(\frac{3}{4} + \frac{i}{2}A_{\mathbf{k}}\right)}{\Gamma\left(\frac{3}{4} - \frac{i}{2}A_{\mathbf{k}}\right)}. \tag{101}$$

Then, from the asymptotic behavior of the Kummer's function [47] we can see that

$$\chi_{\mathbf{k}}(y) \to \chi_{\mathbf{k},0;WKW}(y;\epsilon=1), \quad \textit{when} \quad y \to -\infty, \tag{102}$$

and

$$\chi_{\mathbf{k}}(y) \to B_{\mathbf{k}}^{-1} e^{i\pi/4} 2^{1-iA_{\mathbf{k}}} \chi_{\mathbf{k},0;WKW}(y;\epsilon=1) + C_{\mathbf{k}} B_{\mathbf{k}}^{-1} \chi^{*}_{\mathbf{k},0;WKW}(y;\epsilon=1), \quad \textit{when} \quad y \to \infty, \tag{103}$$

with

$$C_{\mathbf{k}} \equiv \frac{\Gamma\left(\frac{1}{4} + \frac{i}{2}A_{\mathbf{k}}\right)}{\Gamma\left(\frac{1}{4} - \frac{i}{2}A_{\mathbf{k}}\right)} - i\frac{\Gamma\left(\frac{3}{4} + \frac{i}{2}A_{\mathbf{k}}\right)}{\Gamma\left(\frac{3}{4} - \frac{i}{2}A_{\mathbf{k}}\right)}.$$

Then from this last formula we can deduce that, in both cases (diagonalization method and zero order adiabatic vacuum modes), the square of the β-Bogoliubov coefficient is given by

$$|\beta_{\mathbf{k}}|^2 = |C_{\mathbf{k}}/B_{\mathbf{k}}|^2 = e^{2\pi A_{\mathbf{k}}} = \exp\left(-\frac{\pi}{eE}(k_{\perp}^2 + m^2)\right), \tag{104}$$

in agreement with (90).

C. Moving Mirrors

Consider a massless scalar field ϕ in the 2-dimensional Minkowski space-time interacting with a perfect reflecting moving mirror. Assume that the mirror trajectory follows an inertial prescribed trajectory $x = g(t)$, that in light-like coordinates $u \equiv t - z$ and $v \equiv t + z$, we write as $v = V(u)$ or $u = U(v)$.

For a perfectly reflecting mirror the set of "in" and "out" mode functions is [76–78]

$$\begin{cases} \phi^{in}_{\omega,R}(u,v) = \frac{1}{\sqrt{4\pi|\omega|}}\left(e^{-i\omega v} - e^{-i\omega V(u)}\right)\theta(v - V(u)) \\ \phi^{in}_{\omega,L}(u,v) = \frac{1}{\sqrt{4\pi|\omega|}}\left(e^{-i\omega u} - e^{-i\omega U(v)}\right)\theta(u - U(v)) \end{cases} \tag{105}$$

$$\begin{cases} \phi^{out}_{\omega,R}(u,v) = \frac{1}{\sqrt{4\pi|\omega|}}\left(e^{-i\omega u} - e^{-i\omega U(v)}\right)\theta(v - V(u)) \\ \phi^{out}_{\omega,L}(u,v) = \frac{1}{\sqrt{4\pi|\omega|}}\left(e^{-i\omega v} - e^{-i\omega V(u)}\right)\theta(u - U(v)), \end{cases} \tag{106}$$

where $\phi^{in}_{\omega,R}$ (resp. $\phi^{in}_{\omega,L}$) represents particles with frequency ω coming from the right (resp. left) null past infinity domain $\mathcal{J}_R^-$ (resp. $\mathcal{J}_L^-$), and $\phi^{out}_{\omega,R}$ (resp. $\phi^{out}_{\omega,L}$) represents particles with frequency ω going to the right (resp. left) null future infinity domain $\mathcal{J}_R^+$ (resp. $\mathcal{J}_L^+$).

It is a well-known fact that the average number of particles in the ω mode produced from the vacuum in the right hand side (rhs) of the mirror, is [78, 79]

$$N(\omega) = \int d\omega' |\beta^{R,R}_{\omega,\omega'}|^2. \tag{107}$$

Then our main objective is to calculate the beta Bogoliubov coefficient

$$\beta^{R,R}_{\omega,\omega'} \equiv (\phi^{out}_{\omega,R}{}^*; \phi^{in}_{\omega',R})^*, \quad \text{with} \quad \omega, \omega' > 0, \tag{108}$$

where the parenthesis in the right member denotes the usual product for scalar fields [11], that is,

$$\beta^{R,R}_{\omega,\omega'} \equiv i \int_{g(t)}^{\infty} \phi^{out}_{\omega,R}(t,x) \overleftrightarrow{\partial_t} \phi^{in}_{\omega',R}(t,x) dx, \tag{109}$$

that doesn't depend on the space-like hyper-surface chosen.

The best way to perform this calculation is to choose as hyper-surface the right null future infinity $\mathcal{J}^+_R$, then one has

$$\beta^{R,R}_{\omega,\omega'} = 2i \int_{\mathbb{R}} du \phi^{out}_{\omega,R} \partial_u \phi^{in}_{\omega',R} = \frac{-i}{2\pi\sqrt{\omega\omega'}} \int_{\mathbb{R}} du e^{-i\omega u} \partial_u e^{-i\omega' V(u)} = \frac{1}{2\pi} \sqrt{\frac{\omega}{\omega'}} \int_{\mathbb{R}} du e^{-i\omega u} e^{-i\omega' V(u)}. \tag{110}$$

Assuming that the mirror's velocity converges fast enough to a constant when $|u| \to \infty$, and integrating by parts one gets

$$\beta^{R,R}_{\omega,\omega'} = -\frac{1}{2\pi i} \sqrt{\omega\omega'} \int_{\mathbb{R}} du \frac{V''(u)}{(\omega + \omega' V'(u))^2} e^{-i\omega u} e^{-i\omega' V(u)}. \tag{111}$$

For simplicity we assume that the mirror's acceleration is discontinuous at the point $u = a$, after another integration by parts one obtains

$$\begin{aligned} \beta^{R,R}_{\omega,\omega'} \quad &= -\frac{1}{2\pi} \sqrt{\omega\omega'} \frac{1}{(\omega + \omega' V'(a))^3} e^{-i\omega a} e^{-i\omega' V(a)} (V''(a^-) - V''(a^+)) \\ &+ \frac{1}{2\pi} \sqrt{\omega\omega'} \int_{\mathbb{R}} du \left[\frac{V'''(u)}{(\omega + \omega' V'(u))^3} - \frac{3\omega' (V''(u))^2}{(\omega + \omega' V'(u))^4} \right] e^{-i\omega u} e^{-i\omega' V(u)}. \end{aligned} \tag{112}$$

From this formula, assuming that the mirror's trajectory is asymptotically inertial, i.e., $V'(u) > 0 \quad \forall u \in \mathbb{R}$ (see for example [79]), one concludes that $\left|\beta^{R,R}_{\omega,\omega'}\right|^2$ and $\omega \left|\beta^{R,R}_{\omega,\omega'}\right|^2$ are integrable functions in the domain $[0,\infty)^2 \setminus [0,1]^2$.

Finally, we're interested in the production of particles in the infrared domain, i.e., we want to calculate $\left|\beta^{R,R}_{\omega,\omega'}\right|^2$ in $[0,1]^2$. We write the Bogoliubov coefficient as follows

$$\beta^{R,R}_{\omega,\omega'} = \frac{1}{2\pi} \sqrt{\frac{\omega}{\omega'}} \int_{\mathbb{R}} du e^{-i(\omega + B\omega')u} e^{-i\omega'(V(u) - Bu)}, \tag{113}$$

with $B > 0$. After an integration by parts one obtains

$$\beta^{R,R}_{\omega,\omega'} = -\frac{1}{2\pi} \frac{\sqrt{\omega\omega'}}{\omega + B\omega'} \int_{\mathbb{R}} du (V'(u) - B) e^{-i\omega u} e^{-i\omega' V(u)}, \tag{114}$$

and thus, if the function $|V'(u) - B|$ is integrable in $\mathbb{R}$ for some $B > 0$, it can be deduced that $\left|\beta^{R,R}_{\omega,\omega'}\right|^2$ and $\omega \left|\beta^{R,R}_{\omega,\omega'}\right|^2$ are integrable functions in the domain $[0,1]^2$. An example of this kind of trajectories is

$$V(u) = \begin{cases} Bu & u \leq 0 \\ V(u) & 0 \leq u \leq u_0 \\ V(u_0) + B(u - u_0) & u \geq u_0. \end{cases} \tag{115}$$

However if we one is only interested in the convergence of the function $\omega \left|\beta^{R,R}_{\omega,\omega'}\right|^2$ in the domain $[0,1]^2$, one only needs trajectories that satisfy

$$\int_{-\infty}^{0} du|V'(u) - B_1| < \infty \text{ and } \int_{0}^{\infty} du|V'(u) - B_2| < \infty \tag{116}$$

for some no-negatives constants B_1 and B_2, (here it's important to remark that one of these constants can be zero, that is, it's not worth that the trajectory be asymptotically inertial). To prove this statement we write

$$\beta^{R,R}_{\omega,\omega'} = \frac{1}{2\pi}\sqrt{\frac{\omega}{\omega'}}\left[\int_{-\infty}^{0} du e^{-i(\omega+B_1\omega')u}e^{-i\omega'(V(u)-B_1u)} + \int_{0}^{\infty} du e^{-i(\omega+B_2\omega')u}e^{-i\omega'(V(u)-B_2u)}\right], \tag{117}$$

and we assume for simplicity that $V(0) = 0$. After an integration by parts one gets the formula

$$\begin{aligned}\beta^{R,R}_{\omega,\omega'} &= -\frac{1}{2\pi}\sqrt{\omega\omega'}\frac{B_1 - B_2}{(\omega + B_1\omega')(\omega + B_2\omega')} - \frac{1}{2\pi}\frac{\sqrt{\omega\omega'}}{\omega + B_1\omega'}\int_{-\infty}^{0} du(V'(u) - B_1)e^{-i\omega u}e^{-i\omega' V(u)} \\ &\quad -\frac{1}{2\pi}\frac{\sqrt{\omega\omega'}}{\omega + B_2\omega'}\int_{0}^{\infty} du(V'(u) - B_2)e^{-i\omega u}e^{-i\omega' V(u)},\end{aligned} \tag{118}$$

that proves the statement.

In conclusion we've proved that for asymptotically inertial trajectories with continuous velocity the radiated energy is finite. However it is also possible an infinite production of particles with very low frequency (an infrared divergency). To remove this divergency one must assumes that the initial and final mirror's velocity is the same.

Note that particle creation for partially transmitting mirrors is a bit different: At very high frequencies the mirror behaves transparent, and then there are not particle production independently of the mirror's trajectory. On the other hand, at very low frequencies the mirror behaves a perfect reflector and then the same kind of infrared problems as in the perfect reflector case remain. Consequently, if one is only interested in a finite radiated energy, one can considers trajectories with a continuous velocity $V'(u) \quad \forall u \in \mathbb{R}$, which fulfill the condition (116) as, for instance, the no-asymptotically inertial trajectory:

$$v = V(u) \equiv \begin{cases} u & \text{if } \ u \leq 0 \\ \frac{1}{k}(1 - e^{-ku}) & \text{if } \ u \geq 0. \end{cases} \tag{119}$$

1. *Simulating black body collapse*

Now we're interested in a trajectory that simulates a black body collapse [11, 80, 81], that is, with the following form

$$v = V(u) \equiv \begin{cases} u & \text{if} \quad u \leq 0 \\ \frac{1}{k}(1 - e^{-ku}) & \text{if } \ 0 \leq u \leq u_0 \\ V(u_0) + A(u - u_0) & \text{if} \quad u \geq u_0, \end{cases} \tag{120}$$

with $A = e^{-ku_0}$, where k is a frequency and $ku_0 \gg 1$.

Note that this trajectory can be written under the following form, too

$$u = U(v) \equiv \begin{cases} v & \text{if} \quad v \leq 0 \\ -\frac{1}{k}\ln(1 - kv) & \text{if } \ 0 \leq v \leq v_0 \\ U(v_0) + A^{-1}(v - v_0) & \text{if} \quad v \geq v_0. \end{cases} \tag{121}$$

Then,

$$\beta^{R,R}_{\omega,\omega'} = 2i\int_{\mathbb{R}} du\phi^{out}_{\omega,R}\partial_u\phi^{in}_{\omega',R} \;=\; \frac{1}{2\pi i\sqrt{\omega\omega'}}\frac{\omega'}{\omega+\omega'} - \frac{1}{2\pi i\sqrt{\omega\omega'}}e^{-i\omega u_0}e^{-i\omega' V(u_0)}\frac{\omega' A}{\omega+\omega' A} - \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\frac{1}{k}\int_0^{1-A} ds(1-s)^{i\omega/k}e^{-i\omega'/ks}. \tag{122}$$

Assuming for simplicity $\omega \sim k$, if one is interested in the domain of frequencies $1 \ll \omega'/k \ll A^{-1}$, one arrives at

$$\beta^{R,R}_{\omega,\omega'} \;\cong\; \frac{1}{2\pi i\sqrt{\omega\omega'}} - \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\frac{1}{k}\int_0^{1-A} ds(1-s)^{i\omega/k}e^{-i\omega'/ks}. \tag{123}$$

To obtain an explicit expression to the second term on the rhs, we consider the domain

$$D \equiv \{z \in \mathbb{C}/\mathrm{Re}z \in [0, 1-A], \mathrm{Im}z \in [-\epsilon, 0], \text{ with } \frac{k}{\omega'} \ll \epsilon \ll 1\}. \tag{124}$$

and, going through the same steps as in [82], one easily obtains

$$\beta^{R,R}_{\omega,\omega'} \cong \frac{1}{2\pi i\sqrt{\omega\omega'}}e^{-i\omega'/k}\left(\frac{ik}{\omega'}\right)^{i\omega/k}\Gamma\left(1+i\omega/k\right), \tag{125}$$

and thus, using that $|\Gamma\left(1+i\omega/k\right)|^2 = \frac{\pi\omega/k}{\sinh(\pi\omega/k)}$ (see [47]) one gets, for a perfect reflecting mirror, that

$$\left|\beta^{R,R}_{\omega,\omega'}\right|^2 \cong \frac{1}{2\pi\omega' k}\left(e^{2\pi\omega/k}-1\right)^{-1}, \tag{126}$$

in the range $1 \ll \omega'/k \ll A^{-1}$.

Remark II.6. *From that formula, one deduces that the number of radiated particles in the ω mode diverges logarithmically with $u_0 \to \infty$. In this situation the physically relevant quantity is the number of created particles in the ω mode per unit time. This dimensionless quantity is finite and its value is given by [82, 83]*

$$\lim_{u_0\to\infty}\frac{1}{u_0}N(\omega) = \frac{1}{2\pi}\left(e^{2\pi\omega/k}-1\right)^{-1}. \tag{127}$$

Now we'll study what happens when the mirror is partially reflecting. First, we search for the co-moving coordinates (τ, ρ), that is, the coordinates for which the mirror is at rest, τ being i the proper time of the mirror, and we take ρ such that its trajectory is given by $\rho = 0$. Introducing the light-like coordinates $(\bar{u}, \bar{v})$ defined as

$$\bar{u} \equiv \tau - \rho; \quad \bar{v} \equiv \tau + \rho, \tag{128}$$

we will calculate the mirror's trajectory in the coordinates $(\bar{u}, \bar{v})$. Along this trajectory, the length element obeys the identity [84]

$$d\tau^2 = d\bar{u}^2 = d\bar{v}^2 = V'(u)du^2 = U'(v)dv^2. \tag{129}$$

Then, an easy calculation yields the relations

$$\bar{v} = \bar{u}(u) \equiv \begin{cases} u & \text{if} \quad u \le 0 \\ \frac{2}{k}(1-e^{-ku/2}) & \text{if} \;\; 0 \le u \le u_0 \\ \bar{u}(u_0) + \sqrt{A}(u-u_0) & \text{if} \quad u \ge u_0, \end{cases} \tag{130}$$

and,

$$\bar{u} = \bar{v}(v) \equiv \begin{cases} v & \text{if} \quad v \le 0 \\ \frac{2}{k}(1-\sqrt{1-kv}) & \text{if} \;\; 0 \le v \le v_0 \\ \bar{v}(v_0) + A^{-1/2}(v-v_0) & \text{if} \quad v \ge v_0. \end{cases} \tag{131}$$

When the mirror is at rest, scattering is described by the S-matrix (see [85–87] for more details)

$$S(\omega)=\begin{pmatrix} s(\omega) & r(\omega)e^{-2i\omega L} \\ r(\omega)e^{2i\omega L} & s(\omega)\end{pmatrix}, \tag{132}$$

where $x=L$ is the position of the mirror. The S matrix is taken to be real in the temporal domain, causal, unitary, and the identity at high frequencies [86]. Correspondingly, the "in" modes in the coordinates $(\bar{u},\bar{v})$ are [88]

$$g^{in}_{\omega,R}(\bar{u},\bar{v})=\frac{1}{\sqrt{4\pi|\omega|}}s(\omega)e^{-i\omega\bar{v}}\theta(\bar{u}-\bar{v})+\frac{1}{\sqrt{4\pi|\omega|}}\left(e^{-i\omega\bar{v}}+r(\omega)e^{-i\omega\bar{u}}\right)\theta(\bar{v}-\bar{u}). \tag{133}$$

$$g^{in}_{\omega,L}(\bar{u},\bar{v})=\frac{1}{\sqrt{4\pi|\omega|}}\left(e^{-i\omega\bar{u}}+r(\omega)e^{-i\omega\bar{v}}\right)\theta(\bar{u}-\bar{v})+\frac{1}{\sqrt{4\pi|\omega|}}s(\omega)e^{-i\omega\bar{u}}\theta(\bar{v}-\bar{u}). \tag{134}$$

On the other hand, the "in" modes in the coordinates (u,v), namely ϕ^{in}, are defined in the right null past infinity $\mathcal{J}^-_R$ by

$$\phi^{in}_{\omega,R}=\frac{1}{\sqrt{4\pi|\omega|}}e^{-i\omega v};\quad \phi^{in}_{\omega,L}=0, \tag{135}$$

and in the left null past infinity $\mathcal{J}^-_L$ by

$$\phi^{in}_{\omega,R}=0;\quad \phi^{in}_{\omega,L}=\frac{1}{\sqrt{4\pi|\omega|}}e^{-i\omega u}. \tag{136}$$

Writing $\bar{g}^{in}_{\omega,k}(u,v)\equiv g^{in}_{\omega,k}(\bar{u}(u),\bar{v}(v))$ with $k=R,L$, and using that $\bar{g}^{in}_{-\omega,k}=\bar{g}^{in*}_{\omega,k}$, one obtains the following relation

$$\phi^{in}_{\omega,k}=\int_{\mathbb{R}}d\omega'\chi(\omega')(\bar{g}^{in}_{\omega',k};\phi^{in}_{\omega,k})\bar{g}^{in}_{\omega',k},\quad \text{with}\quad k=R,L \tag{137}$$

with $\chi(\omega')$ the sing function. To calculate explicitly the "in" modes, I choose the coefficients $r(\omega)=\frac{-i\gamma}{\omega+i\gamma}$ and $s(\omega)=\frac{\omega}{\omega+i\gamma}$ with $\alpha\geq 0$ that correspond to the so-called Barton-Calogeracos model [88–90]. In this case, on the rhs of mirror one has [91]

$$\phi^{in}_{\omega,R}(u,v)=\frac{1}{\sqrt{4\pi|\omega|}}e^{-i\omega v}+\phi^{refl}_{\omega,R}(u);\qquad \phi^{in}_{\omega,L}(u,v)=\phi^{trans}_{\omega,L}(u), \tag{138}$$

where

$$\phi^{refl}_{\omega,R}(u)=\begin{cases} \frac{1}{\sqrt{4\pi|\omega|}}\frac{-i\gamma}{\omega+i\gamma}e^{-i\omega V(u)}; & u\leq 0 \\ \frac{1}{\sqrt{4\pi|\omega|}}\frac{-i\gamma}{\omega+i\gamma}e^{-\gamma\bar{u}(u)}-\frac{2\gamma}{k\sqrt{4\pi|\omega|}}e^{-i\frac{\omega}{k}}\int_0^{\frac{k}{2}\bar{u}(u)}dse^{\frac{i\omega}{k}\left(s+1-\frac{k}{2}\bar{u}(u)\right)^2}e^{-\frac{2\gamma s}{k}}; & 0\leq u\leq u_0 \\ \frac{1}{\sqrt{4\pi|\omega|}}\frac{-i\gamma}{\omega+i\gamma}e^{-\gamma\bar{u}(u)}-\frac{1}{\sqrt{4\pi|\omega|}}\frac{i\gamma}{\sqrt{A}\omega+i\gamma}\left[e^{-i\omega V(u)}-e^{-i\omega V(u_0)}e^{-\gamma(\bar{u}(u)-\bar{u}(u_0))}\right] & \\ \quad -\frac{2\gamma}{k\sqrt{4\pi|\omega|}}e^{-i\frac{\omega}{k}}e^{-\gamma(\bar{u}(u)-\bar{u}(u_0))}\int_0^{\frac{k}{2}\bar{u}(u_0)}dse^{\frac{i\omega}{4}\left(s+1-\frac{k}{2}\bar{u}(u_0)\right)^2}e^{-\frac{2\gamma s}{k}}; & u\geq u_0 \end{cases} \tag{139}$$

and

$$\phi^{trans}_{\omega,L}(u)=\begin{cases} \frac{1}{\sqrt{4\pi|\omega|}}\frac{\omega}{\omega+i\gamma}e^{-i\omega V(u)}; & u\leq 0 \\ \frac{1}{\sqrt{4\pi|\omega|}}e^{-i\omega u}+\frac{1}{\sqrt{4\pi|\omega|}}\frac{-i\gamma}{\omega+i\gamma}e^{-\alpha\bar{u}(u)}-\frac{2\gamma}{k\sqrt{4\pi|\omega|}}\int_0^{\frac{k}{2}\bar{u}(u)}ds(s+1-\frac{k}{2}\bar{u}(u))^{2i\frac{\omega}{k}}e^{-\frac{2\gamma s}{k}}; & 0\leq u\leq u_0 \\ \frac{1}{\sqrt{4\pi|\omega|}}\frac{-i\gamma}{\omega+i\gamma}e^{-\gamma\bar{u}(u)}+\frac{1}{\sqrt{4\pi|\omega|}}\frac{e^{-i\omega u_0}}{\omega+i\gamma\sqrt{A}}\left[\omega e^{-i\frac{\omega}{\sqrt{A}}(\bar{u}(u)-\bar{u}(u_0))}+i\gamma\sqrt{A}e^{-\gamma(\bar{u}(u)-\bar{u}(u_0))}\right] & \\ \quad -\frac{2\gamma}{k\sqrt{4\pi|\omega|}}e^{-\gamma(\bar{u}(u)-\bar{u}(u_0))}\int_0^{\frac{k}{2}\bar{u}(u_0)}ds(s+1-\frac{k}{2}\bar{u}(u_0))^{2i\frac{\omega}{k}}e^{-\frac{2\gamma s}{k}}; & u\geq u_0 \end{cases} \tag{140}$$

Note that in the case of perfect reflection, that is, when $\gamma \to \infty$ one has

$$\phi^{refl}_{\omega,R}(u) \to -\frac{1}{\sqrt{4\pi|\omega|}} e^{-i\omega V(u)}; \qquad \phi^{trans}_{\omega,L}(u) \to 0, \tag{141}$$

and when the mirror is transparent, i.e., when $\gamma \to 0$ one has

$$\phi^{refl}_{\omega,R}(u) \to 0; \qquad \phi^{trans}_{\omega,L}(u) \to \frac{1}{\sqrt{4\pi|\omega|}} e^{-i\omega u}. \tag{142}$$

Since we're interested in the particle production on the rhs of the mirror, we must now calculate

$$\beta^{R,R}_{\omega,\omega'} \equiv \left(\phi^{out}_{\omega,R}{}^{*}; \phi^{in}_{\omega',R}\right)^{*}, \quad \text{and} \quad \beta^{R,L}_{\omega,\omega'} \equiv \left(\phi^{out}_{\omega,R}{}^{*}; \phi^{in}_{\omega',L}\right)^{*} \quad \omega,\omega' > 0. \tag{143}$$

In order to calculate this products we choose the right null infinity $\mathcal{J}^{+}_{R}$, because here the "out" modes have a very easy form, then

$$\beta^{R,R}_{\omega,\omega'} = \left(\phi^{out}_{\omega,R}{}^{*}; \phi^{refl}_{\omega',R}\right)^{*}, \quad \text{and} \quad \beta^{R,L}_{\omega,\omega'} \equiv \left(\phi^{out}_{\omega,R}{}^{*}; \phi^{trans}_{\omega',L}\right)^{*}. \tag{144}$$

We start calculating $\beta^{R,R}_{\omega,\omega'} = 2i\int_{\mathbb{R}} du \phi^{out}_{\omega,R}\partial_u \phi^{refl}_{\omega',R}$, with the result

$$\begin{aligned}\beta^{R,R}_{\omega,\omega'} \;\cong\; & \frac{1}{2\pi\sqrt{\omega\omega'}}\frac{\gamma}{\omega'+i\gamma}\left[1-\frac{\gamma}{k}\int_A^1 dx x^{i\omega/k-1/2}e^{-2\gamma(1-\sqrt{x})/k}\right] \\ & +\frac{\gamma}{2\pi k i\sqrt{\omega\omega'}}e^{-i\omega'/k}\int_A^1 dx x^{i\omega/k-1/2}e^{i\omega' x/k}\left[1-\frac{2\gamma}{k}\int_0^{1-\sqrt{x}} e^{i\omega'(s^2+2s\sqrt{x})/k}e^{-2\gamma s/k}\right].\end{aligned} \tag{145}$$

Now assuming once again $\omega \sim k$, provided that $1 \ll \frac{\omega'}{k} \ll \frac{\gamma^2}{k^2} \ll A^{-1}$, equation (145) turns into equation (123). Consequently, we precisely obtain the same behavior as for a perfect reflecting mirror. However, in the case $1 \ll \frac{\gamma^2}{k^2} \ll \frac{\omega'}{k} \ll A^{-1}$ we observe that

$$\beta^{R,R}_{\omega,\omega'} \;\cong\; \frac{\alpha}{2\pi k i\sqrt{\omega\omega'}}e^{-i\omega'/k}\left(i\frac{k}{\omega'}\right)^{i\omega/k+1/2}\Gamma\left(1/2+i\omega/k\right), \tag{146}$$

and using the identity $|\Gamma\left(1/2+i\omega/k\right)|^2 = \frac{\pi}{\cosh(\pi\omega/k)}$ (see [47]), we conclude that

$$\left|\beta^{R,R}_{\omega,\omega'}\right|^2 \;\cong\; \frac{1}{2\pi k\omega}\left(\frac{\gamma}{\omega'}\right)^2\left(e^{2\pi\omega/k}+1\right)^{-1}. \tag{147}$$

Finally, a simple but rather cumbersome calculation yields in the first case

$$\left|\beta^{R,L}_{\omega,\omega'}\right|^2 \cong 0, \tag{148}$$

and in the second one

$$\left|\beta^{R,L}_{\omega,\omega'}\right|^2 \sim \frac{1}{\omega\omega'}\mathcal{O}\left[\left(\frac{\gamma}{\omega'}\right)^2\right]. \tag{149}$$

Then we can conclude that the number of produced particles in the mode ω is approximately given by [92, 93]

$$\begin{aligned}N(\omega) \;\cong\; & \int_k^{\gamma^2/k}\frac{1}{2\pi k\omega'}\left(e^{2\pi\omega/k}-1\right)^{-1}+\int_{\gamma^2/k}^{\infty}\frac{1}{2\pi k\omega}\left(\frac{\gamma}{\omega'}\right)^2\left(e^{2\pi\omega/k}+1\right)^{-1} \\ = & \frac{1}{\pi k}\ln(\gamma/k)\left(e^{2\pi\omega/k}-1\right)^{-1}+\frac{1}{2\pi\omega}\left(e^{2\pi\omega/k}+1\right)^{-1} \cong \frac{1}{\pi k}\ln(\gamma/k)\left(e^{2\pi\omega/k}-1\right)^{-1},\end{aligned} \tag{150}$$

because we're assumig $k \sim \omega \ll \gamma$.

That is, the number density of produced particles in the mode ω by a partially transmitting moving mirror is finite when $u_0 \to \infty$, moreover when $\omega \sim k \ll \gamma$ the mirror radiates a thermal flux described by Bose-Einstein statistics. However when $\omega \sim k \sim \gamma$, maybe the contribution of the sector $[\gamma^2/k, \infty)$ could be dominant and another kind of statistics (Fermi-Dirac) would be possible. This is a situation that deserves futher investigation.

III. VACUUM FLUCTUATIONS

The re-normalization of the two point function via adiabatic regularization is given with all the details, and the re-normalization of the stress tensor is also reviewed.

A. Re-normalized two point function

1. *Massless conformally coupled field*

We start this lecture studying the simplest case: a massless conformally coupled scalar field in a Friedman-Robertson-Walker space-time. We'll calculate the re-normalized part of the two-point function $\langle \phi^2(\eta, \vec{x}) \rangle$ using the adiabatic regularization method. This is the simplest example, and it help us to understand all the details of the method.

Consider the quantum scalar field

$$\hat{\phi}(\eta, \mathbf{x}) = \int_{\mathbb{R}^3} d^3\mathbf{k} \left[\hat{a}_{\mathbf{k}} \phi_{\mathbf{k}}(\eta, \mathbf{x}) + \hat{a}^\dagger_{\mathbf{k}} \phi^*_{\mathbf{k}}(\eta, \mathbf{x}) \right], \tag{151}$$

then, using the same notation as in first lecture, one has $\chi_{\mathbf{k}}(\eta) = \frac{e^{-i|\mathbf{k}|\eta}}{\sqrt{2|\mathbf{k}|}}$, and thus, the two-point function is given by

$$\langle \hat{\phi}^2(\eta, \mathbf{x}) \rangle \equiv \int_{\mathbb{R}^3} d^3\mathbf{k} |\phi_{\mathbf{k}}(\eta, \mathbf{x})|^2 = \frac{1}{4\pi^2 C(\eta)} \int_0^\infty |\mathbf{k}| d|\mathbf{k}|. \tag{152}$$

In order to obtain the re-normalized value of the two-point function, we'll follow Bunch's method described in [34]: First, one considers the adiabatic modes obtained in the WKB approximation,

$$\chi_{adi,\mathbf{k}}(\eta) = \frac{1}{\sqrt{2W_{\mathbf{k}}}} e^{-i \int W_{\mathbf{k}} d\eta}, \tag{153}$$

up to order 2. To calculate these adiabatic modes one has to use equation (9), then in the conformally coupled case W_k is given by (see for details [44, 94, 95])

$$W_{\mathbf{k}} = \omega_{\mathbf{k}} - \frac{1}{4} \frac{\omega''_{\mathbf{k}}}{\omega^2_{\mathbf{k}}} + \frac{3}{8} \frac{(\omega'_{\mathbf{k}})^2}{\omega^3_{\mathbf{k}}} \tag{154}$$

with $\omega^2_{\mathbf{k}} = |\mathbf{k}|^2 + C(\eta) m^2$, and a simple calculation yields,

$$W_{\mathbf{k}} = \omega_{\mathbf{k}} - \frac{1}{8} \frac{m^2 C''}{\omega^3_{\mathbf{k}}} + \frac{5}{32} \frac{m^4 (C')^2}{\omega^5_{\mathbf{k}}}. \tag{155}$$

Once the adiabatic modes has been calculated, to obtain the re-normalized expression of $\langle \phi^2(\eta, \mathbf{x}) \rangle$ one has to subtract from (152), the adiabatic terms up to order two (only terms that contain, at most, two derivatives of the scalar factor) that appear in the expression [96, 97]

$$\frac{1}{4\pi^2 C(\eta)} \int_0^\infty \frac{|\mathbf{k}|^{\mathbf{2}}}{W_{\mathbf{k}}} d|\mathbf{k}|, \tag{156}$$

and finally to take the limit $m \to 0$, that is:

$$\langle\hat{\phi}^2(\eta,\mathbf{x})\rangle_{ren} = \lim_{m\to 0}\frac{1}{4\pi^2 C(\eta)}\left[\int_0^\infty\left(|\mathbf{k}|-\frac{|\mathbf{k}|^2}{\omega_{\mathbf{k}}}\right)d|\mathbf{k}| + \frac{m^2C''}{8}\int_0^\infty\frac{|\mathbf{k}|^2}{\omega_{\mathbf{k}}^5}d|\mathbf{k}| - \frac{5m^4(C')^2}{32}\int_0^\infty\frac{|\mathbf{k}|^2}{\omega_{\mathbf{k}}^7}d|\mathbf{k}|\right]. \quad (157)$$

It's not difficult to show that the final result is

$$\langle\hat{\phi}^2(\eta,\mathbf{x})\rangle_{ren} = -\frac{1}{96\pi^2 C}\left[\frac{1}{2}\left(\frac{C'}{C}\right)^2 - \frac{C''}{C}\right] = \frac{1}{48\pi^2}\frac{a''}{a^3} = \frac{1}{288\pi^2}R, \quad (158)$$

which coincides, for the de Sitter phase, with formula (3.19) obtained in [98]. Note also that, in the case of an universe filled by radiation, and consequently $R = 0$, $\langle\phi^2(\eta,\mathbf{x})\rangle_{ren} = 0$ in the massless case independently of the coupling constant value.

2. Massless minimally coupled field

In this Section we consider another simple example, a massless minimally coupled field in the flat chart of the de Sitter space-time, where the modes can also be calculated exactly, they are

$$\chi_{\mathbf{k}}(\eta) = \left(a_{\mathbf{k}}\psi_{\mathbf{k}}(\eta) + b_{\mathbf{k}}\psi^*_{\mathbf{k}}(\eta)\right), \quad (159)$$

where $\psi_{\mathbf{k}}(\eta)$ is given by formula (43) and $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ are some constants.

In general, $\langle\hat{\phi}^2(\eta,\mathbf{x})\rangle = \frac{1}{2\pi^2 C(\eta)}\int_0^\infty |\mathbf{k}|^2|\chi_{\mathbf{k}}(\eta)|^2 d|\mathbf{k}|$, has ultra-violet and infra-red divergencies. To avoid these last ones (see for details [36, 51]), we consider a transition from the radiation dominated phase to the de Sitter one, described by the following scale factor:

$$a(\eta) = \begin{cases} 2-\eta/\eta_0 & \eta < \eta_0 \\ \eta_0/\eta & \eta > \eta_0, \end{cases} \quad (160)$$

with $\eta_0 = -1/H$.

The modes $\chi_{\mathbf{k}}$ for $\eta < \eta_0$, are given by $e^{-i|\mathbf{k}|\eta}$. Note that these modes correspond to the usual choice of the vacuum state for a massless field in the radiation phase, because in that phase the scalar curvature vanishes, and then $\chi_{\mathbf{k}}$ satisfy the equation $\chi''_{\mathbf{k}} + |\mathbf{k}|^2\chi_{\mathbf{k}} = 0$, and consequently, the vacuum state is obtained in the same way as in the Minkowskian case, that is, from the modes $e^{-i|\mathbf{k}|\eta}$.

Matching at the point $\eta = \eta_0$ the modes an their derivatives, one obtains

$$a_{\mathbf{k}} = 1 + \frac{H}{i|\mathbf{k}|} - \frac{H^2}{2|\mathbf{k}|^2}, \qquad b_{\mathbf{k}} = -\frac{H^2}{2|\mathbf{k}|^2}e^{\frac{2i|\mathbf{k}|}{H}} = a_{\mathbf{k}} + \frac{2i|\mathbf{k}|}{3H} + \mathcal{O}\left(|\mathbf{k}|^2/H^2\right). \quad (161)$$

With these coefficients, for small values of $|\mathbf{k}|$, in the de Sitter phase ($\eta > \eta_0$), one has

$$|\chi_{\mathbf{k}}|^2 = \frac{1}{2|\mathbf{k}|}\left[\left(\frac{2}{3H\eta} + 2 + \frac{H^2\eta^2}{6}\right)^2 + \mathcal{O}\left(|\mathbf{k}|^2/H^2\right)\right], \quad (162)$$

what shows that there is not infra-red divergencies.

To analyze ultra-violet divergencies we calculate for large $|\mathbf{k}|$

$$|\chi_{\mathbf{k}}|^2 = \frac{1}{2|\mathbf{k}|}\left[1 + \frac{1}{|\mathbf{k}|^2\eta^2} - \frac{H^2}{|\mathbf{k}|^2}\cos(2|\mathbf{k}|(H^{-1}+\eta) + \mathcal{O}\left(H^3/|\mathbf{k}|^3\right)\right], \quad (163)$$

what shows that the terms that give ultra-violet divergencies in $\langle\hat{\phi}^2(\eta,\mathbf{x})\rangle$ are

$$\frac{\eta^2H^2}{4\pi^2}\int^\infty |\mathbf{k}|d|\mathbf{k}| \quad \text{and} \quad \frac{H^2}{4\pi^2}\int^\infty\frac{1}{|\mathbf{k}|}d|\mathbf{k}|. \quad (164)$$

Once we have separated the divergent terms, we calculate, up to order 2, the adiabatic terms that for $\eta > \eta_0$, are given by

$$W_{\mathbf{k}} = \omega_{\mathbf{k}} - \frac{1}{\eta^2\omega_{\mathbf{k}}} - \frac{1}{8}\frac{m^2C''}{\omega_{\mathbf{k}}^3} + \frac{5}{32}\frac{m^4(C')^2}{\omega_{\mathbf{k}}^5}. \tag{165}$$

From that, one sees that the divergent parts of (156) are, in the de Sitter phase, given by

$$\frac{\eta^2H^2}{4\pi^2}\int^{\infty}\frac{|\mathbf{k}|^2}{\omega_{\mathbf{k}}}d|\mathbf{k}| \quad \text{and} \quad \frac{H^2}{4\pi^2}\int^{\infty}\frac{|\mathbf{k}|^2}{\omega_{\mathbf{k}}^3}d|\mathbf{k}|. \tag{166}$$

Subtracting, for large frequencies, the divergent part of (156) from (164), for example for $|\mathbf{k}| > |\eta|^{-1}$, one gets

$$\lim_{m\to 0}\frac{\eta^2H^2}{4\pi^2}\int_{|\eta|^{-1}}^{\infty}\left(|\mathbf{k}| - \frac{|\mathbf{k}|^2}{\omega_k}\right)d|\mathbf{k}| = 0 \quad \text{and} \quad \lim_{m\to 0}\frac{H^2}{4\pi^2}\int_{|\eta|^{-1}}^{\infty}\left(\frac{1}{|\mathbf{k}|} - \frac{|\mathbf{k}|^2}{\omega_{\mathbf{k}}^3}d|\mathbf{k}|\right) = 0, \tag{167}$$

what shows that the ultra-violet divergencies are canceled.

The problem now is that the subtracted adiabatic term $\frac{H^2}{4\pi^2}\int_0^H\frac{|\mathbf{k}|^2}{\omega_{\mathbf{k}}^3}d|\mathbf{k}|$ contains an infra-red divergency, because it diverges when the mass approaches to zero. It is important to remark that this term can be written as $-\frac{R}{8\pi^2}\int_0^H\left(\xi - \frac{1}{6}\right)\frac{|\mathbf{k}|^2}{\omega_{\mathbf{k}}^3}d|\mathbf{k}|$, that is, it does not appear in the conformal coupled case.

The solution of this infra-red divergency emerges from the following observation: The adiabatic approximation is based on modes of the form (153), and it is clear that this form only has sense if the exact modes, namely $\chi_{\mathbf{k}}$, are oscillating, that is, if $\chi_{\mathbf{k}}$ satisfy the equation (2), with

$$\Omega_{\mathbf{k}}^2(\eta) > 0. \tag{168}$$

In our case this conditions means $|\mathbf{k}| > \sqrt{2}|\eta^{-1}| = \sqrt{2}He^{Ht}$, and thus, we must only subtract adiabatic modes well inside in the Hubble horizon at time t. Consequently, no infra-red divergencies appears.

Here an important remark is in order: Our recipe to eliminate the infra-red divergency does not affect to the conservation of the renormalized stress tensor which for a FRW metric reduces to $(\rho C^{3/2})' + p(C^{3/2})' = 0$, where ρ is the energy density and p is the pressure, because the adiabatic regularization lies in subtracting adiabatic terms up to a given order and then the conservation equation is safisfied for each order, and more important, this subtraction can be performed mode by mode [43, 56]. This means that if one denotes by $\rho_{ad}(\phi_k)$ and $p_{ad}(\phi_k)$ the adiabatic terms of the energy density and pressure for the adiabatic mode defined in equation (153), the conservation equation $(\rho_{ad}(\phi_k)C^{3/2})' + p_{ad}(\phi_k)(C^{3/2})' = 0$ will be satisfied, and since the substraction is performed mode by mode, one can subtracts a given number of modes mantaining the conservation equation.

Summarizing, the re-normalized quantity is given by

$$\langle\hat{\phi}^2(\eta,\mathbf{x})\rangle_{ren} = \lim_{m\to 0}\frac{1}{2\pi^2C(\eta)}\left(\int_0^{\infty}|\mathbf{k}|^2|\chi_{\mathbf{k}}(\eta)|^2d|\mathbf{k}| - \frac{1}{2}\int_{\sqrt{2}/|\eta|}^{\infty}\frac{|\mathbf{k}|^2}{W_{\mathbf{k}}}d|\mathbf{k}|\right). \tag{169}$$

Now since, the ultraviolet divergencies are cancelled, and

$$\lim_{m\to 0}\int_{\sqrt{2}/|\eta|}^{\infty}\frac{m^2|\mathbf{k}|^2}{\omega_{\mathbf{k}}^5}d|\mathbf{k}| = \lim_{m\to 0}\int_{\sqrt{2}/|\eta|}^{\infty}\frac{m^4|\mathbf{k}|^2}{\omega_{\mathbf{k}}^7}d|\mathbf{k}| = 0, \tag{170}$$

using (163) we finally get

$$\begin{aligned}\langle\hat{\phi}^2(\eta,\mathbf{x})\rangle_{ren} &= \frac{\eta^2H^2}{2\pi^2}\int_0^{\sqrt{2}/|\eta|}|\mathbf{k}|^2|\chi_{\mathbf{k}}(\eta)|^2d|\mathbf{k}| \\ &+\frac{\eta^2H^2}{4\pi^2}\int_{\sqrt{2}/|\eta|}^{\infty}\left(-\frac{H^2}{|\mathbf{k}|^2}\cos(2|\mathbf{k}|(H^{-1}+\eta) + \mathcal{O}\left(H^3/|\mathbf{k}|^3\right)\right)|\mathbf{k}|d|\mathbf{k}|.\end{aligned} \tag{171}$$

If we are interested in the late time behavior ($|\eta|H \ll 1$, i.e., $Ht \gg 1$), we can make the following approximation

$$\langle\hat{\phi}^2(\eta,\mathbf{x})\rangle_{ren} \cong \frac{\eta^2H^2}{2\pi^2}\int_0^{\sqrt{2}/|\eta|}|\mathbf{k}|^2|\chi_{\mathbf{k}}(\eta)|^2d|\mathbf{k}| \cong \frac{1}{9\pi^2}\int_0^H|\mathbf{k}|d|\mathbf{k}| + \frac{H^2}{4\pi^2}\int_H^{\sqrt{2}/|\eta|}\frac{1}{|\mathbf{k}|}d|\mathbf{k}|, \tag{172}$$

where, in the first integral we have used the approximation $|\chi_{\mathbf{k}}(\eta)|^2 \cong \frac{2}{9\eta^2 H^2 |\mathbf{k}|}$ (eq. (162)), and in the second one $|\chi_{\mathbf{k}}(\eta)|^2 \cong \frac{1}{2\eta^2 |\mathbf{k}|^3}$ (eq. (163)).

Finally, after integration, at late times we get

$$\langle \hat{\phi}^2(\eta, \mathbf{x}) \rangle_{ren} \cong \frac{H^2}{18\pi^2} + \frac{H^2}{4\pi^2}(\frac{1}{2}\ln 2 + Ht) \cong \frac{H^3}{4\pi^2} t, \tag{173}$$

that coincides with the early result obtained in [26, 35, 36].

Note also that formula (168) justify the prescription given in [27, 99], where the authors assumes that only modes outside to the Hubble horizon at time t contribute to the value of the re-normalized two-point function.

In the opposite case, that is, for a few Hubble times, (for example $t = 1/H$), it is not difficult to show that $\langle \hat{\phi}^2(\eta, \mathbf{x}) \rangle_{ren} \sim \mathcal{O}(H^2)$.

Remark III.1. *In inflationary cosmology the re-normalization of the two point function is sometimes obtained in a different way (see for example [101]). The modes $\chi_{\mathbf{k}}(\eta)$ are given by formula (43), and thus*

$$\langle \hat{\phi}^2(\eta, \mathbf{x}) \rangle = \frac{1}{4\pi^2 C(\eta)} \int_0^\infty |\mathbf{k}| \left(1 + \frac{1}{|\mathbf{k}|^2 \eta^2}\right) d|\mathbf{k}|.$$

To avoid the infra-red divergency one assumes that the initial size of the universe is of the order H^{-1}, then the Fourier expansion shows the modes has to be wave-length smaller than the Hubble horizon, that is, impose the cut-off $|\mathbf{k}| \geq H$. And to avoid the ultra-violet divergency one has to subtract adiabatic modes that satisfy equation (168), i.e., adiabatic modes well inside in the Hubble horizon, and consequently one only has to take into account the modes that leave the horizon, more precisely modes that satisfy $H \leq |\mathbf{k}| \leq \sqrt{2}/|\eta|$, then from (167) and (170) one has

$$\langle \hat{\phi}^2(\eta, \mathbf{x}) \rangle = \frac{1}{4\pi^2 C(\eta)} \int_H^{\sqrt{2}/|\eta|} |\mathbf{k}| \left(1 + \frac{1}{|\mathbf{k}|^2 \eta^2}\right) d|\mathbf{k}|.$$

Finally note that the first term is the usual contribution from vacuum fluctuations in Minkowski space and must be eliminated by re-normalization, which give the following re-normalized two point function

$$\langle \hat{\phi}^2(\eta, \mathbf{x}) \rangle_{ren} = \frac{1}{4\pi^2 C(\eta)} \int_H^{\sqrt{2}/|\eta|} \frac{1}{|\mathbf{k}|\eta^2} d|\mathbf{k}| = \frac{H^2}{4\pi^2}(\frac{1}{2}\ln 2 + Ht) \cong \frac{H^3 t}{4\pi^2},$$

when $Ht \geq 1$.

3. Massive case

First, we study the minimally coupled case with $m \ll H$ in the de Sitter phase. (This situation appears when the inflation fields is in the slow-roll phase, and scalar field fluctuations described by the two-point function are very important in order to understand the self-reproducing universes in inflationary cosmology (see for example [26, 27])). In last Section, we have seen that the re-normalized two-point function is given by the formula (169) without the limit $m \to 0$, because the adiabatic modes that we have to subtract satisfy $|\mathbf{k}| > \frac{\sqrt{2}}{|\eta|}\sqrt{1 - \frac{m^2}{2H^2}} \cong \frac{\sqrt{2}}{|\eta|}$.

The calculation of the two-point in the massive case is more difficult than in the massless one, however, at late times, it is possible to approximate its behavior very well. To do this, note that the adiabatic regularization method guarantees that

$$\frac{1}{2\pi^2 C(\eta)} \int_{\sqrt{2}/|\eta|}^\infty |\mathbf{k}|^2 \left(|\chi_{\mathbf{k}}(\eta)|^2 - \frac{1}{2W_{\mathbf{k}}}\right) d|\mathbf{k}|, \tag{174}$$

is convergent [43]. To perform this integral one can choose mode solutions that correspond to the Bunch-Davies vacuum state, i.e., $\chi_{\mathbf{k}}(\eta) = \sqrt{\frac{\pi\eta}{4}} e^{-i(\frac{\pi\nu}{2} + \frac{\pi}{4})} H_\nu^{(2)}(|\mathbf{k}|\eta)$ with $\nu \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \cong \frac{3}{2} - \frac{m^2}{3H^2}$. Then, at late times, using the asymptotic expansion for large arguments of the Hankel functions (formulae 9.2.8-9.2.10 of [47]) one can shows that the divergent

terms of (174) exactly cancels, and since a easy calculation proves that the convergent ones are of the order $\mathcal{O}(m^2)$, one can disregard its contribution to the two-point function. Thus, at late times, we have

$$\langle\hat{\phi}^2(\eta,\mathbf{x})\rangle_{ren} \cong \frac{\eta^2H^2}{2\pi^2}\int_0^H |\mathbf{k}|^2|\chi_{\mathbf{k}}(\eta)|^2d|\mathbf{k}| + \frac{\eta^2H^2}{2\pi^2}\int_H^{\sqrt{2}/|\eta|} |\mathbf{k}|^2|\chi_{\mathbf{k}}(\eta)|^2d|\mathbf{k}|. \tag{175}$$

In order to avoid infra-red divergency, we can calculate the first integral assuming a phase transition to the radiation dominated universe to a de Sitter phase at time $\eta_0 = -1/H$. In fact, we consider the general mode solutions in the de Sitter phase

$$\chi_{\mathbf{k}}(\eta) = \sqrt{\pi/4}\eta^{1/2}\left(a_{\mathbf{k}}H_\nu^{(2)}(|\mathbf{k}|\eta) + b_{\mathbf{k}}H_\nu^{(1)}(|\mathbf{k}|\eta)\right), \tag{176}$$

and match the modes and their temporal derivatives at $\eta_0 = -1/H$, to obtain

$$\begin{aligned} a_{\mathbf{k}} &= \frac{1}{2i}\sqrt{\frac{\pi|\mathbf{k}|\eta_0}{2}}\left(\left(-i+\frac{H}{2|\mathbf{k}|}\right)H_\nu^{(1)}(|\mathbf{k}|\eta_0) - H_\nu^{(1)\prime}(|\mathbf{k}|\eta_0)\right)e^{i|\mathbf{k}|/H} \\ b_{\mathbf{k}} &= -\frac{1}{2i}\sqrt{\frac{\pi|\mathbf{k}|\eta_0}{2}}\left(\left(-i+\frac{H}{2|\mathbf{k}|}\right)H_\nu^{(2)}(|\mathbf{k}|\eta_0) - H_\nu^{(2)\prime}(|\mathbf{k}|\eta_0)\right)e^{i|\mathbf{k}|/H}. \end{aligned} \tag{177}$$

Then from the small-argument limit

$$H_\nu^{(2)}(|\mathbf{k}|\eta_0) \cong -H_\nu^{(1)}(|\mathbf{k}|\eta_0) \cong \frac{i}{\pi}\Gamma(\nu)\left(\frac{|\mathbf{k}|\eta_0}{2}\right)^{-\nu}, \tag{178}$$

and using that (see [47])

$$H_\nu^{(1,2)\prime}(z) = H_{\nu-1}^{(1,2)}(z) - \frac{\nu}{z}H_\nu^{(1,2)}(z), \tag{179}$$

one arrives at the result $|\chi_{\mathbf{k}}(\eta)|^2 \cong \frac{2}{9|\mathbf{k}|}(H|\eta|)^{1-2\nu}$, and finally one obtains

$$\frac{\eta^2H^2}{2\pi^2}\int_0^H |\mathbf{k}|^2|\chi_{\mathbf{k}}(\eta)|^2d|\mathbf{k}| \cong \frac{H^2}{18\pi^2}e^{-\frac{2m^2t}{3H}} \cong \begin{cases} \frac{H^2}{18\pi^2} & \text{when } \ 1/H \ll t \ll H/m^2 \\ 0 & \text{when } \quad t \gg H/m^2. \end{cases} \tag{180}$$

with agrees with the first term of the right hand side of (173).

The second one, can be done using the following approximation, valid of $|\mathbf{k}| > H$,

$$\chi_{\mathbf{k}}(\eta) \cong \sqrt{\frac{\pi\eta}{4}}e^{-i(\frac{\pi\nu}{2}+\frac{\pi}{4})}H_\nu^{(2)}(|\mathbf{k}|\eta). \tag{181}$$

Effectively, from (177) one can easily obtains, in the range $|\mathbf{k}| > H$, $a_{\mathbf{k}} \cong 1$ and $b_{\mathbf{k}} \cong 0$. Then, since at late time we have $H_\nu^{(2)}(|\mathbf{k}|\eta) \cong -\frac{i}{\pi}\Gamma(\nu)\left(\frac{|\mathbf{k}|\eta}{2}\right)^{-\nu}$, inserting this expression in the second integral of (175), one obtains, for a massive minimally coupled field with $m \ll H$,

$$\frac{\eta^2H^2}{2\pi^2}\int_H^{\sqrt{2}/|\eta|} |\mathbf{k}|^2|\chi_{\mathbf{k}}(\eta)|^2dk \cong \frac{3H^4}{8m^2\pi^2}\left[2^{\frac{m^2}{3H^2}} - (H|\eta|)^{\frac{2m^2}{3H^2}}\right] \cong \frac{3H^4}{8m^2\pi^2}\left[1 - e^{-\frac{2m^2t}{3H}}\right], \tag{182}$$

because we have assumed $m \ll H$.

Then, since the first integral in the right hand side of (175) is smaller than the second one, depending on the value of m^2t/H one has

$$\langle\hat{\phi}^2(\eta,\mathbf{x})\rangle_{ren} \cong \frac{3H^4}{8m^2\pi^2}\left[1 - e^{-\frac{2m^2t}{3H}}\right] \cong \begin{cases} \frac{H^3}{4\pi^2}t & \text{when } \ 1/H \ll t \ll H/m^2 \\ \frac{3H^4}{8m^2\pi^2} & \text{when } \quad t \gg H/m^2, \end{cases} \tag{183}$$

which demonstrates, at late times, the formula (7) of [26].

Remark III.2. *In inflationary cosmology one can chooses $\chi_{\mathbf{k}}(\eta)$ given in formula (181), then if inflation starts at $t = 0$ and the initial size of the universe is of the order $1/H$, after subtracting the adiabatic modes well inside in the Hubble horizon at time t, one only has to take into account the modes that leave the Hubble horizon. Finally one can uses the small-argument limit (179), which is equivalent to eliminate the Minkowskian vacuum fluctuations, to get*

$$\langle\hat{\phi}^2(\eta,\mathbf{x})\rangle_{ren} \cong \frac{3H^4}{8m^2\pi^2}\left[2^{\frac{m^2}{3H^2}} - (H|\eta|)^{\frac{2m^2}{3H^2}}\right] \cong \frac{3H^4}{8m^2\pi^2}\left[1 - e^{-\frac{2m^2 t}{3H}}\right],$$

because we are assuming $m \ll H$.

Finally we calculate the re-normalized two-point function, for a massive conformally coupled field with $m \ll H$ in the de Sitter phase, given by

$$\langle\hat{\phi}^2(\eta,\mathbf{x})\rangle_{ren} = \frac{\eta^2 H^2}{2\pi^2}\left(\int_0^\infty |\mathbf{k}|^2|\chi_{\mathbf{k}}(\eta)|^2 d|\mathbf{k}| - \frac{1}{2}\int_0^\infty \frac{|\mathbf{k}|^2}{W_{\mathbf{k}}} d|\mathbf{k}|\right), \tag{184}$$

where $W_{\mathbf{k}}$ is given by formula (155). (Note that, in the conformally coupled case we do not have to disregard any adiabatic mode.)

At late times, with the same kind of argument used to disregard the terms that appear in equation (174), we can do the following approximation

$$\langle\hat{\phi}^2(\eta,\mathbf{x})\rangle_{ren} \cong \frac{\eta^2 H^2}{2\pi^2}\left(\int_0^{A|\eta|^{-1}} |\mathbf{k}|^2|\chi_{\mathbf{k}}(\eta)|^2 d|\mathbf{k}| - \frac{1}{2}\int_0^{A|\eta|^{-1}} \frac{|\mathbf{k}|^2}{W_{\mathbf{k}}} d|\mathbf{k}|\right), \tag{185}$$

where A is some dimensionless constant of order 1. Actually, one can chooses $A = 1$, and the reasoning does not change, because $|\eta|^{-1}$ is large enough at late times.

Since in this case there is not infra-red divergency, we can choose the modes corresponding to the Bunch-Davies vacuum state, that is, $\chi_{\mathbf{k}}(\eta) = \sqrt{\frac{\pi\eta}{4}} e^{-i(\frac{\pi\nu}{2}+\frac{\pi}{4})} H_\nu^{(2)}(|\mathbf{k}|\eta)$. Then for $k < A/|\eta|$, a good approximation is given by $\chi_{\mathbf{k}}(\eta) \cong \frac{i}{\sqrt{2|\mathbf{k}|}}$, and since

$$\frac{H^2\eta^2}{4\pi^2}\int_0^{A|\eta|^{-1}} \left(|\mathbf{k}| - \frac{|\mathbf{k}|^2}{\omega_{\mathbf{k}}}\right) d|\mathbf{k}| \sim \mathcal{O}\left(m^2 \ln\left(H/m\right)\right), \tag{186}$$

one can disregard this term (it is small compared with the other ones that are of order $\mathcal{O}(H^2)$) and, at late times, we get the same result obtained in formula (158) with $R = 12H^2$, that is,

$$\langle\hat{\phi}^2(\eta,\mathbf{x})\rangle_{ren} = \frac{H^2}{24\pi^2}, \tag{187}$$

because the leading terms in (185) are the same as in (157).

B. Re-normalized stress-tensor

In this section we review some classic results about the re-normalization of the stress-tensor in FRW cosmologies.

The vacuum energy density ρ is given by

$$\rho \equiv \langle\hat{T}_{tt}\rangle = (4\pi^2C^2)^{-1}\int_0^\infty d|\mathbf{k}||\mathbf{k}|^2\left\{(|\chi_{\mathbf{k}}'|^2 + \omega_{\mathbf{k}}^2|\chi_{\mathbf{k}}|^2) + 3(\xi - 1/6)\left[D(\chi_{\mathbf{k}}\chi_{\mathbf{k}}^{*'} + \chi_{\mathbf{k}}^*\chi_{\mathbf{k}}') - \frac{1}{2}D^2|\chi_{\mathbf{k}}|^2\right]\right\}, \tag{188}$$

where $D = C'/C$.

To obtain the re-normalized value, one can uses the adiabatic regularization which consist in subtracting adiabatic modes up to order four. Following [56] one has to subtract the following divergent terms:

$$(4\pi^2C^2)^{-1}\int_0^\infty d|\mathbf{k}||\mathbf{k}|^2\omega_{\mathbf{k}}, \quad \frac{3(\xi-1/6)D^2}{16\pi^2C^2}\int_0^\infty d|\mathbf{k}|\frac{|\mathbf{k}|^2}{\omega_{\mathbf{k}}} - \frac{3(\xi-1/6)D^2m^2}{16\pi^2C}\int_0^\infty d|\mathbf{k}|\frac{|\mathbf{k}|^2}{\omega_{\mathbf{k}}^3} \tag{189}$$

and

$$-\frac{(\xi-1/6)^2}{128\pi^2C^2}(72D''D-36D'^2-27D^4)\int_0^\infty d|\mathbf{k}|\frac{|\mathbf{k}|^2}{\omega_{\mathbf{k}}^3}, \tag{190}$$

and the finite terms:

$$\begin{aligned}\frac{m^2D^2}{384\pi^2C}- &\quad\frac{1}{2880\pi^2C^2}\left(\frac{3}{2}D''D-\frac{3}{4}D'^2-\frac{3}{8}D^4\right)\\ &+\frac{(\xi-1/6)}{256\pi^2C^2}(8D''D-4D'^2-3D^4)+\frac{(\xi-1/6)^2}{64\pi^2C^2}(18D'D^2+9D^4).\end{aligned} \tag{191}$$

The simplest example is when one considers a massless minimally coupled scalar field, in that case the modes are $\chi_{\mathbf{k}}(\eta)=\frac{e^{-i|\mathbf{k}|\eta}}{\sqrt{2|\mathbf{k}|}}$ then one easily obtains [34, 102, 103]

$$\rho_{vac}\equiv\langle\hat{T}_{tt}\rangle_{ren}=\frac{1}{2880\pi^2C^2}\left(\frac{3}{2}D''D-\frac{3}{4}D'^2-\frac{3}{8}D^4\right). \tag{192}$$

In the flat chart of the de-Sitter space-time choosing the modes

$$\chi_{\mathbf{k}}(\eta)=C\sqrt{\frac{\pi\eta}{4}}H_\nu^{(2)}(|\mathbf{k}|\eta), \tag{193}$$

with $\nu\equiv\sqrt{\frac{9}{4}-\frac{m^2}{H^2}-12\xi}$ and $C\equiv e^{-i(\frac{\pi\nu}{2}+\frac{\pi}{4})}$, Bunch and Davies obtained in [104] using point-splitting regularization (which for scalar massive field is equivalent to adiabatic regularization [56, 105, 106])

$$\begin{aligned}\rho_{vac}\equiv\langle\hat{T}_{tt}\rangle_{ren}= &\quad\frac{1}{64\pi^2}\left\{m^2\left[m^2+(12\xi-2)H^2\right]\left[\psi\left(\frac{3}{2}+\nu\right)+\psi\left(\frac{3}{2}-\nu\right)-\ln\left(\frac{m^2}{H^2}\right)\right]\right.\\ &\left.-m^2(12\xi-2)H^2-\frac{2}{3}m^2H^2-\frac{1}{2}(12\xi-2)^2H^4+\frac{H^4}{15}\right\},\end{aligned} \tag{194}$$

where ψ denotes the digamma function. (In exact agreement with the previous resolt obtained in [107] using Schwinger-DeWitt regularization procedure [76])

This result holds for all values of m and ξ except in the massless minimally coupled case, because when m and ξ are close to zero one has [108]

$$\frac{1}{64\pi^2}m^2\left[m^2+(12\xi-2)H^2\right]\psi\left(\frac{3}{2}-\nu\right)\cong\frac{1}{128\pi^2}\frac{H^2}{1+\frac{12\xi H^2}{m^2}}, \tag{195}$$

and thus the limit in (194) gives different answers depending the way that m and ξ approach to the origin. In order to calculate ρ_{vac} in the massless minimally coupled case using adiabatic regularization on has to consider the $\xi=0$, use the modes given in (43) and finally take $m\to0$ [34, 56], gettin

$$\rho_{vac}=-\frac{29H^4}{960\pi^2}, \tag{196}$$

which can be also obtained taking $\xi=0$ in (194) and afterwards $m\to0$.

Remark III.3. *It's well-known that the trace of the stress tensor corresponding to a massless conformally coupled field is zero, however after regularization its value is not necessarily zero. It was showed in [56] that the trace anomaly provided by adiabatic regularization, in the massless conformally coupled case is given by*

$$T_{vac}\equiv\langle T_\alpha^\alpha\rangle_{ren}=\frac{1}{960\pi^2C^2}(D'''-D'D^2). \tag{197}$$

Then from the relation $T_{vac}=\rho_{vac}-3p_{vac}$ one gets the value of the vacuum pressure $p_{vac}\equiv\langle T_{xx}\rangle_{ren}$. In fact, if one knows one of these values the others come from the trace anomaly relation $T_{vac}=\rho_{vac}-3p_{vac}$ and the conservation equation $\left(\rho_{vac}C^{3/2}\right)'+p_{vac}\left(C^{3/2}\right)'=0$.

IV. AVOIDANCE OF COSMOLOGICAL SINGULARITIES

The vacuum quantum effects due to a massless conformally couped field are taken into account in order to avoid classical cosmological singularities.

A. Review of classical and quantum cosmology

Classical cosmology: We will use the following notation: $\kappa^2 = 16\pi G = 16\pi/m_p^2$, being G Newton's constant, ρ energy density, p pressure, ω a dimensionless parameter and $H = \dot{a}/a$ the Hubble parameter, being a the scale factor and the dot denotes denotes the derivative with respect to the cosmic time t. Then the Friedmann equation and conservation equation, for a flat Friedmann-Robertson-Walker cosmology, can be written respectively as:

$$H^2 = \frac{\kappa^2}{6}\rho, \quad \dot{\rho} = -3H(\rho + p), \tag{198}$$

and the equation of state for a barotropic perfect fluid, that we will consider in the paper, has the form $p = \omega\rho$.

With the derivative of the Friedmann equation, and the other two equation one easily obtains the "acceleration" equation

$$\dot{H} = -\frac{\kappa^2}{4}(1+\omega)\rho. \tag{199}$$

Combining (198) and (199), one can deletes ρ, and obtains the equation $\dot{H} = -\frac{3}{2}(1+\omega)H^2$, then integrating one gets

$$H(t) = \frac{2}{3(1+\omega)}\frac{1}{t - t_s}, \tag{200}$$

where $t_s \equiv t_0 - \frac{2}{3H_0(1+\omega)}$, being $H_0 = H(t_0)$ the initial condition.

From the definition of the Hubble parameter, the following behavior for the scale factor is obtained

$$a(t) = a_0 \left(\frac{t - t_s}{t_0 - t_s}\right)^{\frac{2}{3(1+\omega)}}, \tag{201}$$

and from the Friedmann equation, one has

$$\rho(t) = \frac{8}{3\kappa^2(1+\omega)^2}\frac{1}{(t-t_s)^2}. \tag{202}$$

The following remark is in order: if one assumes $H_0 > 0$, then for $\omega > -1$ one has $t_s < t_0$, that is, the singularity is at early times (Big Bang singularity), on the other hands, for $\omega < -1$ one has $t_s > t_0$, that is, the singularity is at late times (Big Rip singularity) [110].

Quantum Effects: Using the same notation as [39], for a massless field conformally coupled with gravity, one has the following expression for the trace anomaly

$$T_{vac} = 6\alpha(\dddot{H} + 12H^2\dot{H} + 7H\ddot{H} + 4\dot{H}^2) - 12\beta(H^4 + H^2\dot{H}) \tag{203}$$

with (see for example [109])

$$\alpha = \frac{1}{2880\pi^2}(N_0 + 6N_{1/2} + 12N_1), \quad \beta = \frac{-1}{2880\pi^2}(N_0 + 11N_{1/2} + 62N_1) \tag{204}$$

where N_0 is the number of scalar fields, $N_{1/2}$ is the number of four components neutrinos and N_1 is the number of electromagnetic fields.

Remark IV.1. *The constants α and β come from the regularization process. For example dimensional regularization gives formula (204), and point-splitting gives (see [39]) $\alpha = \frac{1}{2880\pi^2}(N_0+3N_{1/2}-18N_1)$, $\beta = \frac{-1}{2880\pi^2}(N_0+11/2N_{1/2}+62N_1)$. Then, since the method of regularization influences these values, and it is uncertain what fields are present in our universe, one can considers all values of both parameters.*

Remark IV.2. *The relation between the notation of this section and the one of [21] that we used in section (II.A.3) is*

$$M^2 = -\frac{2}{\kappa^2\alpha}, \qquad H_0^2 = -\frac{2}{\kappa^2\beta}.$$

As explained in Remark III.1, to obtain the vacuum energy density one can uses the trace anomaly $T_{vac} = \rho_{vac} - 3p_{vac}$, and the conservation equation. The result is

$$\rho_{vac} = 6\alpha(3H^2\dot{H} + H\ddot{H} - \frac{1}{2}\dot{H}^2) - 3\beta H^4, \tag{205}$$

which coincides with eq. (192) if one only considers scalar fields. Then taking into account this vacuum energy density, the modified Friedmann equation behaves

$$H^2 = \frac{\kappa^2}{6}(\rho + \rho_{vac}). \tag{206}$$

With the derivative of this last equation, the conservation equation and the trace anomaly, one obtains the modified acceleration equation

$$\dot{H} = -\frac{\kappa^2}{4}\left((1+\omega)\rho + \rho_{vac} + \frac{1}{3}(\rho_{vac} - T_{vac})\right). \tag{207}$$

From both equations, one can deletes ρ, and one obtains the following third order differential equation

$$-\frac{4}{\kappa^2}\dot{H} - \rho_{vac} - \frac{1}{3}(\rho_{vac} - T_{vac}) = (1+\omega)\frac{6}{\kappa^2}H^2 - (1+\omega)\rho_{vac}, \tag{208}$$

that in terms of the Hubble parameter is given by

$$\begin{aligned} -\frac{4}{\kappa^2}\dot{H} - (1+\omega)\frac{6}{\kappa^2}H^2 \; - \; & 3\beta(\omega+1)H^4 + (18\alpha(\omega+1) - 4\beta)H^2\dot{H} \\ & +6\alpha(\omega+2)H\ddot{H} + 3\alpha(\omega+3)\dot{H}^2 + 2\alpha\dddot{H} = 0. \end{aligned} \tag{209}$$

B. $\alpha = 0$

In this section we consider the simplest case ($\alpha = 0$), and we will see that, quantum effects don't avoid the singularities. Equation (12) reduces to the following first order differential equation

$$\dot{H} = -\frac{1+\omega}{4}\left(\frac{\frac{6}{\kappa^2}H^2 + 3\beta H^4}{\frac{1}{\kappa^2} + \beta H^2}\right). \tag{210}$$

1. First, we consider the case $\alpha = 0$ and $\beta > 0$. Integrating equation (13) one obtains

$$-1/H(t) + \sqrt{\frac{\beta\kappa^2}{2}}\arctan\left(\sqrt{\frac{\beta\kappa^2}{2}}H(t)\right) = -\frac{3}{2}(1+\omega)(t-t_s) + \sqrt{\frac{\beta\kappa^2}{2}}\arctan\left(\sqrt{\frac{\beta\kappa^2}{2}}H_0\right) \tag{211}$$

- For $\omega < -1$, when $t \to -\infty$ one has $H \to 0$ (that is quantum effects are small at early times), on the other hands, when $t \to \bar{t}_s \equiv t_s - \frac{2}{3(1+\omega)}\sqrt{\frac{\beta\kappa^2}{2}}\left(\pi - \arctan\left(\sqrt{\frac{\beta\kappa^2}{2}}H_0\right)\right)$ one has $H \to \infty$, that is, the quantum effects don't avoid the Big Rip singularity that appears at $\bar{t}_s$.
- For $\omega > -1$, when $t \to \infty$ one has $H \to 0$ (that is quantum effects are small at late times), on the other hands, when $t \to \bar{t}_s \equiv t_s - \frac{2}{3(1+\omega)}\sqrt{\frac{\beta\kappa^2}{2}}\left(\pi - \arctan\left(\sqrt{\frac{\beta\kappa^2}{2}}H_0\right)\right)$ one has $H \to \infty$, that is, the quantum effects don't avoid the Big Bang singularity that appears at $\bar{t}_s$.

2. Now we consider the case $\alpha = 0$ and $\beta < 0$. The solution of equation (13) is given by

$$-1/H(t) + \frac{1}{2H_+}\ln\left|\frac{H(t)-H_+}{H_0-H_+}\frac{H_0+H_+}{H(t)+H_+}\right| = -\frac{3}{2}(1+\omega)(t-t_s), \tag{212}$$

with $H_+ \equiv \sqrt{\frac{-2}{\beta\kappa^2}}$.

- In the case $\omega < -1$, for $H_0 \in (0, H_+/\sqrt{2})$ when $t \to -\infty$ on has $H \to 0$, and when $H \to H_+/\sqrt{2}$ at $\bar{t}_s \equiv t_s + \frac{2}{3(1+\omega)}\left(\sqrt{2}/H_+ - \frac{1}{2H_+}\ln\left|\frac{H_+/\sqrt{2}-H_+}{H_0-H_+}\frac{H_0+H_+}{H_+/\sqrt{2}+H_+}\right|\right)$ one has $\dot{H}(\bar{t}_s) = +\infty$, that is, these solutions are singular (the scalar curvature, $R \equiv 6(2H^2+\dot{H})$, diverges). For $H_0 \in (H_+/\sqrt{2}, H_+)$ when $t \to -\infty$ on has $H \to H_+$, however at $\bar{t}_s$ $\dot{H}$ diverges. And finally for $H_0 \in (H_+, \infty)$ there is a singularity at finite time. Effectively , when $t \to -\infty$ on has $H \to H_+$, and when $t \to t_s$ on has $H \to \infty$.
- In the opposite case $\omega > -1$, for $H_0 \in (0, H_+/\sqrt{2})$ when $t \to \infty$ on has $H \to 0$, and when $H \to H_+/\sqrt{2}$ at $\bar{t}_s \equiv t_s + \frac{2}{3(1+\omega)}\left(\sqrt{2}/H_+ - \frac{1}{2H_+}\ln\left|\frac{H_+/\sqrt{2}-H_+}{H_0-H_+}\frac{H_0+H_+}{H_+/\sqrt{2}+H_+}\right|\right)$ one has $\dot{H}(\bar{t}_s) = +\infty$, that is, these solutions are singular. For $H_0 \in (H_+/\sqrt{2}, H_+)$ when $t \to \infty$ on has $H \to H_+$, however at $\bar{t}_s$ $\dot{H}$ diverges. And finally for $H_0 \in (H_+, \infty)$ there is a singularity at finite time. Effectively, when $t \to \infty$ on has $H \to H_+$, and when $t \to t_s$ on has $H \to \infty$.

C. Empty universe

Another simple case is the restriction to the invariant manifold $\rho(t) \equiv 0$. In that case, one only needs the modified Friedmann equation, that is, the following second order differential equation

$$H^2 = \kappa^2\alpha(3H^2\dot{H} + H\ddot{H} - \frac{1}{2}\dot{H}^2) - \frac{\kappa^2\beta}{2}H^4 \tag{213}$$

Remark IV.3. *The solutions with $H > 0$ and those with $H < 0$ decouple. To see this, we perform the change of variable $Z \equiv \dot{H}/H$ to make the system no-singular at $H = 0$, then at $H = 0$ the system behaves $\dot{Z} = Z^2/2$, this means that the solutions can't cross the axis $H = 0$.*

Equation (213) is an autonomous second order differential equation, then since solutions are invariant under time translations, the general solution is a one-parameter family of solutions. Taking this into account, we will prove that there is a one-parameter family of singular solutions. First we look for a particular singular solution, with the following behavior $H(t) = \frac{C}{t-t_s}$ near the singularity. Inserting this expression in (213), and retaining only the leading singular terms one obtains $C_\pm = \frac{3\alpha}{\beta}\left(-1 \pm \sqrt{1+\frac{\beta}{3\alpha}}\right)$. Here it is clear that we have to impose the condition $\frac{\beta}{3\alpha} \geq -1$. In terms of the scale factor one has $a(t) = a_0\left(\frac{t-t_s}{t_0-t_s}\right)^{C_\pm}$. Then, for $\frac{\beta}{3\alpha} > 0$, the solution with C_+ has a singularity of the type $a(t_s) = 0$, and the other one of the type $a(t_s) = \infty$. However, when $-1 \leq \frac{\beta}{3\alpha} < 0$, for both values of C, the solution satisfy $a(t_s) = 0$.

Now we can prove that there is a one-parameter family of singular solutions whose leading term is $H(t) = \frac{C_\pm}{t-t_s}$. To do this, we first transform the differential equation (213) in a first order one performing the change $u(H) = \dot{H}(t)$, then the equation becomes

$$H^2 = \kappa^2\alpha(3H^2u + Huu' - \frac{1}{2}\dot{H}^2) - \frac{\kappa^2\beta}{2}H^4, \tag{214}$$

where $u'(H) \equiv du/dH$. Using the new variables $H(t) = \frac{C_\pm}{t-t_s}$ has the form $u = -\frac{1}{C_\pm}H^2$, then if the equation is linearized about this point one obtains, for $H \to \pm\infty$, the following general solution

$$u_{linearized} = -\frac{1}{C_\pm}H^2 + K|H|^{1-3C_\pm} - \frac{C_\pm}{\kappa^2\alpha(3C_\pm - 1)}, \qquad \text{when} \qquad C_\pm \neq 1/3 \tag{215}$$

and

$$u_{linearized} = -\frac{1}{C_\pm}H^2 + K - \frac{1}{3\kappa^2}\ln|H|, \qquad \text{when} \qquad C_\pm = 1/3, \tag{216}$$

where K is an arbitrary constant.

It's clear that we have to impose $C_\pm > -1/3$. Then when $-1 < \frac{\beta}{3\alpha} < 0$ or $\frac{\beta}{3\alpha} > 15$, one has $C_\pm > -1/3$, and then for both values one has a one-parameter family of solutions. When $0 < \frac{\beta}{3\alpha} < 15$, for C_+, one has a one-parameter family, however for C_- one has to choose $K = 0$, that is, one only has a particular solution.

Now we can perform the qualitative analysis. Firstly, note that when $\beta < 0$ there exist two de Sitter solutions $H_\pm = \pm\sqrt{-\frac{2}{\kappa^2\beta}}$. Making the change of variable $p \equiv \sqrt{|H|}$ (see [41, 42]), one obtains

$$\frac{d}{dt}\left(\dot{p}^2/2 + V(p)\right) = -3\epsilon p^2\dot{p}^2 \tag{217}$$

where $\epsilon = sing(H)$ and $V(p) = -\frac{p^2}{4\kappa^2\alpha}\left(1 + \frac{\kappa^2\beta}{6}p^4\right)$. In in the phase-space one has

$$\begin{cases} \dot{p} &= y \\ \dot{y} &= -3\epsilon p^2 y - V'(p). \end{cases} \tag{218}$$

We only consider the domain $H > 0$, because solutions with $H > 0$ and $H < 0$ decouple. The point $p_+ \equiv \sqrt{H_+}$ is an extremum of the potential V, then, linearizing the system (21), one obtains that for $\alpha < 0$ the critical point $(p_+, 0)$ is a saddle point, and for $\alpha > 0$ is a node stable. From the form of the potential, and taking into account that the system is dissipative in $H > 0$, the following results in the phase-space (p, y), with $H > 0$, are obtained:

1. Case $\alpha > 0$, $\beta > 0$

   We have $V < 0$ in $(0, \infty)$ and $V(0) = 0$. The $(0, 0)$ is an unstable critical point. The solutions are singular at early and late times ($p \to \infty$). Only a solution is not singular at late times, it is the trajectory that arrives at $p = 0$ with zero energy (it arrives at the point $(0, 0)$), and only one if not singular at early times, it starts from $p = 0$ with zero energy (it starts at $(0, 0)$).

2. Case $\alpha < 0$, $\beta > 0$

   Now, $V > 0$ in $(0, \infty)$ and $V(0) = 0$. The $(0, 0)$ is an stable critical point, and solutions are only singular at early times. At late times they approach to the stable critical point.

3. Case $\alpha > 0$, $\beta < 0$

   In this case, the system has two critical points. $(0, 0)$ is an unstable critical point, and $(p_+, 0)$ is stable. Solutions are only singular at early times. At late times they oscillate and shrink around to the stable point, that is, $(p_+, 0)$ is a global attractor. Moreover, there is a solution that ends at $(0, 0)$, and only a no singular solution that starts at $(0, 0)$ (starts with zero energy) and ends at $(p_+, 0)$.

4. Case $\alpha < 0$, $\beta < 0$

   This is the Starobinski model [20]. The system has two critical points. $(0, 0)$ is an stable critical point, and $(p_+, 0)$ is a saddle point. There are solutions that don't cross the axe $p = p_+$, these solutions are singular at early and late times, they correspond to the trajectories that can't pass the top of the potential. There are other solutions that cross twice the axe $p = p_+$, they are also singular at early and late times, these trajectories pass the top of the potential bounce at $p = 0$ and pass once again the top of the potential. There are solutions that cross once the axe $p = p_+$, these solutions are singular at early times, however at late times the solutions spiral and shrink to the origin, these solutions pass the top of the potential once and then bounce some times in $p = 0$, shrinking to $p = 0$. These last solutions has the asymptotic behavior described in the beginning of section 2.1.3. Finally, there is only two unstable no-singular solution, one goes from $(p_+, 0)$ to $(0, 0)$, and the other one is the de Sitter solution $(p_+, 0)$.

Remark IV.4. *Note that equation (16) remains the same with the change $H(t) \to -H(-t)$, this means that solutions with $H < 0$ are the time reversal of the studied above.*

Remark IV.5. *For $\frac{\beta}{3\alpha} < -1$ the values of $C_\pm$ are complexes, this is due to the fact that the system cannot go to (or come from) $p \to \infty$ monotonically, because the dissipation effect is not large enough compared with the potential force [41].*

Remark IV.6. *The case $\omega = -1$ ($\rho =$ constant are invariant manifolds), is equivalent to the case of an empty universe with a cosmological constant. This case was studied with great detail in [41].*

D. The general case

The best way to study the general case is to consider the system

$$\begin{cases} \dot{H} &= & Y \\ \dot{Y} &= & \frac{1}{2\alpha H}\left(2H^2/\kappa^2 - \rho/3 - 6\alpha H^2 Y + \alpha Y^2 + \beta H^4\right) \\ \dot{\rho} &= & -3H\rho(1+\omega). \end{cases} \tag{219}$$

When $\beta < 0$, the system has two critical points $(H_\pm, 0, 0)$ whit $H_\pm = \pm\sqrt{-\frac{2}{\beta\kappa^2}}$. The semi-plane $\rho = 0, H > 0$ (resp. $\rho = 0, H < 0$) is an attractor (resp. a "repeller") when $\omega > -1$, and the roles are interchanged when $\omega < -1$.

What is important is to stress that in the case $\alpha < 0$ there isn't bouncing solutions, because at bouncing time, namely t_b, one has $-\rho(t_b)/3 + \alpha Y^2(t_b) = 0$, what means $\rho(t_b) = 0$, but as we have seen in last section, $\rho = 0$ is an invariant manifold where trajectories with $H > 0$ and those with $H < 0$ decouple. For this reason in the case $\alpha < 0$ there isn't stable no-singular trajectories, the only unstable no-singular solutions are the ones that appear in the Starobinski model.

From this last paragraph one can concludes that, to obtain no-singular solutions, the interesting case is $\alpha > 0$. In fact, the interesting one is $\alpha > 0$ and $\beta < 0$.

In that case we can use the dimensionless variables $\bar{t} = H_+ t$, $\bar{H} = H/H_+$ $\bar{Y} = Y/H_+^2$ and $\bar{\rho} = \frac{\kappa^2\rho}{6H_+^2}$, then the system becomes

$$\begin{cases} \bar{H}' &= & \bar{Y} \\ \bar{Y}' &= & \frac{1}{2\alpha H}\left(-\beta\bar{H}^2\beta\bar{\rho} - 6\alpha\bar{H}^2\bar{Y} + \alpha\bar{Y}^2 + \beta\bar{H}^4\right) \\ \bar{\rho}' &= & -3\bar{H}\bar{\rho}(1+\omega), \end{cases} \tag{220}$$

where $'$ denotes the derivative with respect the time $\bar{t}$.

In these variables the critical points are $(\pm 1, 0, 0)$. The linearized system at $(1, 0, 0)$ has eigenvalues $\lambda_\pm = -3/2\left(1 \pm \sqrt{1 + \frac{4\beta}{3\alpha}}\right)$ and $\lambda_3 = -3(1+\omega)$. Since $\mathrm{Re}\lambda_\pm < 0$ and $\bar{\rho} \equiv 0$ is an invariant manifold, all solutions in that semi-plane with $\bar{H} > 0$ go asymptotically towards this critical point. The eigenvector $\vec{v}_3 = (1, -3(1+\omega), 18\omega(1+\omega)\alpha/\beta - 2)$ corresponds to the eigenvalue λ_3, then for $\omega < -1$, there is a solution that escapes to the de Sitter expanding phase following the direction of the vector $\vec{v}_3$. On the other hands, when $\omega > -1$ the critical point is an attractor. At the other critical point the eigenvalues are $\lambda_\pm = -3/2\left(1 \pm \sqrt{-1 + \frac{4\beta}{3\alpha}}\right)$ and $\lambda_3 = 3(1+\omega)$. Since $\mathrm{Re}\lambda_\pm > 0$ and $\bar{\rho} \equiv 0$ is an invariant manifold, all solutions in that semi-plane with $\bar{H} < 0$ escape from this critical point. The eigenvector $\vec{v}_3 = (1, 3(1+\omega), -18\omega(1+\omega)\alpha/\beta + 2)$ corresponds to the eigenvalue λ_3, then for $\omega < -1$, there is a particular solution that goes asymptotically towards the de Sitter contracting phase following the direction of the vector $\vec{v}_3$. On the other hands, when $\omega > -1$ the critical point is a repeller.

Now we look for singular solutions of the form $\bar{H} = C/(\bar{t} - \bar{t}_s)$, $\bar{t} \to \bar{t}_s^{\pm}$. Inserting this value of the Hubble parameter in the conservation equation one obtains $\bar{\rho}(t) = \bar{\rho}_0|\bar{t} - \bar{t}_s|^{-3C(1+\omega)}$, where $\bar{\rho}_0$ has to be a positive parameter. Now , when $\omega < -1$, inserting the Hubble parameter and the energy density in the modified Friedmann equation one gets, once again, the values of C obtained in section 4.3, that is, $C_\pm = \frac{3\alpha}{\beta}\left(-1 \pm \sqrt{1 + \frac{\beta}{3\alpha}}\right)$, because in that case the energy goes to zero when $\bar{t} \to \bar{t}_s^{\pm}$, and then, one obtains the same kind of result that in the case $\bar{\rho} \equiv 0$, the only difference is that now one has a two-parameter family of singular solutions ($\bar{\rho}_0$ is a free parameter, and the general solution of the system (23) is a two-parameter family due to the time invariance under translations).

The case $\omega > -1$ is more involved. We summarize the results:

1. When $-1 < \frac{\beta}{3\alpha} < 0$

- If $C_- < \frac{4}{3(1+\omega)}$, there are two two-parameter families with $\bar{H} = C_\pm/(\bar{t}-\bar{t}_s)$ and $\bar{\rho}_0$ free parameter.
- If $C_- = \frac{4}{3(1+\omega)}$, there is a two-parameter family with $\bar{H} = C_+/(\bar{t}-\bar{t}_s)$ and $\bar{\rho}_0$ free parameter, and a one-parameter family with $\bar{H} = C_-/(\bar{t}-\bar{t}_s)$ and $\bar{\rho}_0 \equiv 0$.
- If $C_+ < \frac{4}{3(1+\omega)} < C_-$, there is a two-parameter family with $\bar{H} = C_+/(\bar{t}-\bar{t}_s)$ and $\bar{\rho}_0$ free parameter, a one-parameter family with $\bar{H} = C_-/(\bar{t}-\bar{t}_s)$ and $\bar{\rho}_0 \equiv 0$, and a one-parameter family with $\bar{H} = \frac{4}{3(1+\omega)(t-\bar{t}_s)}$ and $\bar{\rho}_0 = -\frac{16}{9(1+\omega)^2}\left(\frac{16}{9(1+\omega)^2} + \frac{6\alpha}{\beta}\frac{4}{3(1+\omega)} - \frac{3\alpha}{\beta}\right)$.
- If $\frac{4}{3(1+\omega)} \leq C_+$, there are two one-parameter families with $\bar{H} = C_\pm/(\bar{t}-\bar{t}_s)$ and $\bar{\rho}_0 = 0$.

2. When $-1 = \frac{\beta}{3\alpha}$
   - If $1 < \frac{4}{3(1+\omega)}$, there is a two-parameter family with $\bar{H} = 1/(\bar{t}-\bar{t}_s)$ and $\bar{\rho}_0$ free parameter.
   - If $1 \geq \frac{4}{3(1+\omega)}$, there is a one-parameter family with $\bar{H} = 1/(\bar{t}-\bar{t}_s)$ and $\bar{\rho}_0 = 0$.
3. When $-1 > \frac{\beta}{3\alpha}$, there aren't singular solutions of the form $\bar{H} = C/(\bar{t}-\bar{t}_s)$.

Remark IV.7. *This result can be obtained in an equivalent way inserting in equation (209) the function $H = C/(t-t_s)$. Then, retaining the leading singular term, one obtains the values $C_\pm$ and $\frac{4}{3(1+\omega)}$. Finally, transforming the differential equation in a second order one in the same way as we have done in Section IV, and linearizing around the singular behaviors obtained above, one can see the form and the number of parameters that depend the different families of singular solutions.*

To understand this summary we perform the change of variable $p \equiv \sqrt{|H|}$. Then the modified Friedmann equation becomes [42]

$$\frac{d}{dt}\left(\dot{p}^2/2 + \widetilde{V}(p)\right) = -3\epsilon p^2\dot{p}^2 + \frac{3\epsilon}{24\alpha}(1+\omega)\rho, \tag{221}$$

where $\widetilde{V}(p) = -\frac{p^2}{4\kappa^2\alpha}\left(1 + \frac{\kappa^2\beta}{6}p^4\right) - \frac{\rho}{24\alpha p^2}$, and $\epsilon \equiv sgn(H)$.

The case $\omega < -1$ is clear. Since $\bar{\rho} \to 0$ at $\bar{t} = \bar{t}_s$, one essentially obtains the same results as Section IV. However, for $\omega > -1$, on the right hand side of equation (221) one term is dissipative and the other one is anti-dissipative, moreover, in this case both terms diverge at $\bar{t} = \bar{t}_s$. Then if one looks for singular solutions of the form $\bar{H} = C/(\bar{t}-\bar{t}_s)$, the first term in the right hand side of (24) has to be dominant. And since this term is of the order $1/(\bar{t}-\bar{t}_s)^4$, and the other one is of the order $1/|\bar{t}-\bar{t}_s|^{3C(1+\omega)}$, they will appear all the situations described above.

It is also interesting to understand the form of the potential $\widetilde{V}$ (its picture appears in figure 3 of ref. [42]). It only has a zero at the point $p_0 = (3/2)^{1/4}\left(1 + \sqrt{1+\frac{4}{3}\bar{\rho}}\right)^{1/4}$, and two critical points at $p_\pm = \left(\frac{1\pm\sqrt{1-4\bar{\rho}}}{2}\right)^{1/4}$ ($p_- < p_+$). Then for $\bar{\rho} > 1/4$ there aren't critical points, and the potential is strictly increasing from $-\infty$ to ∞. For $\bar{\rho} < 1/4$, the potential satisfy $\widetilde{V}(0) = -\infty$, $\widetilde{V}(\infty) = \infty$ and has a relative maximum at p_- and a relative minimum at p_+ (a hollow). For very small values of $\bar{\rho}$ at p_- one has $\bar{H}^2 \cong \bar{\rho}$, that is, the system is nearly to the Friedmann phase, and at p_+ one has $|\bar{H}| \cong 1$, that is, the system is near to the de Sitter phase.

Next step is to find solutions that approximate to the Friedmann one when $|t| \to \infty$ (see [54] for the radiation case, i.e., $\omega = 1/3$). To do that, we consider equation (209) in the dimensionless variables introduced above, and we reduce the order performing the change of variable $u(y) = \bar{H}(\bar{t})$ where $y = \bar{H}$. The obtained equation is:

$$2\beta u + 3(1+\omega)\beta(y^2 - y^4) + (18\alpha(1+\omega) - 4\beta)y^2 u + 6\alpha(2+\omega)y\dot{u}u + 3\alpha(3+\omega)u^2 + 2\alpha(\ddot{u}u^2 + \dot{u}^2 u) = 0, \tag{222}$$

where now $\dot{u} \equiv du/dy$. Since the Friedmann solution in these variables is $u_F = -\frac{3}{2}(1+\omega)y^2$, the linearized equation about this point ($u_{linearized} = u_F + h$) is

$$\ddot{h} + 2y^{-1}\frac{\omega}{1+\omega}\dot{h} + \left(\frac{4\beta}{9\alpha(1+\omega)^2}y^{-4} + Ay^{-2}\right)h + B = 0, \tag{223}$$

where A and B are some constants depending on the parameters α, β and ω.

The idea to solve this equation is to take into account that for large values of $|t|$ (small values of y), one can disregard the term Ay^{-2}. The homogeneous equation is solved performing the change $h = |y|^{-\frac{\omega}{1+\omega}} z$, then one obtains:

$$\ddot{z} + \frac{4\beta}{9\alpha(1+\omega)^2} y^{-4} z = 0, \tag{224}$$

that we solve using the WKB approximation (see [55] page 276). Consequently the homogeneous equation has the two independent solutions

$$h_{homogeneous,\pm}(y) = y^{1/(1+\omega)} exp\left(\pm \frac{2}{3(1+\omega)} \sqrt{-\beta/\alpha} \frac{1}{y} \right). \tag{225}$$

A particular solution is obtained using power series. It leading term is

$$h_{particular}(y) = \frac{9\alpha(1+\omega)^2}{4\beta} B y^4. \tag{226}$$

Then the general solution of the linearized equation is approximately

$$u_{linearized} = u_F + K h_{homogeneous,\pm}(y) + h_{particular}(y) \quad \text{for} \quad \mp y > 0, \tag{227}$$

where K is an arbitrary parameter, that is, we have proved that there is a one-parameter family of solutions that approximate to the Friedmann one for large values of $|t|$.

Once we have seen these preliminary results, we can describe qualitatively the behavior of the no-singular solutions. We start with the case $w < -1$. We have seen that there is a one-parameter family of solutions that at early times are in the expanding Friedmann phase, and we have to look for no-singular solutions that match, at late time, with that family. The only no-singular solutions at early times are: a one-parameter family that approaches asymptotically to the contracting Friedmann phase, and a particular solution that goes asymptotically towards the contracting de Sitter phase, following the direction $(1, 3(1+\omega), -18\omega(1+\omega)\alpha/\beta + 2)$. From the system (24) it is easier to understand the dynamics. First, at early times the system is at the point p_- with $\bar{H} > 0$. Then it leaves this expanding Friedmann state and rolls down either to the right or to the left. In the former case, the universe approaches to an expanding de Sitter phase (the relative minimum p_+). However since $\bar{\rho}$ is an increasing function with time, the critical points will disappear and the potential will be an increasing function with p, this means that the universe rolls down to $p = 0$, that is, it bounces and enters in a decreasing phase $\bar{H} < 0$. Then it can arrive asymptotically at the points p_- or p_+, (the no-singular solutions at late time), or it bounces many times in order to have enough energy in $\bar{H} < 0$, to arrive at $p = \infty$ (singular solution). This last behavior can be easily understood, if one takes first into account that for $\bar{H} > 0$ (resp. $\bar{H} < 0$) the system is dissipative (resp. anti-dissipative), and second that the energy of the system changes its sing when it bounces (see equation (12) of [41]).

Finally, from the behavior of the no-singular solutions at late time, one can deduce that one has to very fine tune the initial conditions and the parameters α and β in order to obtain no-singular solutions that match these late time no-singular behaviors with the expanding Friedmann stage at early times, because these families of solutions aren't general solutions (a two-parameter family).

On the other hand when $\omega > -1$, we also have a one-parameter family of solutions that at late times are in the expanding Friedmann phase (in terms of the variable p this corresponds to the point p_- and $H > 0$), and we have to look for no-singular solutions that match, at early time, with that family. The only no-singular solutions at early times are: a one-parameter family that leaves the contracting Friedmann phase, in terms of the variable p, this means, that the system leave the relative maximum p_- with $\bar{H} < 0$, and rolls down to the right or to the left, but in all cases, since the energy density is an increasing function of time in this region, the system goes to $p = 0$, i.e., it bounce and starts an expanding phase. The other no-singular solution at early times is a two-parameter solution (a general solution) that leaves the contracting de Sitter phase, in terms of the variable p, this means, that the system stars at p_+ with $\bar{H} < 0$, and then due to the anti-dissipation (at early times in $\bar{H} < 0$ the energy density is very small and the dominant term is the first one on the right hand side of equation (221)) the system is released from the hollow and rolls down towards the region $\bar{H} > 0$.

From these no-singular early time behavior one can conclude that, in order to match these early time no-singular solutions with the expanding Friedmann phase at late time, one has to fine tune the initial condition. And depending on the values of the parameters α and β we will obtain different kinds of connections. For example, in [55], different numerical calculations have been done in the radiation case, and they show the different connections in terms of both parameters.

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