A Reformulation of Classical Mechanics

Alejandro A. Torassa

(2014) Buenos Aires, Argentina
atorassa@gmail.com

Abstract

This paper presents a reformulation of classical mechanics which is invariant under transformations between reference frames and which can be applied in any reference frame (rotating or non-rotating) (inertial or non-inertial) without the necessity of introducing fictitious forces.

Introduction

The reformulation of classical mechanics presented in this paper is obtained starting from a general equation of motion. This paper considers that any observer S uses a reference frame S and a dynamic reference frame Š. The general equation of motion is a transformation equation between the reference frame S and the dynamic reference frame Š.

The dynamic position $\breve{\mathbf{r}}_a$, the dynamic velocity $\breve{\mathbf{v}}_a$, and the dynamic acceleration $\breve{\mathbf{a}}_a$ of a particle A of mass m_a relative to the dynamic reference frame Š are given by:

$$\breve{\mathbf{r}}_a = \int\int (\mathbf{F}_a/m_a)\, dt\, dt$$

$$\breve{\mathbf{v}}_a = \int (\mathbf{F}_a/m_a)\, dt$$

$$\breve{\mathbf{a}}_a = (\mathbf{F}_a/m_a)$$

where $\mathbf{F}_a$ is the net force acting on particle A.

The dynamic angular velocity $\breve{\omega}_S$ and the dynamic angular acceleration $\breve{\alpha}_S$ of the reference frame S fixed to a particle S relative to the dynamic reference frame Š are given by:

$$\breve{\omega}_S = \pm \left| (\mathbf{F}_1/m_s - \mathbf{F}_0/m_s) \cdot (\mathbf{r}_1 - \mathbf{r}_0)/(\mathbf{r}_1 - \mathbf{r}_0)^2 \right|^{1/2}$$

$$\breve{\alpha}_S = d(\breve{\omega}_S)/dt$$

where $\mathbf{F}_0$ and $\mathbf{F}_1$ are the net forces acting on the reference frame S in the points 0 and 1, $\mathbf{r}_0$ and $\mathbf{r}_1$ are the positions of the points 0 and 1 relative to the reference frame S, and m_s is the mass of particle S (the point 0 is the origin of the reference frame S and the center of mass of particle S) (the point 0 belongs to the axis of dynamic rotation, and the segment 01 is perpendicular to the axis of dynamic rotation) (the vector $\breve{\omega}_S$ is along the axis of dynamic rotation)

General Equation of Motion

The general equation of motion for two particles A and B relative to an observer S is:

$$m_a\, m_b \left[\mathbf{r}_a - \mathbf{r}_b\right] - m_a\, m_b \left[\breve{\mathbf{r}}_a - \breve{\mathbf{r}}_b\right] = 0$$

where m_a and m_b are the masses of particles A and B, $\mathbf{r}_a$ and $\mathbf{r}_b$ are the positions of particles A and B, $\breve{\mathbf{r}}_a$ and $\breve{\mathbf{r}}_b$ are the dynamic positions of particles A and B.

Differentiating the above equation with respect to time, we obtain:

$$m_a\, m_b \left[(\mathbf{v}_a - \mathbf{v}_b) + \breve{\omega}_S \times (\mathbf{r}_a - \mathbf{r}_b)\right] - m_a\, m_b \left[\breve{\mathbf{v}}_a - \breve{\mathbf{v}}_b\right] = 0$$

Differentiating again with respect to time, we obtain:

$$m_a\, m_b \left[(\mathbf{a}_a - \mathbf{a}_b) + 2\,\breve{\omega}_S \times (\mathbf{v}_a - \mathbf{v}_b) + \breve{\omega}_S \times (\breve{\omega}_S \times (\mathbf{r}_a - \mathbf{r}_b)) + \breve{\alpha}_S \times (\mathbf{r}_a - \mathbf{r}_b)\right] - m_a\, m_b \left[\breve{\mathbf{a}}_a - \breve{\mathbf{a}}_b\right] = 0$$

Reference Frames

Applying the above equation to two particles A and S, we have:

$$m_a\, m_s \left[(\mathbf{a}_a - \mathbf{a}_s) + 2\,\breve{\omega}_S \times (\mathbf{v}_a - \mathbf{v}_s) + \breve{\omega}_S \times (\breve{\omega}_S \times (\mathbf{r}_a - \mathbf{r}_s)) + \breve{\alpha}_S \times (\mathbf{r}_a - \mathbf{r}_s)\right] - m_a\, m_s \left[\breve{\mathbf{a}}_a - \breve{\mathbf{a}}_s\right] = 0$$

If we divide by m_s and if the reference frame S fixed to particle S ($\mathbf{r}_s = 0, \mathbf{v}_s = 0$ and $\mathbf{a}_s = 0$) is rotating relative to the dynamic reference frame Š ($\breve{\omega}_S \neq 0$) then we obtain:

$$m_a \left[\mathbf{a}_a + 2\,\breve{\omega}_S \times \mathbf{v}_a + \breve{\omega}_S \times (\breve{\omega}_S \times \mathbf{r}_a) + \breve{\alpha}_S \times \mathbf{r}_a\right] - m_a \left[\breve{\mathbf{a}}_a - \breve{\mathbf{a}}_s\right] = 0$$

If the reference frame S is non-rotating relative to the dynamic reference frame Š ($\breve{\omega}_S = 0$) then we obtain:

$$m_a\, \mathbf{a}_a - m_a \left[\breve{\mathbf{a}}_a - \breve{\mathbf{a}}_s\right] = 0$$

If the reference frame S is inertial relative to the dynamic reference frame Š ($\breve{\omega}_S = 0$ and $\breve{\mathbf{a}}_s = 0$) then we obtain:

$$m_a\, \mathbf{a}_a - m_a\, \breve{\mathbf{a}}_a = 0$$

that is:

$$m_a\, \mathbf{a}_a - \mathbf{F}_a = 0$$

or else:

$$\mathbf{F}_a = m_a\, \mathbf{a}_a$$

where this equation is Newton's second law.

Equation of Motion

From the general equation of motion it follows that the acceleration $\mathbf{a}_a$ of a particle A of mass m_a relative to a reference frame S fixed to a particle S of mass m_s is given by:

$$\mathbf{a}_a = \frac{\mathbf{F}_a}{m_a} - 2\,\breve{\omega}_S \times \mathbf{v}_a - \breve{\omega}_S \times (\breve{\omega}_S \times \mathbf{r}_a) - \breve{\alpha}_S \times \mathbf{r}_a - \frac{\mathbf{F}_s}{m_s}$$

where $\mathbf{F}_a$ is the net force acting on particle A, $\breve{\omega}_S$ is the dynamic angular velocity of the reference frame S, $\mathbf{v}_a$ is the velocity of particle A, $\mathbf{r}_a$ is the position of particle A, $\breve{\alpha}_S$ is the dynamic angular acceleration of the reference frame S, and $\mathbf{F}_s$ is the net force acting on particle S.

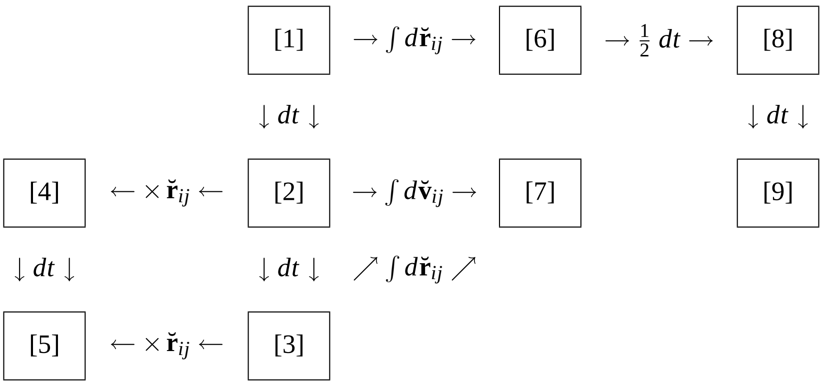
In contradiction with Newton's first and second laws, from the above equation it follows that particle A can have a non-zero acceleration even if there is no force acting on particle A, and also that particle A can have zero acceleration (state of rest or of uniform linear motion) even if there is an unbalanced force acting on particle A.

Therefore, in order to apply Newton's first and second laws in a non-inertial reference frame it is necessary to introduce fictitious forces.

However, this paper considers that Newton's first and second laws are false. Therefore, in this paper there is no need to introduce fictitious forces.

System of Equations

If we consider a system of N particles (of total mass M and center of mass CM) and a single particle J relative to a reference frame S (fixed to a particle S) then from the general equation of motion the following equations are obtained:



The equations [1, 2, 3, 4 and 5] are vector equations, and the equations [6, 7, 8 and 9] are scalar equations. The principles of conservation are obtained from the equations [2, 4, 7 and 9]

Equation [1]

$$\sum_{i=1}^{N} m_i \left[(\mathbf{r}_{ij}) - (\breve{\mathbf{r}}_{ij}) \right] = 0$$

Equation [2]

$$\sum_{i=1}^{N} m_i \left[(\mathbf{v}_{ij} + \breve{\omega}_S \times \mathbf{r}_{ij}) - (\breve{\mathbf{v}}_{ij}) \right] = 0$$

Equation [3]

$$\sum_{i=1}^{N} m_i \left[(\mathbf{a}_{ij} + 2\breve{\omega}_S \times \mathbf{v}_{ij} + \breve{\omega}_S \times (\breve{\omega}_S \times \mathbf{r}_{ij}) + \breve{\alpha}_S \times \mathbf{r}_{ij}) - (\breve{\mathbf{a}}_{ij}) \right] = 0$$

Equation [4]

$$\sum_{i=1}^{N} m_i \left[(\mathbf{v}_{ij} + \breve{\omega}_S \times \mathbf{r}_{ij}) \times \mathbf{r}_{ij} - (\breve{\mathbf{v}}_{ij}) \times \breve{\mathbf{r}}_{ij} \right] = 0$$

Equation [5]

$$\sum_{i=1}^{N} m_i \left[(\mathbf{a}_{ij} + 2\breve{\omega}_S \times \mathbf{v}_{ij} + \breve{\omega}_S \times (\breve{\omega}_S \times \mathbf{r}_{ij}) + \breve{\alpha}_S \times \mathbf{r}_{ij}) \times \mathbf{r}_{ij} - (\breve{\mathbf{a}}_{ij}) \times \breve{\mathbf{r}}_{ij} \right] = 0$$

Equation [6]

$$\sum_{i=1}^{N} {}^1\!/_2\, m_i \left[(\mathbf{r}_{ij})^2 - (\breve{\mathbf{r}}_{ij})^2 \right] = 0$$

Equation [7]

$$\sum_{i=1}^{N} {}^1\!/_2\, m_i \left[(\mathbf{v}_{ij} + \breve{\omega}_S \times \mathbf{r}_{ij})^2 - (\breve{\mathbf{v}}_{ij})^2 \right] = 0$$

Equation [8]

$$\sum_{i=1}^{N} {}^1\!/_2\, m_i \left[(\mathbf{r}_{ij} \cdot \mathbf{v}_{ij}) - (\breve{\mathbf{r}}_{ij} \cdot \breve{\mathbf{v}}_{ij}) \right] = 0$$

Equation [9]

$$\sum_{i=1}^{N} {}^1\!/_2\, m_i \left[(\mathbf{v}_{ij} \cdot \mathbf{v}_{ij} + \mathbf{a}_{ij} \cdot \mathbf{r}_{ij}) - (\breve{\mathbf{v}}_{ij} \cdot \breve{\mathbf{v}}_{ij} + \breve{\mathbf{a}}_{ij} \cdot \breve{\mathbf{r}}_{ij}) \right] = 0$$

The i-th particle (of mass m_i) relative to particle J, to particle S, and to the center of mass CM

$\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ $\breve{\mathbf{r}}_{ij} = \breve{\mathbf{r}}_i - \breve{\mathbf{r}}_j$ $\mathbf{r}_{is} = \mathbf{r}_i - \mathbf{r}_s$ $\breve{\mathbf{r}}_{is} = \breve{\mathbf{r}}_i - \breve{\mathbf{r}}_s$ $\mathbf{r}_{icm} = \mathbf{r}_i - \mathbf{r}_{cm}$ $\breve{\mathbf{r}}_{icm} = \breve{\mathbf{r}}_i - \breve{\mathbf{r}}_{cm}$

$\mathbf{v}_{ij} = \mathbf{v}_i - \mathbf{v}_j$ $\breve{\mathbf{v}}_{ij} = \breve{\mathbf{v}}_i - \breve{\mathbf{v}}_j$ $\mathbf{v}_{is} = \mathbf{v}_i - \mathbf{v}_s$ $\breve{\mathbf{v}}_{is} = \breve{\mathbf{v}}_i - \breve{\mathbf{v}}_s$ $\mathbf{v}_{icm} = \mathbf{v}_i - \mathbf{v}_{cm}$ $\breve{\mathbf{v}}_{icm} = \breve{\mathbf{v}}_i - \breve{\mathbf{v}}_{cm}$

$\mathbf{a}_{ij} = \mathbf{a}_i - \mathbf{a}_j$ $\breve{\mathbf{a}}_{ij} = \breve{\mathbf{a}}_i - \breve{\mathbf{a}}_j$ $\mathbf{a}_{is} = \mathbf{a}_i - \mathbf{a}_s$ $\breve{\mathbf{a}}_{is} = \breve{\mathbf{a}}_i - \breve{\mathbf{a}}_s$ $\mathbf{a}_{icm} = \mathbf{a}_i - \mathbf{a}_{cm}$ $\breve{\mathbf{a}}_{icm} = \breve{\mathbf{a}}_i - \breve{\mathbf{a}}_{cm}$

Δ Equation [2]

$$\textstyle\sum_{i=1}^{N} \Delta\, m_i \left[(\mathbf{v}_{ij} + \breve{\omega}_S \times \mathbf{r}_{ij}) - (\breve{\mathbf{v}}_{ij}) \right] = 0$$

Now, replacing particle J by particle S and distributing $(\Delta\, m_i)$ we have:

$$\textstyle\sum_{i=1}^{N} \left[\Delta\, m_i\, (\mathbf{v}_{is} + \breve{\omega}_S \times \mathbf{r}_{is}) - \Delta\, m_i\, (\breve{\mathbf{v}}_{is}) \right] = 0$$

If the reference frame S $(\mathbf{v}_s = 0)$ is inertial $(\breve{\omega}_S = 0$ and $\breve{\mathbf{v}}_s = \text{constant})$ then:

$$\textstyle\sum_{i=1}^{N} \left[\Delta\, m_i\, \mathbf{v}_i - \Delta\, m_i\, \breve{\mathbf{v}}_i \right] = 0$$

Since $\left[\Delta\, m_i\, \breve{\mathbf{v}}_i = \int_1^2 m_i\, \breve{\mathbf{a}}_i\, dt = \int_1^2 \mathbf{F}_i\, dt \right]$ we obtain:

$$\textstyle\sum_{i=1}^{N} \left[\Delta\, m_i\, \mathbf{v}_i - \int_1^2 \mathbf{F}_i\, dt \right] = 0$$

If the system of particles is isolated and if the internal forces obey Newton's third law in its weak form $(\sum_{i=1}^{N} \mathbf{F}_i = 0)$ then:

$$\textstyle\sum_{i=1}^{N} m_i\, \mathbf{v}_i = \mathbf{P} = \text{constant}$$

Therefore, if the system of particles is isolated and if the internal forces obey Newton's third law in its weak form then the total linear momentum **P** of the system of particles remains constant relative to an inertial reference frame.

Δ Equation [4]

$$\textstyle\sum_{i=1}^{N} \Delta\, m_i \left[(\mathbf{v}_{ij} + \breve{\omega}_S \times \mathbf{r}_{ij}) \times \mathbf{r}_{ij} - (\breve{\mathbf{v}}_{ij}) \times \breve{\mathbf{r}}_{ij} \right] = 0$$

Now, replacing particle J by the center of mass CM and distributing $(\Delta\, m_i)$ we have:

$$\textstyle\sum_{i=1}^{N} \left[\Delta\, m_i\, (\mathbf{v}_{icm} + \breve{\omega}_S \times \mathbf{r}_{icm}) \times \mathbf{r}_{icm} - \Delta\, m_i\, (\breve{\mathbf{v}}_{icm}) \times \breve{\mathbf{r}}_{icm} \right] = 0$$

Since $\left[\Delta\, m_i\, (\breve{\mathbf{v}}_{icm}) \times \breve{\mathbf{r}}_{icm} = \Delta\, m_i\, \breve{\mathbf{v}}_{icm} \times \breve{\mathbf{r}}_{icm} = \int_1^2 (m_i\, \breve{\mathbf{a}}_{icm} \times \breve{\mathbf{r}}_{icm})\, dt = \int_1^2 (m_i\, \breve{\mathbf{a}}_{icm} \times \mathbf{r}_{icm})\, dt \right]$ we obtain:

$$\textstyle\sum_{i=1}^{N} \left[\Delta\, m_i\, (\mathbf{v}_{icm} + \breve{\omega}_S \times \mathbf{r}_{icm}) \times \mathbf{r}_{icm} - \int_1^2 (m_i\, \breve{\mathbf{a}}_{icm} \times \mathbf{r}_{icm})\, dt \right] = 0$$

Given that $\left[\sum_{i=1}^{N} \int_1^2 (m_i\, \breve{\mathbf{a}}_{icm} \times \mathbf{r}_{icm})\, dt = \sum_{i=1}^{N} \int_1^2 (m_i\, \breve{\mathbf{a}}_i \times \mathbf{r}_{icm})\, dt = \sum_{i=1}^{N} \int_1^2 (\mathbf{F}_i \times \mathbf{r}_{icm})\, dt \right]$ we get:

$$\textstyle\sum_{i=1}^{N} \left[\Delta\, m_i\, (\mathbf{v}_{icm} + \breve{\omega}_S \times \mathbf{r}_{icm}) \times \mathbf{r}_{icm} - \int_1^2 (\mathbf{F}_i \times \mathbf{r}_{icm})\, dt \right] = 0$$

If the system of particles is isolated and if the internal forces obey Newton's third law in its strong form $(\sum_{i=1}^{N} \mathbf{F}_i \times \mathbf{r}_{icm} = 0)$ then:

$$\textstyle\sum_{i=1}^{N} m_i\, (\mathbf{v}_{icm} + \breve{\omega}_S \times \mathbf{r}_{icm}) \times \mathbf{r}_{icm} = \mathbf{L} = \text{constant}$$

Therefore, if the system of particles is isolated and if the internal forces obey Newton's third law in its strong form then the total angular momentum **L** of the system of particles remains constant.

Δ Equation [7]

$$\textstyle\sum_{i=1}^{N} \Delta \, ^1\!/_2 \, m_i \left[(\mathbf{v}_{ij} + \breve{\omega}_S \times \mathbf{r}_{ij})^2 - (\breve{\mathbf{v}}_{ij})^2 \right] = 0$$

Now, replacing particle J by the center of mass CM and distributing $(\Delta \, ^1\!/_2 \, m_i)$ we have:

$$\textstyle\sum_{i=1}^{N} \left[\Delta \, ^1\!/_2 \, m_i \, (\mathbf{v}_{icm} + \breve{\omega}_S \times \mathbf{r}_{icm})^2 - \Delta \, ^1\!/_2 \, m_i \, (\breve{\mathbf{v}}_{icm})^2 \right] = 0$$

Since $\left[\Delta \, ^1\!/_2 \, m_i \, (\breve{\mathbf{v}}_{icm})^2 = \Delta \, ^1\!/_2 \, m_i \, \breve{\mathbf{v}}_{icm} \cdot \breve{\mathbf{v}}_{icm} = \int_1^2 m_i \, \breve{\mathbf{a}}_{icm} \cdot d\breve{\mathbf{r}}_{icm} = \int_1^2 m_i \, \breve{\mathbf{a}}_{icm} \cdot d\mathbf{r}_{icm} \right]$ [Eq. A] we obtain:

$$\textstyle\sum_{i=1}^{N} \left[\Delta \, ^1\!/_2 \, m_i \, (\mathbf{v}_{icm} + \breve{\omega}_S \times \mathbf{r}_{icm})^2 - \int_1^2 m_i \, \breve{\mathbf{a}}_{icm} \cdot d\mathbf{r}_{icm} \right] = 0$$

Given that $\left[\sum_{i=1}^{N} \int_1^2 m_i \, \breve{\mathbf{a}}_{icm} \cdot d\mathbf{r}_{icm} = \sum_{i=1}^{N} \int_1^2 m_i \, \breve{\mathbf{a}}_i \cdot d\mathbf{r}_{icm} = \sum_{i=1}^{N} \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_{icm} \right]$ [Eq. B] we get:

$$\textstyle\sum_{i=1}^{N} \left[\Delta \, ^1\!/_2 \, m_i \, (\mathbf{v}_{icm} + \breve{\omega}_S \times \mathbf{r}_{icm})^2 - \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_{icm} \right] = 0$$

Therefore, we can consider that the total work W done by the forces acting on the system of particles, the total kinetic energy K of the system of particles and the total potential energy U of the system of particles are as follows:

$$W = \textstyle\sum_{i=1}^{N} \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_{icm}$$

$$\Delta K = \textstyle\sum_{i=1}^{N} \Delta \, ^1\!/_2 \, m_i \, (\mathbf{v}_{icm} + \breve{\omega}_S \times \mathbf{r}_{icm})^2$$

$$\Delta U = \textstyle\sum_{i=1}^{N} - \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_{icm}$$

If the system of particles is isolated and if the internal forces obey Newton's third law in its weak form $(\sum_{i=1}^{N} \mathbf{F}_i = 0)$ then:

$$W = \textstyle\sum_{i=1}^{N} \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i$$

$$\Delta U = \textstyle\sum_{i=1}^{N} - \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i$$

The total work W done by the forces acting on the system of particles is equal to the change in the total kinetic energy K of the system of particles.

$$W = \Delta K$$

The total work W done by the conservative forces acting on the system of particles is equal and opposite in sign to the change in the total potential energy U of the system of particles.

$$W = -\Delta U$$

Therefore, if the system of particles is exclusively subject to conservative forces then the total mechanical energy E of the system of particles remains constant.

$$E = K + U = \text{constant}$$

Δ Equation [9]

$$\textstyle\sum_{i=1}^{N} \Delta\, {}^1\!/_2\, m_i \left[(\mathbf{v}_{ij} \cdot \mathbf{v}_{ij} + \mathbf{a}_{ij} \cdot \mathbf{r}_{ij}) - (\check{\mathbf{v}}_{ij} \cdot \check{\mathbf{v}}_{ij} + \breve{\mathbf{a}}_{ij} \cdot \check{\mathbf{r}}_{ij}) \right] = 0$$

Now, replacing particle J by the center of mass CM and distributing $(\Delta\, {}^1\!/_2\, m_i)$ we have:

$$\textstyle\sum_{i=1}^{N} \left[\Delta\, {}^1\!/_2\, m_i\, (\mathbf{v}_{icm} \cdot \mathbf{v}_{icm} + \mathbf{a}_{icm} \cdot \mathbf{r}_{icm}) - (\Delta\, {}^1\!/_2\, m_i\, \check{\mathbf{v}}_{icm} \cdot \check{\mathbf{v}}_{icm} + \Delta\, {}^1\!/_2\, m_i\, \breve{\mathbf{a}}_{icm} \cdot \check{\mathbf{r}}_{icm}) \right] = 0$$

Since $[\text{Eq. A}]$ and $[\Delta\, {}^1\!/_2\, m_i\, \breve{\mathbf{a}}_{icm} \cdot \check{\mathbf{r}}_{icm} = \Delta\, {}^1\!/_2\, m_i\, \breve{\mathbf{a}}_{icm} \cdot \mathbf{r}_{icm}]$ we obtain:

$$\textstyle\sum_{i=1}^{N} \left[\Delta\, {}^1\!/_2\, m_i\, (\mathbf{v}_{icm} \cdot \mathbf{v}_{icm} + \mathbf{a}_{icm} \cdot \mathbf{r}_{icm}) - (\int_1^2 m_i\, \breve{\mathbf{a}}_{icm} \cdot d\mathbf{r}_{icm} + \Delta\, {}^1\!/_2\, m_i\, \breve{\mathbf{a}}_{icm} \cdot \mathbf{r}_{icm}) \right] = 0$$

Given that $[\text{Eq. B}]$ and $[\sum_{i=1}^{N} \Delta\, {}^1\!/_2\, m_i\, \breve{\mathbf{a}}_{icm} \cdot \mathbf{r}_{icm} = \sum_{i=1}^{N} \Delta\, {}^1\!/_2\, m_i\, \breve{\mathbf{a}}_i \cdot \mathbf{r}_{icm} = \sum_{i=1}^{N} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \mathbf{r}_{icm}]$ we get:

$$\textstyle\sum_{i=1}^{N} \left[\Delta\, {}^1\!/_2\, m_i\, (\mathbf{v}_{icm} \cdot \mathbf{v}_{icm} + \mathbf{a}_{icm} \cdot \mathbf{r}_{icm}) - (\int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_{icm} + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \mathbf{r}_{icm}) \right] = 0$$

Therefore, we can consider that the total work W' done by the forces acting on the system of particles, the total kinetic energy K' of the system of particles and the total potential energy U' of the system of particles are as follows:

$$W' = \textstyle\sum_{i=1}^{N} (\int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_{icm} + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \mathbf{r}_{icm})$$

$$\Delta K' = \textstyle\sum_{i=1}^{N} \Delta\, {}^1\!/_2\, m_i\, (\mathbf{v}_{icm} \cdot \mathbf{v}_{icm} + \mathbf{a}_{icm} \cdot \mathbf{r}_{icm})$$

$$\Delta U' = \textstyle\sum_{i=1}^{N} -(\int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_{icm} + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \mathbf{r}_{icm})$$

If the system of particles is isolated and if the internal forces obey Newton's third law in its weak form $(\sum_{i=1}^{N} \mathbf{F}_i = 0)$ then:

$$W' = \textstyle\sum_{i=1}^{N} (\int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \mathbf{r}_i)$$

$$\Delta U' = \textstyle\sum_{i=1}^{N} -(\int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \mathbf{r}_i)$$

The total work W' done by the forces acting on the system of particles is equal to the change in the total kinetic energy K' of the system of particles.

$$W' = \Delta K'$$

The total work W' done by the conservative forces acting on the system of particles is equal and opposite in sign to the change in the total potential energy U' of the system of particles.

$$W' = -\Delta U'$$

Therefore, if the system of particles is exclusively subject to conservative forces then the total mechanical energy E' of the system of particles remains constant.

$$E' = K' + U' = \text{constant}$$

General Observations

The magnitudes $\breve{\mathbf{r}}$, $\breve{\mathbf{v}}$, $\breve{\mathbf{a}}$, $\breve{\omega}$ and $\breve{\alpha}$ are invariant under transformations between reference frames.

In any reference frame $\mathbf{r}_{ij} = \breve{\mathbf{r}}_{ij}$. Therefore, $\mathbf{r}_{ij}$ is invariant under transformations between reference frames.

In any non-rotating reference frame $\mathbf{v}_{ij} = \breve{\mathbf{v}}_{ij}$ and $\mathbf{a}_{ij} = \breve{\mathbf{a}}_{ij}$. Therefore, $\mathbf{v}_{ij}$ and $\mathbf{a}_{ij}$ are invariant under transformations between non-rotating reference frames.

In any inertial reference frame $\mathbf{a}_i = \breve{\mathbf{a}}_i$. Therefore, $\mathbf{a}_i$ is invariant under transformations between inertial reference frames. Any inertial reference frame is a non-rotating reference frame.

In the universal reference frame $\mathbf{r}_i = \breve{\mathbf{r}}_i$, $\mathbf{v}_i = \breve{\mathbf{v}}_i$ and $\mathbf{a}_i = \breve{\mathbf{a}}_i$. Therefore, the universal reference frame is an inertial reference frame.

The total angular momentum $\mathbf{L}$ of a system of particles is invariant under transformations between reference frames.

The total kinetic energy K and the total potential energy U of a system of particles are invariant under transformations between reference frames. Therefore, the total mechanical energy E of a system of particles is invariant under transformations between reference frames.

The total kinetic energy K' and the total potential energy U' of a system of particles are invariant under transformations between reference frames. Therefore, the total mechanical energy E' of a system of particles is invariant under transformations between reference frames.

The total mechanical energy E of a system of particles is equal to the total mechanical energy E' of the system of particles $(E = E')$

Bibliography

A. Einstein, Relativity: The Special and General Theory.

E. Mach, The Science of Mechanics.

R. Resnick and D. Halliday, Physics.

J. Kane and M. Sternheim, Physics.

H. Goldstein, Classical Mechanics.

L. Landau and E. Lifshitz, Mechanics.

Appendix

Definitions and Relations

$$\mathbf{r}_i = \mathbf{r}_i \qquad \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$$

$$\mathbf{v}_i = d\mathbf{r}_i/dt \qquad \mathbf{v}_{ij} = d\mathbf{r}_{ij}/dt$$

$$\mathbf{a}_i = d\mathbf{v}_i/dt \qquad \mathbf{a}_{ij} = d\mathbf{v}_{ij}/dt$$

$$\mathbf{v}_i = \int \mathbf{a}_i \, dt \qquad \mathbf{v}_{ij} = \int \mathbf{a}_{ij} \, dt$$

$$\Delta \mathbf{v}_i = \int_1^2 \mathbf{a}_i \, dt \qquad \Delta \mathbf{v}_{ij} = \int_1^2 \mathbf{a}_{ij} \, dt$$

$$1/2 \, \mathbf{v}_i \cdot \mathbf{v}_i = \int \mathbf{a}_i \cdot d\mathbf{r}_i \qquad 1/2 \, \mathbf{v}_{ij} \cdot \mathbf{v}_{ij} = \int \mathbf{a}_{ij} \cdot d\mathbf{r}_{ij}$$

$$\Delta \, 1/2 \, \mathbf{v}_i \cdot \mathbf{v}_i = \int_1^2 \mathbf{a}_i \cdot d\mathbf{r}_i \qquad \Delta \, 1/2 \, \mathbf{v}_{ij} \cdot \mathbf{v}_{ij} = \int_1^2 \mathbf{a}_{ij} \cdot d\mathbf{r}_{ij}$$

$$\mathbf{v}_i \times \mathbf{r}_i = \int (\mathbf{a}_i \times \mathbf{r}_i) \, dt \qquad \mathbf{v}_{ij} \times \mathbf{r}_{ij} = \int (\mathbf{a}_{ij} \times \mathbf{r}_{ij}) \, dt$$

$$\Delta \mathbf{v}_i \times \mathbf{r}_i = \int_1^2 (\mathbf{a}_i \times \mathbf{r}_i) \, dt \qquad \Delta \mathbf{v}_{ij} \times \mathbf{r}_{ij} = \int_1^2 (\mathbf{a}_{ij} \times \mathbf{r}_{ij}) \, dt$$

Invariant Equations

$$\mathbf{r}_{ij} \cdot \mathbf{r}_{ij} = \acute{\mathbf{r}}_{ij} \cdot \acute{\mathbf{r}}_{ij}$$

$$\mathbf{r}_{ij} \cdot \mathbf{v}_{ij} = \acute{\mathbf{r}}_{ij} \cdot \acute{\mathbf{v}}_{ij}$$

$$\mathbf{v}_{ij} \cdot \mathbf{v}_{ij} + \mathbf{a}_{ij} \cdot \mathbf{r}_{ij} = \acute{\mathbf{v}}_{ij} \cdot \acute{\mathbf{v}}_{ij} + \acute{\mathbf{a}}_{ij} \cdot \acute{\mathbf{r}}_{ij}$$

$$\mathbf{r}_{ij} = \acute{\mathbf{r}}_{ij}$$

$$\mathbf{v}_{ij} + \breve{\omega}_S \times \mathbf{r}_{ij} = \acute{\mathbf{v}}_{ij} + \breve{\omega}_{\acute{S}} \times \acute{\mathbf{r}}_{ij}$$

$$\mathbf{a}_{ij} + 2\,\breve{\omega}_S \times \mathbf{v}_{ij} + \breve{\omega}_S \times (\breve{\omega}_S \times \mathbf{r}_{ij}) + \breve{\alpha}_S \times \mathbf{r}_{ij} = \acute{\mathbf{a}}_{ij} + 2\,\breve{\omega}_{\acute{S}} \times \acute{\mathbf{v}}_{ij} + \breve{\omega}_{\acute{S}} \times (\breve{\omega}_{\acute{S}} \times \acute{\mathbf{r}}_{ij}) + \breve{\alpha}_{\acute{S}} \times \acute{\mathbf{r}}_{ij}$$

Alternative Equations

$$\mathbf{L} = \sum_{i=1}^{N} m_i \, (\mathbf{v}_i + \breve{\omega}_S \times \mathbf{r}_i) \times \mathbf{r}_i - \mathrm{M} \, (\mathbf{v}_{cm} + \breve{\omega}_S \times \mathbf{r}_{cm}) \times \mathbf{r}_{cm}$$

$$\mathbf{L} = \sum_{i=1}^{N} \sum_{j>i}^{N} m_i \, m_j \, \mathrm{M}^{-1} (\mathbf{v}_{ij} + \breve{\omega}_S \times \mathbf{r}_{ij}) \times \mathbf{r}_{ij}$$

$$K = \sum_{i=1}^{N} 1/2 \, m_i \, (\mathbf{v}_i + \breve{\omega}_S \times \mathbf{r}_i)^2 - 1/2 \, \mathrm{M} \, (\mathbf{v}_{cm} + \breve{\omega}_S \times \mathbf{r}_{cm})^2$$

$$K = \sum_{i=1}^{N} \sum_{j>i}^{N} 1/2 \, m_i \, m_j \, \mathrm{M}^{-1} (\mathbf{v}_{ij} + \breve{\omega}_S \times \mathbf{r}_{ij})^2$$

$$K' = \sum_{i=1}^{N} 1/2 \, m_i \, (\mathbf{v}_i \cdot \mathbf{v}_i + \mathbf{a}_i \cdot \mathbf{r}_i) - 1/2 \, \mathrm{M} \, (\mathbf{v}_{cm} \cdot \mathbf{v}_{cm} + \mathbf{a}_{cm} \cdot \mathbf{r}_{cm})$$

$$K' = \sum_{i=1}^{N} \sum_{j>i}^{N} 1/2 \, m_i \, m_j \, \mathrm{M}^{-1} (\mathbf{v}_{ij} \cdot \mathbf{v}_{ij} + \mathbf{a}_{ij} \cdot \mathbf{r}_{ij})$$