On the Hamiltonicity of random bipartite graphs

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Abstract

We prove that if $p \gg \ln n/n$, then a.a.s. every subgraph of random bipartite graph $G(n,n,p)$ with minimum degree at least $(1/2+o(1))np$ is Hamiltonian. The range of p and the constant $1/2$ involved are both asymptotically best possible. The result can be viewed as a generalization of the Dirac theorem within the context of bipartite graphs. The proof uses Pósa's rotation and extension method and is closely related to a recent work of Lee and Sudakov.

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1 Introduction

An undirected graph $G=(V,E)$ on n vertices is represented by the vertex set V with $|V|=n$ and edge set E. G is called bipartite if its vertex set V can be divided into two disjoint sets (classes) V_0 and V_1 such that no two vertices within the same set are adjacent. Bipartite graphs have interesting characteristics (e.g. contains no odd-length cycles) and are widely used in computational sciences [1]. The random bipartite graph model $G(n,n,p)$ with $2n$ vertices is defined as follows (see e.g. [5])

Definition 1. *Let n be a positive integer and $0 \le p \le 1$. The random bipartite graph $G(n,n,p)$ is a probability space over the set of bipartite graphs on the vertex set $V=V_0 \cup V_1$ with $|V_0|=|V_1|=n$ where each pair of vertices between V_0 and V_1 forms an edge randomly and independently with probability p.*

Here, we focus on finding general sufficient conditions for Hamiltonicity in $G(n,n,p)$. It is known that it is NP-complete to determine whether a Hamiltonian cycle exists in a graph [12]. A remarkable theorem of Dirac [7] asserts that every graph (bipartite or not) on n vertices of minimum degree at least $\lceil n/2 \rceil$ is Hamiltonian.

Many classical graph properties can be naturally extended to the Erdős-Rényi random graph model $G(n,p)$ (see e.g. [5]). If a graph property holds for $G(n,p)$ (or $G(n,n,p)$) with probability tends to 1 as n goes to infinity, then we say that this property holds asymptotically almost surely (a.a.s.). The above Dirac's theorem has recently been investigated in the context of random graphs [20, 11, 2, 3, 16]. For example, Sudakov and Vu [20] proved that if $p > (\ln n)^4/n$, then a.a.s. every subgraph of $G(n,p)$ with minimum degree at least $(1/2 + o(1))np$ is Hamiltonian. In [16], Lee and Sudakov further proved the following result.

Theorem 1.([16]) *If $p \gg \ln n/n$, then a.a.s. every subgraph of $G(n,p)$ with minimum degree at least $(1/2+o(1))np$ is Hamiltonian. Furthermore, both the range of p and the value of the constant $1/2$ are asymptotically best possible.*

Here, we show the following theorem analogously for bipartite graphs.

Theorem 2. *For any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$ such that for $p \geq C\ln n/n$, a.a.s. every subgraph of $G(n,n,p)$ with minimum degree at least $(1/2+\varepsilon)np$ is Hamiltonian. Furthermore, both the range of p and the constant $1/2$ are asymptotically best possible.*

Frieze [10] proved the following limit distribution for $G(n,n,p)$.

Theorem 3.([10]) *If $p = (\ln n + \ln\ln n + c_n)/n$, then*

$$\lim_{n\to\infty} \mathrm{P}(G(n,n,p) \text{ is Hamiltonian}) = \begin{cases} 0 & c_n \to -\infty \\ e^{-2e^{-c}} & c_n \to c \\ 1 & c_n \to \infty \end{cases}$$

This result implies that the range of p in Theorem 2 is asymptotically tight. To see the value of the constant $1/2$ is also asymptotically best possible, we consider a random bipartite graph with vertex set $V = V_0 \cup V_1$, and further partition the vertex classes V_0 and V_1 into two equivalent parts, respectively. We obtain $V_0 = V_{00} \cup V_{01}$ and $V_1 = V_{10} \cup V_{11}$ with all the four subsets having size $n/2$. Now deleting all the edges between V_{00} and V_{11}, and those between V_{01} and V_{10}. This results in removing roughly half of the edges incident with each vertex in the random bipartite graph and leaving it disconnected. Thus the graph is no longer Hamiltonian.

Note that the average degree for each vertex in $G(n,n,p)$ is np. Hence, our Theorem 2 indeed complements Theorem 3 in the sense of fault tolerance issues, namely allowing deletion of some edges incident to each vertex while preserving the Hamiltonicity. This property is very appealing in many information and computer systems [19]. Towards deriving the Hamiltonicity of $G(n,n,p)$ we apply Pósa's rotation-extension method ([17, Chapter 10, Problem 20] and [18]) to bipartite graphs with some necessary modifications. Since our argument closely follows the line of [16], we will only sketch/outline the similarities and focus on the differences.

2 Preliminaries

Throughout this work all bipartite graphs we consider are defined on the same vertex set and have the same partition, i.e., $V = V_0 \cup V_1$ and $|V_0| = |V_1| = n$, unless mentioned otherwise. Denote with $K_{n,n}$ the complete bipartite graph on V. For $v \in V$, let $\deg(v)$ be the degree of vertex v. Given a vertex set $X \subseteq V$, denote with $e_G(X)$ (or simply $e(X)$) the number of edges in X. Similarly, for two vertex sets X and Y, denote by $e_G(X,Y)$ (or $e(X,Y)$) the number of edges $\{x,y\}$ with $x \in X$ and $y \in Y$. We use $N_G(X)$ (or $N(X)$) to denote the neighborhood of set X in a graph G, namely, the vertices of $V \backslash X$ which are adjacent to some vertex in X.

The following lemma is a useful concentration inequality (a.k.a. Chernoff's inequality) for independent random variables (see e.g. [6]).

Lemma 1. *Let $\varepsilon > 0$. Let $X_1, \cdots, X_n$ be independent random variables with*

$$\mathrm{P}(X_i = 1) = p_i, \quad \mathrm{P}(X_i = 0) = 1 - p_i.$$

Define the sum $X = \sum_{i=1}^n X_i$ and its expectation $\mathrm{E}(X) = \sum_{i=1}^n p_i$. We have

$$\mathrm{P}(X \le \mathrm{E}(X) - \varepsilon) \le e^{-\varepsilon^2/2\,\mathrm{E}(X)},$$

and

$$\mathrm{P}(X \ge \mathrm{E}(X) + \varepsilon) \le e^{-\frac{\varepsilon^2}{2(\mathrm{E}(X)+\varepsilon/3)}}.$$

By using the above concentration inequality, we can readily obtain the following two results on random bipartite graphs.

Lemma 2. *For any $\varepsilon > 0$, there exists a constant C such that for $p \geq C \ln n/n$ a.a.s. $G = G(n, n, p)$ contains $e(G) = (1 + o(1))n^2 p$ edges, and for any $v \in V$, $(1 - \varepsilon)np \leq \deg(v) \leq (1 + \varepsilon)np$.*

Lemma 3. *Let $\omega(n)$ be any function which tends to infinity with n. If $p \geq \ln n/n$, then a.a.s. for every two subsets of vertices X and Y belonging to different vertex classes in $G(n, n, p)$,*

$$e(X, Y) = |X||Y|p + o(|X||Y|p + \omega(n)n).$$

The next lemma presents some expansion properties for random bipartite graphs in the sense of fault tolerance. Results of the same flavor for Erdős-Rényi random graphs can be found in e. g. [16, 20, 14].

Lemma 4. *For any $0 < \varepsilon < 1/4$, there exists a constant C such that for $p \geq C \ln n/n$ a.a.s. $G = G(n, n, p)$ has the following property. For every subgraph H of maximum degree at most $(1/2 - 2\varepsilon)np$, the graph $G' = G \backslash H$ satisfies:*

(i) For any $X \subseteq V$, $|X| \leq (\ln n)^{-1/4} p^{-1}$, $|N_{G'}(X)| \geq (1/2 + \varepsilon)np|X|$,

(ii) For any $X \subseteq V$, $n(\ln n)^{-1/2} \leq |X| \leq \varepsilon n$, $|N_{G'}(X)| \geq (1/2 + \varepsilon)n$,

(iii) G' is connected.

Proof. Let H and G' be defined as stated in the Lemma 4.

To show (i) we only need to prove that a.a.s for any $X \subseteq V$, $|X| \leq (\ln n)^{-1/4} p^{-1}$,

$$|N_G(X)| \geq (1 - \varepsilon)|X|np, \tag{1}$$

as this would imply that

$$|N_{G'}(X)| \geq |N_G(X)| - \left(\frac{1}{2} - 2\varepsilon\right) np|X| \geq \left(\frac{1}{2} + \varepsilon\right) |X|np.$$

It remains to show (1). Fix a set $X \subseteq V$ with $|X| \leq (\ln n)^{-1/4} p^{-1}$. Decompose X as $X = X_0 \cup X_1$ such that $X_0 \subseteq V_0$ and $X_1 \subseteq V_1$. For any vertex $v \in V \backslash X$, let $I_v = 1_{[v \in N_G(X)]}$ be the indicator. Then we have $Y := |N_G(X)| = \sum_{v \in V_1 \backslash X_1} I_v + \sum_{v \in V_0 \backslash X_0} I_v$. Since by definition $\max\{|X_0|p, |X_1|p\} \leq |X|p = o(1)$, we obtain

$$\begin{aligned}
\mathrm{E}(Y) &= \sum_{v \in V_1 \setminus X_1} \mathrm{P}(I_v = 1) + \sum_{v \in V_0 \setminus X_0} \mathrm{P}(I_v = 1) \\
&= (n - |X_1|)(1 - (1-p)^{|X_0|}) + (n - |X_0|)(1 - (1-p)^{|X_1|}) \\
&= (n - |X_1|)(1 + o(1))|X_0|p + (n - |X_0|)(1 + o(1))|X_1|p \\
&= (1 + o(1))n|X|p - 2|X_0||X_1|p.
\end{aligned}$$

Note that $|X| \leq (\ln n)^{-1/4}p^{-1} \leq n/(C(\ln n)^{5/4})$ and $|X_0||X_1| \leq (|X|/2)^2$. Inserting these estimations into the above equation we derive that $\mathrm{E}(Y) = (1+o(1))n|X|p$. Using Lemma 1, we can bound the probability

$$\begin{aligned}
\mathrm{P}(Y \leq (1-\varepsilon)|X|np) &\leq \mathrm{P}\left(Y \leq \mathrm{E}(Y) - \frac{\varepsilon}{2}\mathrm{E}(Y)\right) \\
&\leq e^{-\frac{\varepsilon^2}{8}\mathrm{E}Y} = e^{-\frac{\varepsilon^2}{8}(1+o(1))|X|np} = n^{-C'|X|}
\end{aligned}$$

for some C' sufficiently large. Taking the union bound over all X similarly as in [16, Proposition 2.5] gives our conclusion.

To show (ii) we first show that a.a.s. for any disjoint pair of sets $X = X_0 \cup X_1$ and $Y = Y_0 \cup Y_1$ with $X_i, Y_i \subseteq V_i$, $i = 0, 1$, $n(\ln n)^{-1/2} \leq |X| \leq \varepsilon n$, $|Y_0| \geq n - |X_0| - a$, $|Y_1| \geq n - |X_1| - b$ and $a + b \leq (1/2 + \varepsilon)n$, $a, b \geq 0$, we have

$$\begin{aligned}
e_G(X, Y) &\geq (1-\varepsilon)(|X_0||Y_1| + |X_1||Y_0|)p \qquad (2)\\
&\geq (1-\varepsilon)(n|X| - 2|X_0||X_1| - b|X_0| - a|X_1|)p \\
&\geq (1-\varepsilon)\left(n|X| - \frac{|X|^2}{2} - |X|\left(\frac{1}{2} + \varepsilon\right)n\right)p \\
&= (1-\varepsilon)\left(\frac{n}{2} - \varepsilon n - \frac{|X|}{2}\right)|X|p \\
&> \left(\frac{1}{2} - 2\varepsilon\right)n|X|p. \qquad (3)
\end{aligned}$$

Indeed, to prove (3) it suffices to prove the first inequality (2). Let X and Y be a fixed pair of sets as defined above. It follows from the expectation $\mathrm{E}(e_G(X,Y)) = |X_0||Y_1|p + |X_1||Y_0|p$ and Lemma 1 that

$$\begin{aligned}
\mathrm{P}(e_G(X,Y) < (1-\varepsilon)(|X_0||Y_1| + |X_1||Y_0|)p) &\leq 2e^{-\frac{\varepsilon^2}{3}(|X_0||Y_1|p + |X_1||Y_0|p)} \\
&\leq 2e^{-\frac{C\varepsilon^2}{3}\left(\frac{1-4\varepsilon}{2-2\varepsilon}\right)n(\ln n)^{1/2}} \\
&\ll 2^{-4n}.
\end{aligned}$$

But there are at most 2^{4n} possible choices of X and Y. Taking the union bound yields our assertion.

Next, we condition on the event that (3) holds. Assume that there exists a set $X \subseteq V$ with $n(\ln n)^{-1/2} \le |X| \le \varepsilon n$ which has less than $(1/2+\varepsilon)n$ neighbors in G'. Therefore, there exists a set $Y = Y_0 \cup Y_1$ satisfying $Y_0 \subseteq V_0$, $Y_1 \subseteq V_1$, $|Y_0| \ge n - |X_0| - N_{G'}(X_1)$, $|Y_1| \ge n - |X_1| - N_{G'}(X_0)$ and $N_{G'}(X_1) + N_{G'}(X_0) \le (1/2+\varepsilon)n$ such that Y is disjoint from X and there are no edges between X and Y in G'. Furthermore, using (3) we obtain

$$0 = e_{G'}(X,Y) \ge e_G(X,Y) - \left(\frac{1}{2} - 2\varepsilon\right) np|X| > 0,$$

which is a clear contradiction.

To show (iii) we condition on the event that both (i) and (ii) hold. Now assume that G' is disconnected. We have a set $X = X_0 \cup X_1$ with $X_0 \subseteq V_0$ and $X_1 \subseteq V_1$, and X induces a connected component in G'. From (i) we see that $|X_i| \ge (np/2)(\ln n)^{-1/4}p^{-1} = n(\ln n)^{-1/4}/2$ for $i = 0, 1$. And then from (ii) we know that $|X_i| > n/2$ for $i = 0, 1$. Hence we have $|X| > n$. Let $Y = V \backslash X$ and we know that Y also contains a connected component. Therefore, $|Y| > n$ and thus $2n = |V| = |X| + |Y| > 2n$, a contradiction. $\square$

3 Proof of Theorem 2: rotation and extension

In this section we establish our main result Theorem 2 by applying a dramatic tool: rotation and extension method [18]. This technique and its variants have been intensively developed in finding long paths and cycles in the random graph setting, see e. g. [16, 3, 9, 13, 11, 15, 4]. When treating it in bipartite graphs, more attention should be paid simply because some edges are forbidden and thus can not be used to "extend" the path.

Consider a connected bipartite graph $G = (V, E)$ with $V = V_0 \cup V_1$ and $|V_0| = |V_1| = n$. Let l be an odd number and $P = (v_0, \cdots, v_l) \subseteq K_{n,n}$ be a path on V, which is not necessarily a subgraph of G. Consider the following two situations:

- If $\{v_0, v_l\}$ is an edge of G, we can use it to close P into a cycle. We change a notation $P' := P$.

- If $\{v_0, v_i\}$ is an edge of G for some $i < l$, we can rotate P, by breaking the edge $\{v_{i-1}, v_i\}$, to a new path $P' = (v_{i-1}, \cdots, v_0, v_i, v_{i+1}, \cdots, v_l)$ which is also of length l

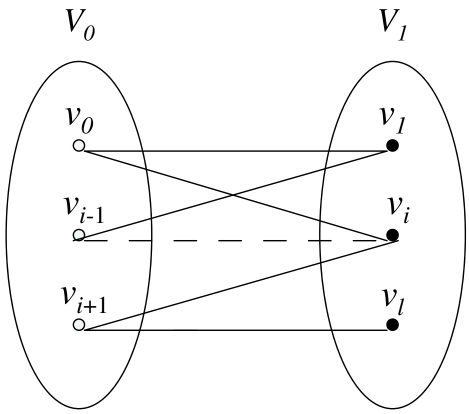

Figure 1: Rotating an odd-length path $(v_0, v_1, \cdots, v_l)$ in a bipartite graph.

in $G \cup P$ (see Fig. 1). Now, if the edge $\{v_{i-1}, v_l\} \in G$, we can use it to close P' into a cycle.

Now that G is connected, either $G \cup P'$ is Hamiltonian or there exists a longer path (not necessarily having odd length) in $G \cup P'$ (hence extending P').

Following the idea of [16] we will show that random bipartite graph $G(n, n, p)$ contains two subgraphs, one of which takes the role of rotation and the other of which takes the role of extension.

Definition 2. *Let $\delta > 0$. A connected bipartite graph $G = (V, E)$ has property $REB(\delta)$ if for every odd-length path $P \subseteq K_{n,n}$ (not necessarily a subgraph of G) either*

- *there exists a path longer than P in $G \cup P$;*

  or

- *there exists a set $S_P \subseteq V(P)$ with $|S_P| \geq \delta n$ such that for any vertex $v \in S_P$, there exists a set $T_v \subseteq V(P)$ with $|T_v| \geq \delta n$ such that for any $w \in T_v$, there exists a path from v to w in $G \cup P$ over $V(P)$, which is of the same length of P.*

Definition 2 roughly means that if G satisfies $REB(\delta)$, then every short odd-length path is extendable and every long odd-length path can be rotated in a number of ways. The next result shows that we can find a subgraph of $G(n, n, p)$ which has property $REB(1 + 2\varepsilon)$.

Proposition 1. *For any $0 < \varepsilon < 1/4$, there exists a constant C such that for $p \geq C \ln n/n$ a.a.s. $G = G(n, n, p)$ has the following property. For every subgraph H of* *maximum degree at most* $(1/2 - 2\varepsilon)np$*, the graph* $G' = G\backslash H$ *satisfies* $REB(1 + 2\varepsilon)$*.*

Proof. Let C be large enough so that the assertions of Lemma 2 and Lemma 4 hold a.a.s.. Let H and G' be defined as stated in Proposition 1. It follows from Lemma 2 and Lemma 4 (iii) that G' has minimum degree at least $(1 - \varepsilon)np - (1/2 - 2\varepsilon)np = (1/2 + \varepsilon)np$ and it is connected. Consider an odd-length path $P = (v_0, \cdots, v_l)$. Without loss of generality we let $v_0 \in V_0$ and $v_l \in V_1$. Suppose that there does not exist a path longer than P in $G' \cup P$. We want to show that the second condition in Definition 2 holds.

We first show that there exists a set $S_{P,0} \subseteq V_0 \cap V(P)$ with $|S_{P,0}| \geq (1/2 + \varepsilon)n$ such that for any $v \in S_{P,0}$, there exists a set $T_{v,1} \subseteq V_1 \cap V(P)$ with $|T_{v,1}| \geq (1/2 + \varepsilon)n$ such that for any $w \in T_{v,1}$, there exists a path from v to w in $G' \cup P$ over $V(P)$, which is of length l. Indeed, we can construct the set $S_{P,0}$ iteratively exactly in the same way as Steps 1 and 2 in [16, Lemma 3.2]. We sketch the procedure briefly as follows. Let $X^{(0)} = \{v_0\}$. By using Lemma 4 (i), (ii) we can show recursively that $|X^{(i)}| \geq (np/4)^i$ and for any $v \in X^{(i)}$ there exists a path of length l in $G' \cup P$ over $V(P)$ connecting v to v_l. We can get sets $X^{(t)}$, $X^{(t+1)}$ and $X^{(t+2)}$ satisfying $|X^{(t)}| = \max\{1, (\ln n)^{-1/4}p^{-1}\}$, $|X^{(t+1)}| = n/(\ln n)^{1/2}$ and $|X^{(t+2)}| \geq n/4$. According to the construction in [16], each vertex in $X^{(i)}$ is obtained by i rotations (breaking i edges of P) and what's more, $X^{(i)} \subseteq V_0 \cap V(P)$ for all i since $X^{(0)} = \{v_0\} \subseteq V_0$. Consider $X^{(t+3)}$ obtained by another round of rotation. Calculating the number of edges incident to $X^{(t+2)}$ which we need to remove gives [16, Lemma 3.2, Step 2]

$$\left|X^{(t+2)}\right| \left(\frac{1}{2} - 2\varepsilon\right) np \geq \left|X^{(t+2)}\right| \left|V_0\backslash X^{(t+3)}\right| p + o(n^2p).$$

(We remark that Lemma 3 will be needed here.) Hence, we obtain $|X^{(t+3)}| \geq (1/2 + \varepsilon)n$. Set $S_{P,0} = X^{(t+3)}$. Now, for each $v \in S_{P,0}$ there exists a path of length l over $V(P)$ connecting v to v_l. Since $v_l \in V_1$, for each such path we perform the same procedure as above by letting $X^{(0)} = \{v_l\}$, we obtain a set $T_{v,1} \subseteq V_1 \cap V(P)$ with $|T_{v,1}| \geq (1/2 + \varepsilon)n$ such that for any $w \in T_{v,1}$, there exists a path of length l connecting w to v in $G' \cup P$ over $V(P)$.

Next, we show that there exists a set $S_{P,1} \subseteq V_1 \cap V(P)$ with $|S_{P,1}| \geq (1/2 + \varepsilon)n$ such that for any $v \in S_{P,1}$, there exists a set $T_{v,0} \subseteq V_0 \cap V(P)$ with $|T_{v,0}| \geq (1/2 + \varepsilon)n$ such that for any $w \in T_{v,0}$, there exists a path from v to w in $G' \cup P$ over $V(P)$, which is of length l. Indeed, since $v_l \in V_1$, this statement can be shown by constructing the sets iteratively with $X^{(0)} = \{v_l\}$ analogously as above.

Finally, we set $S_P = S_{P,0} \cup S_{P,1}$ and $T_v = T_{v,0} \cup T_{v,1}$. The proof is then complete. □

The next proposition shows that we actually can find a small subgraph of $G(n, n, p)$ which satisfies $REB(1 + 2\varepsilon)$.

Proposition 2. *For any $\varepsilon > 0$ and $0 < \delta < 1$, there exists a constant C such that for $p \geq C \ln n/n$ a.a.s. $G = G(n, n, p)$ has the following property. For every subgraph H of maximum degree at most $(1/2 - 3\varepsilon)np$, the graph $G' = G \backslash H$ contains a subgraph with at most $2\delta n^2 p$ edges satisfying $REB(1 + 2\varepsilon)$.*

Proof. Take C large enough such that for $p \geq \delta C \ln n/n$, the assertions of Lemma 2 and Proposition 1 hold a.a.s.. Let $p' = \delta p$ and let $\hat{G}$ be a thinning of G by taking each edge of G independently with probability δ. Therefore, $\hat{G}$ is equivalent to $G(n, n, p')$ in the sense of edge distribution.

$\hat{G}$ is said to be good if it contains at most $2n^2 p' = 2\delta n^2 p$ edges and all subgraphs obtained by deleting at most $(1/2 - 2\varepsilon)np'$ edges incident to each vertex have $REB(1+2\varepsilon)$. It follows from Lemma 2 and Proposition 1 that the probability that $\hat{G}$ is good is $1 - o(1)$. Let H be defined as stated in Proposition 2. We may argue exactly as in [16, Lemma 4.1] to derive that there exists a choice of $\hat{G}$ such that $\hat{G} \backslash H \subseteq G'$ becomes the subgraph that we are looking for. □

Definition 3. *Let $\delta > 0$ and a bipartite graph G_1 satisfy property $REB(\delta)$. A bipartite graph G_2 is said to complement G_1, if for every odd-length path $P \subseteq K_{n,n}$ (not necessarily a subgraph of G_1) either*

- *there exists a path longer than P in $G_1 \cup P$;*

  or

- *there exists $v \in S_P$ and $w \in T_v$ such that $\{v, w\}$ is an edge of $G_1 \cup G_2$, where S_P and T_v are defined as in Definition 2.*

The following result can be proved similarly as [16, Proposition 3.4].

Proposition 3. *Let $\delta > 0$. For every G_1 satisfying $REB(\delta)$ and G_2 complementing G_1, the union graph $G_1 \cup G_2$ is Hamiltonian.*

The next proposition shows that random bipartite graph can take the role of extension by complementing all its small subgraphs satisfying property $REB(1 + 2\varepsilon)$.

Proposition 4. *For any $\varepsilon > 0$, there exist constants δ and C such that for $p \geq C \ln n/n$ a.a.s. $G = G(n, n, p)$ has the following property. For every subgraph H of maximum degree at most $(1/2 - \varepsilon)np$, the graph $G' = G \backslash H$ complements all subgraphs $R \subseteq G$ with at most $\delta n^2 p$ edges satisfying $REB(1 + 2\varepsilon)$.*

Proof. Let $\mathcal{G}$ be the collection of all subgraphs of G obtained by deleting at most $(1/2 - \varepsilon)np$ edges incident to each vertex. The probability $\mathbb{P}$ that the proposition fails can be estimated as [16, Lemma 3.5]

$$\mathbb{P} \leq \sum_{\substack{R \in REB(1+2\varepsilon) \\ |E(R)| \leq \delta n^2 p}} \mathrm{P}(\text{some } G' \in \mathcal{G} \text{ does not complement } R | R \subseteq G) \cdot \mathrm{P}(R \subseteq G). \tag{4}$$

Let R be a fixed subgraph satisfying $REB(1+2\varepsilon)$ and $P \subseteq K_{n,n}$ be a fixed odd-length path. A coarse upper bound of the number of such paths is $(2n)(2n)!$ (which will be enough for our purpose). Since $R \in REB(1 + 2\varepsilon)$, we obtain sets S_P and T_v (for every $v \in S_P$) as specified in Definition 2. We choose an arbitrary subset $S'_P \subseteq S_P$ with $|S'_P| = \varepsilon n$. For every $v \in S'_P$, define $T'_v = T_v \backslash S'_P$. Hence, $|T'_v| \geq (1 + \varepsilon)n$. Fix a vertex $v \in S'_P$. Let $X = \sum_{w \in T'_v} 1_{[\{v,w\} \text{ is an edge in } G]}$. A key ingredient towards bounding the probability $\mathrm{P}(\text{some } G' \in \mathcal{G} \text{ does not complement } R | R \subseteq G)$ in (4) is to estimate $\mathrm{P}(X < (1/2)np)$ (i.e., the probability that the number of neighbors of v in T'_v is less than $(1/2)np$) [16, Lemma 3.5]. By using Lemma 1 and noting that $(1 + \varepsilon)np \leq \mathrm{E}(X) \leq (1 + 2\varepsilon)np$, we obtain

$$\begin{aligned}
\mathrm{P}\left(X < \frac{np}{2}\right) &\leq P(X - \mathrm{E}(X) < -\varepsilon np) \\
&\leq \mathrm{P}\left(X - \mathrm{E}(X) < -\frac{\varepsilon}{2}\mathrm{E}(X)\right) \\
&= e^{-c(\varepsilon)np},
\end{aligned}$$

where $c(\varepsilon) > 0$ is some constant depending on ε. Reasoning as [16, Lemma 3.5] we have the estimation

$$\mathrm{P}(\text{some } G' \in \mathcal{G} \text{ does not complement } R | R \subseteq G) \leq 2n(2n)! e^{-c'(\varepsilon)np},$$

where $c'(\varepsilon) > 0$ is another constant depending on ε. The other terms in (4) can be bounded similarly as in [16, Lemma 3.5] and then we finally get $\mathbb{P} = o(1)$ which completes the proof. □

Putting the above pieces together we are ready to show our main result.

Proof of Theorem 2. Let C be large enough and δ be small enough so that $G(n, n, p)$ with $p \geq C \ln n/n$ a.a.s. satisfies Lemma 2 with $\varepsilon/2$ instead of ε, satisfies Proposition 4 with $\varepsilon/6$ instead of ε, and satisfies Proposition 2 with $\delta/2$ instead of δ and $\varepsilon/6$ instead of ε.

Lemma 2 implies that $G(n,n,p)$ has maximum degree at most $(1+\varepsilon/2)np$. Hence, it suffices to prove that for each subgraph H with maximum degree at most $(1+\varepsilon/2)np - (1/2+\varepsilon)np = (1/2-\varepsilon/2)np$, the graph $G(n,n,p)\backslash H$ is Hamiltonian.

Let H be such a graph. It follows from Proposition 2 that there exists a subgraph of $G(n,n,p)\backslash H$ with at most $\delta n^2 p$ edges satisfying $REB(1+\varepsilon/3)$. Proposition 4 then indicates that $G(n,n,p)\backslash H$ complements this subgraph. By Proposition 3 we see that $G(n,n,p)\backslash H$ is indeed Hamiltonian, which concludes the proof. □

4 Concluding remarks

In this paper, we showed that if $p \gg \ln n/n$, then a.a.s every subgraph of $G(n,n,p)$ with minimum degree at least $(1/2+o(1))np$ is Hamiltonian. Taking into account of the limit distribution for Hamilton cycles given in [10], a direction worthy of further investigation would be obtaining a Dirac-type result near the threshold $\ln n/n$. Another perhaps more demanding open problem is to study this issue for random intersection graphs $G_{n,m,p}$, which also possess underlying bipartite structures. The threshold for Hamiltonicity in $G_{n,m,p}$ is recently established in [8].

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