A New Theory in Relational Mechanics

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(Paper I)

In relational mechanics, a new theory is presented, which is invariant under transformations between inertial and non-inertial reference frames and which can be applied in any reference frame without introducing fictitious forces. In addition to the above, in this paper, we assume that all forces always obey Newton's third law.

Introduction

The new theory in relational mechanics presented in this paper is obtained starting from an auxiliary system of particles (called Universe) that is used to obtain kinematic magnitudes (such as universal position, universal velocity, etc.) that are invariant under transformations between inertial and non-inertial reference frames.

The universal position $\mathbf{r}_i$, the universal velocity $\mathbf{v}_i$ and the universal acceleration $\mathbf{a}_i$ of a particle i are given by:

$$\mathbf{r}_i \doteq (\vec{r}_i - \vec{R})$$

$$\mathbf{v}_i \doteq (\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R})$$

$$\mathbf{a}_i \doteq (\vec{a}_i - \vec{A}) - 2\,\vec{\omega} \times (\vec{v}_i - \vec{V}) + \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,] - \vec{\alpha} \times (\vec{r}_i - \vec{R})$$

($\mathbf{v}_i \doteq d(\mathbf{r}_i)/dt$) and ($\mathbf{a}_i \doteq d^2(\mathbf{r}_i)/dt^2$) where $\vec{r}_i$ is the position vector of particle i, $\vec{R}$ is the position vector of the center of mass of the Universe, and $\vec{\omega}$ is the angular velocity vector of the Universe (see Appendix I)

A reference frame S is non-rotating if the angular velocity $\vec{\omega}$ of the Universe relative to S is equal to zero, and the reference frame S is also inertial if the acceleration $\vec{A}$ of the center of mass of the Universe relative to S is equal to zero.

The New Dynamics

[1] A force is always caused by the interaction between two or more particles.

[2] The net force $\mathbf{F}_i$ acting on a particle i of mass m_i produces a universal acceleration $\mathbf{a}_i$ according to the following equation: [$\mathbf{F}_i = m_i\,\mathbf{a}_i$]

[3] In this paper, we assume that all forces always obey Newton's third law in its weak form and in its strong form.

The Definitions

For a system of N particles, the following definitions are applicable:

Mass	$\mathrm{M} \doteq \sum_i^{\mathrm{N}} m_i$
Position CM 1	$\vec{R}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \vec{r}_i$
Velocity CM 1	$\vec{V}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \vec{v}_i$
Acceleration CM 1	$\vec{A}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \vec{a}_i$
Position CM 2	$\mathbf{R}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \mathbf{r}_i$
Velocity CM 2	$\mathbf{V}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \mathbf{v}_i$
Acceleration CM 2	$\mathbf{A}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \mathbf{a}_i$
Linear Momentum 1	$\mathbf{P}_1 \doteq \sum_i^{\mathrm{N}} m_i \, \mathbf{v}_i$
Angular Momentum 1	$\mathbf{L}_1 \doteq \sum_i^{\mathrm{N}} m_i \left[\mathbf{r}_i \times \mathbf{v}_i \right]$
Angular Momentum 2	$\mathbf{L}_2 \doteq \sum_i^{\mathrm{N}} m_i \left[(\mathbf{r}_i - \mathbf{R}_{cm}) \times (\mathbf{v}_i - \mathbf{V}_{cm}) \right]$
Work 1	$\mathrm{W}_1 \doteq \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i = \Delta \mathrm{K}_1$
Kinetic Energy 1	$\Delta \mathrm{K}_1 \doteq \sum_i^{\mathrm{N}} \Delta \, {}^1\!/_2 \, m_i \, (\mathbf{v}_i)^2$
Potential Energy 1	$\Delta \mathrm{U}_1 \doteq - \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i$
Mechanical Energy 1	$\mathrm{E}_1 \doteq \mathrm{K}_1 + \mathrm{U}_1$
Lagrangian 1	$\mathrm{L}_1 \doteq \mathrm{K}_1 - \mathrm{U}_1$
Work 2	$\mathrm{W}_2 \doteq \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d(\mathbf{r}_i - \mathbf{R}_{cm}) = \Delta \mathrm{K}_2$
Kinetic Energy 2	$\Delta \mathrm{K}_2 \doteq \sum_i^{\mathrm{N}} \Delta \, {}^1\!/_2 \, m_i \, (\mathbf{v}_i - \mathbf{V}_{cm})^2$
Potential Energy 2	$\Delta \mathrm{U}_2 \doteq - \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d(\mathbf{r}_i - \mathbf{R}_{cm})$
Mechanical Energy 2	$\mathrm{E}_2 \doteq \mathrm{K}_2 + \mathrm{U}_2$
Lagrangian 2	$\mathrm{L}_2 \doteq \mathrm{K}_2 - \mathrm{U}_2$

Work 3	$\mathrm{W}_3 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \mathbf{r}_i = \Delta\, \mathrm{K}_3$
Kinetic Energy 3	$\Delta\, \mathrm{K}_3 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i\, \mathbf{a}_i \cdot \mathbf{r}_i$
Potential Energy 3	$\Delta\, \mathrm{U}_3 \doteq -\sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \mathbf{r}_i$
Mechanical Energy 3	$\mathrm{E}_3 \doteq \mathrm{K}_3 + \mathrm{U}_3$
Work 4	$\mathrm{W}_4 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) = \Delta\, \mathrm{K}_4$
Kinetic Energy 4	$\Delta\, \mathrm{K}_4 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i \left[(\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) \right]$
Potential Energy 4	$\Delta\, \mathrm{U}_4 \doteq -\sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm})$
Mechanical Energy 4	$\mathrm{E}_4 \doteq \mathrm{K}_4 + \mathrm{U}_4$
Work 5	$\mathrm{W}_5 \doteq \sum_i^{\mathrm{N}} \left[\int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}) \right] = \Delta\, \mathrm{K}_5$
Kinetic Energy 5	$\Delta\, \mathrm{K}_5 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i \left[(\vec{v}_i - \vec{V})^2 + (\vec{a}_i - \vec{A}) \cdot (\vec{r}_i - \vec{R}) \right]$
Potential Energy 5	$\Delta\, \mathrm{U}_5 \doteq -\sum_i^{\mathrm{N}} \left[\int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}) \right]$
Mechanical Energy 5	$\mathrm{E}_5 \doteq \mathrm{K}_5 + \mathrm{U}_5$
Work 6	$\mathrm{W}_6 \doteq \sum_i^{\mathrm{N}} \left[\int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}_{cm}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}_{cm}) \right] = \Delta\, \mathrm{K}_6$
Kinetic Energy 6	$\Delta\, \mathrm{K}_6 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i \left[(\vec{v}_i - \vec{V}_{cm})^2 + (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) \right]$
Potential Energy 6	$\Delta\, \mathrm{U}_6 \doteq -\sum_i^{\mathrm{N}} \left[\int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}_{cm}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}_{cm}) \right]$
Mechanical Energy 6	$\mathrm{E}_6 \doteq \mathrm{K}_6 + \mathrm{U}_6$

The Relations

From the above definitions, the following relations can be obtained (see Appendix II)

$$\mathrm{K}_1 = \mathrm{K}_2 + {}^1\!/_2\, \mathrm{M}\, \mathbf{V}_{cm}^2$$

$$\mathrm{K}_3 = \mathrm{K}_4 + {}^1\!/_2\, \mathrm{M}\, \mathbf{A}_{cm} \cdot \mathbf{R}_{cm}$$

$$\mathrm{K}_5 = \mathrm{K}_6 + {}^1\!/_2\, \mathrm{M} \left[(\vec{V}_{cm} - \vec{V})^2 + (\vec{A}_{cm} - \vec{A}) \cdot (\vec{R}_{cm} - \vec{R}) \right]$$

$$\mathrm{K}_5 = \mathrm{K}_1 + \mathrm{K}_3 \quad \& \quad \mathrm{U}_5 = \mathrm{U}_1 + \mathrm{U}_3 \quad \& \quad \mathrm{E}_5 = \mathrm{E}_1 + \mathrm{E}_3$$

$$\mathrm{K}_6 = \mathrm{K}_2 + \mathrm{K}_4 \quad \& \quad \mathrm{U}_6 = \mathrm{U}_2 + \mathrm{U}_4 \quad \& \quad \mathrm{E}_6 = \mathrm{E}_2 + \mathrm{E}_4$$

The Principles

The linear momentum $[\,\mathbf{P}_1\,]$ of an isolated system of N particles remains constant if the internal forces obey Newton's third law in its weak form.

$$\mathbf{P}_1 \;=\; \text{constant} \qquad \Big[\; d(\mathbf{P}_1)/dt \;=\; \textstyle\sum_i^{\rm N} m_i\,\mathbf{a}_i \;=\; \sum_i^{\rm N} \mathbf{F}_i \;=\; 0 \;\Big]$$

The angular momentum $[\,\mathbf{L}_1\,]$ of an isolated system of N particles remains constant if the internal forces obey Newton's third law in its strong form.

$$\mathbf{L}_1 \;=\; \text{constant} \qquad \Big[\; d(\mathbf{L}_1)/dt \;=\; \textstyle\sum_i^{\rm N} m_i\,\big[\,\mathbf{r}_i \times \mathbf{a}_i\,\big] \;=\; \sum_i^{\rm N} \mathbf{r}_i \times \mathbf{F}_i \;=\; 0 \;\Big]$$

The angular momentum $[\,\mathbf{L}_2\,]$ of an isolated system of N particles remains constant if the internal forces obey Newton's third law in its strong form.

$$\mathbf{L}_2 \;=\; \text{constant} \qquad \Big[\; d(\mathbf{L}_2)/dt \;=\; \textstyle\sum_i^{\rm N} m_i\,\big[\,(\mathbf{r}_i - \mathbf{R}_{cm}) \times (\mathbf{a}_i - \mathbf{A}_{cm})\,\big] \;=$$

$$\textstyle\sum_i^{\rm N} m_i\,\big[\,(\mathbf{r}_i - \mathbf{R}_{cm}) \times \mathbf{a}_i\,\big] \;=\; \sum_i^{\rm N} (\mathbf{r}_i - \mathbf{R}_{cm}) \times \mathbf{F}_i \;=\; 0 \;\Big]$$

The mechanical energy $[\,\mathrm{E}_1\,]$ and the mechanical energy $[\,\mathrm{E}_2\,]$ of a system of N particles remain constant if the system is only subject to conservative forces.

$$\mathrm{E}_1 \;=\; \text{constant} \qquad \Big[\; \Delta\,\mathrm{E}_1 \;=\; \Delta\,\mathrm{K}_1 + \Delta\,\mathrm{U}_1 \;=\; 0 \;\Big]$$

$$\mathrm{E}_2 \;=\; \text{constant} \qquad \Big[\; \Delta\,\mathrm{E}_2 \;=\; \Delta\,\mathrm{K}_2 + \Delta\,\mathrm{U}_2 \;=\; 0 \;\Big]$$

The mechanical energy $[\,\mathrm{E}_3\,]$ and the mechanical energy $[\,\mathrm{E}_4\,]$ of a system of N particles are always zero (and therefore they always remain constant)

$$\mathrm{E}_3 \;=\; \text{constant} \qquad \Big[\; \mathrm{E}_3 \;=\; \textstyle\sum_i^{\rm N} 1/2\,\big[\,m_i\,\mathbf{a}_i \cdot \mathbf{r}_i - \mathbf{F}_i \cdot \mathbf{r}_i\,\big] \;=\; 0 \;\Big]$$

$$\mathrm{E}_4 \;=\; \text{constant} \qquad \Big[\; \mathrm{E}_4 \;=\; \textstyle\sum_i^{\rm N} 1/2\,\big[\,m_i\,\mathbf{a}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) - \mathbf{F}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm})\,\big] \;=\; 0 \;\Big]$$

$$\textstyle\sum_i^{\rm N} 1/2\; m_i\,\big[\,(\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm})\,\big] \;=\; \sum_i^{\rm N} 1/2\; m_i\,\mathbf{a}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm})$$

The mechanical energy $[\,\mathrm{E}_5\,]$ and the mechanical energy $[\,\mathrm{E}_6\,]$ of a system of N particles remain constant if the system is only subject to conservative forces.

$$\mathrm{E}_5 \;=\; \text{constant} \qquad \Big[\; \Delta\,\mathrm{E}_5 \;=\; \Delta\,\mathrm{K}_5 + \Delta\,\mathrm{U}_5 \;=\; 0 \;\Big]$$

$$\mathrm{E}_6 \;=\; \text{constant} \qquad \Big[\; \Delta\,\mathrm{E}_6 \;=\; \Delta\,\mathrm{K}_6 + \Delta\,\mathrm{U}_6 \;=\; 0 \;\Big]$$

Observations

All equations of this paper can be applied in any inertial reference frame and also in any non-inertial reference frame.

Additionally, inertial reference frames and non-inertial reference frames must not introduce fictitious forces into $\mathbf{F}_i$.

In this paper, the magnitudes [m, $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$, M, $\mathbf{R}$, $\mathbf{V}$, $\mathbf{A}$, $\mathbf{F}$, $\mathbf{P}_1$, $\mathbf{L}_1$, $\mathbf{L}_2$, W_1, K_1, U_1, E_1, L_1, W_2, K_2, U_2, E_2, L_2, W_3, K_3, U_3, E_3, W_4, K_4, U_4, E_4, W_5, K_5, U_5, E_5, W_6, K_6, U_6 and E_6] are invariant under transformations between inertial and non-inertial reference frames.

The mechanical energy E_3 of a system of particles is always zero [$E_3 = K_3 + U_3 = 0$]

Therefore, the mechanical energy E_5 of a system of particles is always equal to the mechanical energy E_1 of the system of particles [$E_5 = E_1$]

The mechanical energy E_4 of a system of particles is always zero [$E_4 = K_4 + U_4 = 0$]

Therefore, the mechanical energy E_6 of a system of particles is always equal to the mechanical energy E_2 of the system of particles [$E_6 = E_2$]

If the potential energy U_1 of a system of particles is a homogeneous function of degree k then the potential energy U_3 and the potential energy U_5 of the system of particles are given by: [$U_3 = (\frac{k}{2})\, U_1$] and [$U_5 = (1+\frac{k}{2})\, U_1$]

If the potential energy U_2 of a system of particles is a homogeneous function of degree k then the potential energy U_4 and the potential energy U_6 of the system of particles are given by: [$U_4 = (\frac{k}{2})\, U_2$] and [$U_6 = (1+\frac{k}{2})\, U_2$]

If the potential energy U_1 of a system of particles is a homogeneous function of degree k and if the kinetic energy K_5 of the system of particles is equal to zero, then we obtain: [$K_1 = -\,K_3 = U_3 = (\frac{k}{2})\, U_1 = (\frac{k}{2+k})\, E_1$]

If the potential energy U_2 of a system of particles is a homogeneous function of degree k and if the kinetic energy K_6 of the system of particles is equal to zero, then we obtain: [$K_2 = -\,K_4 = U_4 = (\frac{k}{2})\, U_2 = (\frac{k}{2+k})\, E_2$]

If the potential energy U_1 of a system of particles is a homogeneous function of degree k and if the average kinetic energy $\langle K_5 \rangle$ of the system of particles is equal to zero, then we obtain: [$\langle K_1 \rangle = -\,\langle K_3 \rangle = \langle U_3 \rangle = (\frac{k}{2})\, \langle U_1 \rangle = (\frac{k}{2+k})\, \langle E_1 \rangle$]

If the potential energy U_2 of a system of particles is a homogeneous function of degree k and if the average kinetic energy $\langle K_6 \rangle$ of the system of particles is equal to zero, then we obtain: [$\langle K_2 \rangle = -\,\langle K_4 \rangle = \langle U_4 \rangle = (\frac{k}{2})\, \langle U_2 \rangle = (\frac{k}{2+k})\, \langle E_2 \rangle$]

The average kinetic energy $\langle \mathrm{K}_5 \rangle$ and the average kinetic energy $\langle \mathrm{K}_6 \rangle$ of a system of particles with bounded motion (in $\langle \mathrm{K}_5 \rangle$ relative to $\vec{R}$ and in $\langle \mathrm{K}_6 \rangle$ relative to $\vec{R}_{cm}$) are always zero.

The kinetic energy K_5 and the kinetic energy K_6 of a system of N particles can also be expressed as follows : $[\ \mathrm{K}_5 = \sum_i^{\mathrm{N}} 1/2\ m_i\,(\,\dot{r}_i\,\dot{r}_i + \ddot{r}_i\,r_i\,)\ ]$ where $r_i \doteq |\,\vec{r}_i - \vec{R}\,|$ and $[\ \mathrm{K}_6 = \sum_{j>i}^{\mathrm{N}} 1/2\, m_i\, m_j\, \mathrm{M}^{-1}(\,\dot{r}_{ij}\,\dot{r}_{ij} + \ddot{r}_{ij}\,r_{ij}\,)\ ]$ where $r_{ij} \doteq |\,\vec{r}_i - \vec{r}_j\,|$

The kinetic energy K_5 and the kinetic energy K_6 of a system of N particles can also be expressed as follows : $[\ \mathrm{K}_5 = \sum_i^{\mathrm{N}} 1/2\ m_i\,(\,\ddot{\tau}_i\,)\ ]$ where $\tau_i \doteq 1/2\ (\vec{r}_i - \vec{R}) \cdot (\vec{r}_i - \vec{R})$ and $[\ \mathrm{K}_6 = \sum_{j>i}^{\mathrm{N}} 1/2\, m_i\, m_j\, \mathrm{M}^{-1}(\,\ddot{\tau}_{ij}\,)\ ]$ where $\tau_{ij} \doteq 1/2\ (\vec{r}_i - \vec{r}_j) \cdot (\vec{r}_i - \vec{r}_j)$

The kinetic energy K_6 is the only kinetic energy that can be expressed without the necessity of introducing any magnitude that is related to the Universe [such as: $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$, $\vec{\omega}$, $\vec{R}$, etc.]

In an isolated system of particles, the potential energy U_2 is equal to the potential energy U_1 if the internal forces obey Newton's third law in its weak form $[\,\mathrm{U}_2 = \mathrm{U}_1\,]$

In an isolated system of particles, the potential energy U_4 is equal to the potential energy U_3 if the internal forces obey Newton's third law in its weak form $[\,\mathrm{U}_4 = \mathrm{U}_3\,]$

In an isolated system of particles, the potential energy U_6 is equal to the potential energy U_5 if the internal forces obey Newton's third law in its weak form $[\,\mathrm{U}_6 = \mathrm{U}_5\,]$

A reference frame S is non-rotating if the angular velocity $\vec{\omega}$ of the Universe relative to S is equal to zero, and the reference frame S is also inertial if the acceleration $\vec{A}$ of the center of mass of the Universe relative to S is equal to zero.

If the origin of a non-rotating reference frame S $[\,\vec{\omega} = 0\,]$ always coincides with the center of mass of the Universe $[\,\vec{R} = \vec{V} = \vec{A} = 0\,]$ then relative to S: $[\,\mathbf{r}_i = \vec{r}_i,\ \mathbf{v}_i = \vec{v}_i$ and $\mathbf{a}_i = \vec{a}_i\,]$ Therefore, it is easy to see that always: $[\,\mathbf{v}_i = d(\mathbf{r}_i)/dt$ and $\mathbf{a}_i = d^2(\mathbf{r}_i)/dt^2\,]$

This paper does not contradict Newton's first and second laws since these two laws are valid in all inertial reference frames. The equation $[\ \mathbf{F}_i = m_i\,\mathbf{a}_i\ ]$ is a simple reformulation of Newton's second law.

In this paper, the equation $[\ \mathbf{F}_i = m_i\,\mathbf{a}_i\ ]$ would be false in all reference frames (inertial or non-inertial) if a new force were always disobeyed Newton's third law in its strong form or in its weak form.

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Appendix I

The Universe

The Universe is a system that contains all particles, that is always free of external forces, and that all internal forces always obey Newton's third law in its weak form and in its strong form.

The position $\vec{R}$, the velocity $\vec{V}$ and the acceleration $\vec{A}$ of the center of mass of the Universe relative to a reference frame S (and the angular velocity $\vec{\omega}$ and the angular acceleration $\vec{\alpha}$ of the Universe relative to the reference frame S) are given by:

$$\mathrm{M} \doteq \textstyle\sum_i^{All} m_i$$

$$\vec{R} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{All} m_i\, \vec{r}_i$$

$$\vec{V} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{All} m_i\, \vec{v}_i$$

$$\vec{A} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{All} m_i\, \vec{a}_i$$

$$\vec{\omega} \doteq \overleftrightarrow{I}^{-1} \cdot \vec{L}$$

$$\vec{\alpha} \doteq d(\vec{\omega})/dt$$

$$\overleftrightarrow{I} \doteq \textstyle\sum_i^{All} m_i \, [\, |\, \vec{r}_i - \vec{R}\, |^2\, \overleftrightarrow{1} - (\vec{r}_i - \vec{R}) \otimes (\vec{r}_i - \vec{R})\,]$$

$$\vec{L} \doteq \textstyle\sum_i^{All} m_i \, (\vec{r}_i - \vec{R}) \times (\vec{v}_i - \vec{V})$$

where M is the mass of the Universe, $\overleftrightarrow{I}$ is the inertia tensor of the Universe (relative to $\vec{R}$) and $\vec{L}$ is the angular momentum of the Universe relative to the reference frame S.

The Transformations

$$(\vec{r}_i - \vec{R}) \doteq \mathbf{r}_i = \mathbf{r}'_i$$

$$(\vec{r}'_i - \vec{R}') \doteq \mathbf{r}'_i = \mathbf{r}_i$$

$$(\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R}) \doteq \mathbf{v}_i = \mathbf{v}'_i$$

$$(\vec{v}'_i - \vec{V}') - \vec{\omega}' \times (\vec{r}'_i - \vec{R}') \doteq \mathbf{v}'_i = \mathbf{v}_i$$

$$(\vec{a}_i - \vec{A}) - 2\,\vec{\omega} \times (\vec{v}_i - \vec{V}) + \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,] - \vec{\alpha} \times (\vec{r}_i - \vec{R}) \doteq \mathbf{a}_i = \mathbf{a}'_i$$

$$(\vec{a}'_i - \vec{A}') - 2\,\vec{\omega}' \times (\vec{v}'_i - \vec{V}') + \vec{\omega}' \times [\,\vec{\omega}' \times (\vec{r}'_i - \vec{R}')\,] - \vec{\alpha}' \times (\vec{r}'_i - \vec{R}') \doteq \mathbf{a}'_i = \mathbf{a}_i$$

Appendix II

The Relations

In a system of particles, these relations can be obtained (The magnitudes $\mathbf{R}_{cm}$, $\mathbf{V}_{cm}$, $\mathbf{A}_{cm}$, $\vec{R}_{cm}$, $\vec{V}_{cm}$ and $\vec{A}_{cm}$ can be replaced by the magnitudes $\mathbf{R}$, $\mathbf{V}$, $\mathbf{A}$, $\vec{R}$, $\vec{V}$ and $\vec{A}$, or by the magnitudes $\mathbf{r}_j$, $\mathbf{v}_j$, $\mathbf{a}_j$, $\vec{r}_j$, $\vec{v}_j$ and $\vec{a}_j$, respectively. On the other hand, $\mathbf{R} = \mathbf{V} = \mathbf{A} = 0$)

$$\mathbf{r}_i \doteq (\vec{r}_i - \vec{R})$$

$$\mathbf{R}_{cm} \doteq (\vec{R}_{cm} - \vec{R})$$

$$\longrightarrow \quad (\mathbf{r}_i - \mathbf{R}_{cm}) = (\vec{r}_i - \vec{R}_{cm})$$

$$\mathbf{v}_i \doteq (\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R})$$

$$\mathbf{V}_{cm} \doteq (\vec{V}_{cm} - \vec{V}) - \vec{\omega} \times (\vec{R}_{cm} - \vec{R})$$

$$\longrightarrow \quad (\mathbf{v}_i - \mathbf{V}_{cm}) = (\vec{v}_i - \vec{V}_{cm}) - \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})$$

$$(\mathbf{v}_i - \mathbf{V}_{cm}) \cdot (\mathbf{v}_i - \mathbf{V}_{cm}) = \big[(\vec{v}_i - \vec{V}_{cm}) - \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] \cdot \big[(\vec{v}_i - \vec{V}_{cm}) - \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] =$$

$$(\vec{v}_i - \vec{V}_{cm}) \cdot (\vec{v}_i - \vec{V}_{cm}) - 2\,(\vec{v}_i - \vec{V}_{cm}) \cdot \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] + \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] \cdot \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] =$$

$$(\vec{v}_i - \vec{V}_{cm}) \cdot (\vec{v}_i - \vec{V}_{cm}) + 2\,(\vec{r}_i - \vec{R}_{cm}) \cdot \big[\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] + \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] \cdot \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] =$$

$$(\vec{v}_i - \vec{V}_{cm}) \cdot (\vec{v}_i - \vec{V}_{cm}) + \big[2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] \cdot \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] =$$

$$(\vec{v}_i - \vec{V}_{cm})^2 + \big[2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big]^2$$

$$(\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) = \big\{ (\vec{a}_i - \vec{A}_{cm}) - 2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) + \vec{\omega} \times [\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})] -$$

$$\vec{\alpha} \times (\vec{r}_i - \vec{R}_{cm}) \big\} \cdot (\vec{r}_i - \vec{R}_{cm}) = (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) - \big[2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) +$$

$$\big\{ \vec{\omega} \times [\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})] \big\} \cdot (\vec{r}_i - \vec{R}_{cm}) - \big[\vec{\alpha} \times (\vec{r}_i - \vec{R}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) = (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) -$$

$$\big[2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big\{ \big[\vec{\omega} \cdot (\vec{r}_i - \vec{R}_{cm}) \big] \vec{\omega} - (\vec{\omega} \cdot \vec{\omega})\,(\vec{r}_i - \vec{R}_{cm}) \big\} \cdot (\vec{r}_i - \vec{R}_{cm}) =$$

$$(\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) - \big[2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big[\vec{\omega} \cdot (\vec{r}_i - \vec{R}_{cm}) \big]^2 - (\vec{\omega})^2\,(\vec{r}_i - \vec{R}_{cm})^2$$

$$\longrightarrow \quad (\mathbf{v}_i - \mathbf{V}_{cm})^2 + (\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) = (\vec{v}_i - \vec{V}_{cm})^2 + (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm})$$

Appendix III

The Magnitudes

The magnitudes $\mathbf{L}_2$, W_2, K_2, U_2, W_4, K_4, U_4, W_6, K_6 and U_6 of a system of N particles can also be expressed as follows:

$$\mathbf{L}_2 \;=\; \textstyle\sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\, (\mathbf{r}_i - \mathbf{r}_j) \times (\mathbf{v}_i - \mathbf{v}_j) \,\big]$$

$$W_2 \;=\; \textstyle\sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\mathbf{r}_i - \mathbf{r}_j) \,\big]$$

$$\Delta\, K_2 \;=\; \textstyle\sum_{j>i}^{N} \Delta\, {}^1\!/_2\, m_i\, m_j\, M^{-1}\, (\mathbf{v}_i - \mathbf{v}_j)^2 \;=\; W_2$$

$$\Delta\, U_2 \;=\; -\, \textstyle\sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\mathbf{r}_i - \mathbf{r}_j) \,\big]$$

$$W_4 \;=\; \textstyle\sum_{j>i}^{N} \Delta\, {}^1\!/_2\, m_i\, m_j\, M^{-1} \big[\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\mathbf{r}_i - \mathbf{r}_j) \,\big]$$

$$\Delta\, K_4 \;=\; \textstyle\sum_{j>i}^{N} \Delta\, {}^1\!/_2\, m_i\, m_j\, M^{-1} \big[\, (\mathbf{a}_i - \mathbf{a}_j) \cdot (\mathbf{r}_i - \mathbf{r}_j) \,\big] \;=\; W_4$$

$$\Delta\, U_4 \;=\; -\, \textstyle\sum_{j>i}^{N} \Delta\, {}^1\!/_2\, m_i\, m_j\, M^{-1} \big[\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\mathbf{r}_i - \mathbf{r}_j) \,\big]$$

$$W_6 \;=\; \textstyle\sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\vec{r}_i - \vec{r}_j) + \Delta\, {}^1\!/_2\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\vec{r}_i - \vec{r}_j) \,\big]$$

$$\Delta\, K_6 \;=\; \textstyle\sum_{j>i}^{N} \Delta\, {}^1\!/_2\, m_i\, m_j\, M^{-1} \big[\, (\vec{v}_i - \vec{v}_j)^2 + (\vec{a}_i - \vec{a}_j) \cdot (\vec{r}_i - \vec{r}_j) \,\big] \;=\; W_6$$

$$\Delta\, U_6 \;=\; -\, \textstyle\sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\vec{r}_i - \vec{r}_j) + \Delta\, {}^1\!/_2\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\vec{r}_i - \vec{r}_j) \,\big]$$

The magnitudes $W_{(1 \text{ to } 6)}$ and $U_{(1 \text{ to } 6)}$ of an isolated system of N particles, whose internal forces obey Newton's third law in its weak form, can be reduced to:

$$W_1 \;=\; W_2 \;=\; \textstyle\sum_i^N \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i$$

$$\Delta\, U_1 \;=\; \Delta\, U_2 \;=\; -\, \textstyle\sum_i^N \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i$$

$$W_3 \;=\; W_4 \;=\; \textstyle\sum_i^N \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \vec{r}_i$$

$$\Delta\, U_3 \;=\; \Delta\, U_4 \;=\; -\, \textstyle\sum_i^N \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \vec{r}_i$$

$$W_5 \;=\; W_6 \;=\; \textstyle\sum_i^N \big[\, \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \vec{r}_i \,\big]$$

$$\Delta\, U_5 \;=\; \Delta\, U_6 \;=\; -\, \textstyle\sum_i^N \big[\, \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \vec{r}_i \,\big]$$

A New Theory in Relational Mechanics

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(Paper II)

In relational mechanics, a new theory is presented, which is invariant under transformations between inertial and non-inertial reference frames, which can be applied in any reference frame without introducing fictitious forces and which establishes the existence of a new universal force of interaction, called kinetic force.

Introduction

The new theory in relational mechanics presented in this paper is obtained starting from an auxiliary system of particles (called Universe) that is used to obtain kinematic magnitudes (such as universal position, universal velocity, etc.) that are invariant under transformations between inertial and non-inertial reference frames.

The universal position $\mathbf{r}_i$, the universal velocity $\mathbf{v}_i$ and the universal acceleration $\mathbf{a}_i$ of a particle i are given by:

$$\mathbf{r}_i \doteq (\vec{r}_i - \vec{R})$$

$$\mathbf{v}_i \doteq (\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R})$$

$$\mathbf{a}_i \doteq (\vec{a}_i - \vec{A}) - 2\,\vec{\omega} \times (\vec{v}_i - \vec{V}) + \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,] - \vec{\alpha} \times (\vec{r}_i - \vec{R})$$

($\mathbf{v}_i \doteq d(\mathbf{r}_i)/dt$) and ($\mathbf{a}_i \doteq d^2(\mathbf{r}_i)/dt^2$) where $\vec{r}_i$ is the position vector of particle i, $\vec{R}$ is the position vector of the center of mass of the Universe, and $\vec{\omega}$ is the angular velocity vector of the Universe (see Appendix I)

A reference frame S is non-rotating if the angular velocity $\vec{\omega}$ of the Universe relative to S is equal to zero, and the reference frame S is also inertial if the acceleration $\vec{A}$ of the center of mass of the Universe relative to S is equal to zero.

The New Dynamics

[1] A force is always caused by the interaction between two or more particles.

[2] The total force $\mathbf{T}_i$ acting on a particle i is always zero [$\mathbf{T}_i = 0$]

[3] In this paper, we assume that all non-kinetic forces always obey Newton's third law in its weak form and in its strong form.

The Kinetic Force

The kinetic force $\mathbf{K}_{ij}$ exerted on a particle i of mass m_i by another particle j of mass m_j, caused by the interaction between particle i and particle j, is given by:

$$\mathbf{K}_{ij} \;=\; -\, m_i\, m_j\, \mathrm{M}^{-1}\, (\mathbf{a}_i - \mathbf{a}_j)$$

where $\mathbf{a}_i$ is the universal acceleration of particle i, $\mathbf{a}_j$ is the universal acceleration of particle j, and M is the mass of the Universe.

From the above equation it follows that the net kinetic force $\mathbf{K}_i$ ($= \sum_j^{All} \mathbf{K}_{ij}$) acting on a particle i of mass m_i is given by:

$$\mathbf{K}_i \;=\; -\, m_i\, (\mathbf{a}_i - \mathbf{A}_{cm})$$

where $\mathbf{a}_i$ is the universal acceleration of particle i and $\mathbf{A}_{cm}$ is the universal acceleration of the center of mass of the Universe.

Since the universal acceleration of the center of mass of the Universe $\mathbf{A}_{cm}$ is always zero, then the net kinetic force $\mathbf{K}_i$ acting on a particle i of mass m_i is certainly given by:

$$\mathbf{K}_i \;=\; -\, m_i\, \mathbf{a}_i$$

where $\mathbf{a}_i$ is the universal acceleration of particle i.

The kinetic force $\mathbf{K}_{ij}$ is considered in the new dynamics, mainly in the [2] principle, as a new universal force of interaction.

Finally, the kinetic force $\mathbf{K}_{ij}$ always obey Newton's third law in its weak form.

The [2] Principle

The second principle of the new dynamics establishes that the total force $\mathbf{T}_i$ acting on a particle i is always zero.

$$\mathbf{T}_i \;=\; 0$$

If the total force $\mathbf{T}_i$ is divided into the following two parts: the net kinetic force $\mathbf{K}_i$ and the net non-kinetic force $\mathbf{F}_i$ ($\sum$ of gravitational forces, electrostatic forces, etc.) then we have:

$$\mathbf{K}_i + \mathbf{F}_i \;=\; 0$$

Now, substituting ($\mathbf{K}_i \;=\; -\, m_i\, \mathbf{a}_i$) and rearranging, we finally obtain:

$$\mathbf{F}_i \;=\; m_i\, \mathbf{a}_i$$

This equation (similar to Newton's second law) will be used throughout this paper.

On the other hand, in this paper a system of particles is isolated when the system is free of external non-kinetic forces.

The Definitions

For a system of N particles, the following definitions are applicable:

Mass	$\mathrm{M} \doteq \sum_i^{\mathrm{N}} m_i$
Position CM 1	$\vec{R}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \vec{r}_i$
Velocity CM 1	$\vec{V}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \vec{v}_i$
Acceleration CM 1	$\vec{A}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \vec{a}_i$
Position CM 2	$\mathbf{R}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \mathbf{r}_i$
Velocity CM 2	$\mathbf{V}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \mathbf{v}_i$
Acceleration CM 2	$\mathbf{A}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \mathbf{a}_i$
Linear Momentum 1	$\mathbf{P}_1 \doteq \sum_i^{\mathrm{N}} m_i \, \mathbf{v}_i$
Angular Momentum 1	$\mathbf{L}_1 \doteq \sum_i^{\mathrm{N}} m_i \left[\mathbf{r}_i \times \mathbf{v}_i \right]$
Angular Momentum 2	$\mathbf{L}_2 \doteq \sum_i^{\mathrm{N}} m_i \left[(\mathbf{r}_i - \mathbf{R}_{cm}) \times (\mathbf{v}_i - \mathbf{V}_{cm}) \right]$
Work 1	$\mathrm{W}_1 \doteq \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i = \Delta \mathrm{K}_1$
Kinetic Energy 1	$\Delta \mathrm{K}_1 \doteq \sum_i^{\mathrm{N}} \Delta \, {}^1\!/_2 \, m_i \, (\mathbf{v}_i)^2$
Potential Energy 1	$\Delta \mathrm{U}_1 \doteq - \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i$
Mechanical Energy 1	$\mathrm{E}_1 \doteq \mathrm{K}_1 + \mathrm{U}_1$
Lagrangian 1	$\mathrm{L}_1 \doteq \mathrm{K}_1 - \mathrm{U}_1$
Work 2	$\mathrm{W}_2 \doteq \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d(\mathbf{r}_i - \mathbf{R}_{cm}) = \Delta \mathrm{K}_2$
Kinetic Energy 2	$\Delta \mathrm{K}_2 \doteq \sum_i^{\mathrm{N}} \Delta \, {}^1\!/_2 \, m_i \, (\mathbf{v}_i - \mathbf{V}_{cm})^2$
Potential Energy 2	$\Delta \mathrm{U}_2 \doteq - \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d(\mathbf{r}_i - \mathbf{R}_{cm})$
Mechanical Energy 2	$\mathrm{E}_2 \doteq \mathrm{K}_2 + \mathrm{U}_2$
Lagrangian 2	$\mathrm{L}_2 \doteq \mathrm{K}_2 - \mathrm{U}_2$

Work 3 $\quad \mathrm{W}_3 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \mathbf{r}_i = \Delta\,\mathrm{K}_3$

Kinetic Energy 3 $\quad \Delta\,\mathrm{K}_3 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i\, \mathbf{a}_i \cdot \mathbf{r}_i$

Potential Energy 3 $\quad \Delta\,\mathrm{U}_3 \doteq -\sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \mathbf{r}_i$

Mechanical Energy 3 $\quad \mathrm{E}_3 \doteq \mathrm{K}_3 + \mathrm{U}_3$

Work 4 $\quad \mathrm{W}_4 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) = \Delta\,\mathrm{K}_4$

Kinetic Energy 4 $\quad \Delta\,\mathrm{K}_4 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i \left[(\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) \right]$

Potential Energy 4 $\quad \Delta\,\mathrm{U}_4 \doteq -\sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm})$

Mechanical Energy 4 $\quad \mathrm{E}_4 \doteq \mathrm{K}_4 + \mathrm{U}_4$

Work 5 $\quad \mathrm{W}_5 \doteq \sum_i^{\mathrm{N}} \left[\int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}) \right] = \Delta\,\mathrm{K}_5$

Kinetic Energy 5 $\quad \Delta\,\mathrm{K}_5 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i \left[(\vec{v}_i - \vec{V})^2 + (\vec{a}_i - \vec{A}) \cdot (\vec{r}_i - \vec{R}) \right]$

Potential Energy 5 $\quad \Delta\,\mathrm{U}_5 \doteq -\sum_i^{\mathrm{N}} \left[\int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}) \right]$

Mechanical Energy 5 $\quad \mathrm{E}_5 \doteq \mathrm{K}_5 + \mathrm{U}_5$

Work 6 $\quad \mathrm{W}_6 \doteq \sum_i^{\mathrm{N}} \left[\int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}_{cm}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}_{cm}) \right] = \Delta\,\mathrm{K}_6$

Kinetic Energy 6 $\quad \Delta\,\mathrm{K}_6 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i \left[(\vec{v}_i - \vec{V}_{cm})^2 + (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) \right]$

Potential Energy 6 $\quad \Delta\,\mathrm{U}_6 \doteq -\sum_i^{\mathrm{N}} \left[\int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}_{cm}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}_{cm}) \right]$

Mechanical Energy 6 $\quad \mathrm{E}_6 \doteq \mathrm{K}_6 + \mathrm{U}_6$

The Relations

From the above definitions, the following relations can be obtained (see Appendix II)

$$\mathrm{K}_1 = \mathrm{K}_2 + {}^1\!/_2\, \mathrm{M}\, \mathbf{V}_{cm}^2$$

$$\mathrm{K}_3 = \mathrm{K}_4 + {}^1\!/_2\, \mathrm{M}\, \mathbf{A}_{cm} \cdot \mathbf{R}_{cm}$$

$$\mathrm{K}_5 = \mathrm{K}_6 + {}^1\!/_2\, \mathrm{M} \left[(\vec{V}_{cm} - \vec{V})^2 + (\vec{A}_{cm} - \vec{A}) \cdot (\vec{R}_{cm} - \vec{R}) \right]$$

$$\mathrm{K}_5 = \mathrm{K}_1 + \mathrm{K}_3 \quad \& \quad \mathrm{U}_5 = \mathrm{U}_1 + \mathrm{U}_3 \quad \& \quad \mathrm{E}_5 = \mathrm{E}_1 + \mathrm{E}_3$$

$$\mathrm{K}_6 = \mathrm{K}_2 + \mathrm{K}_4 \quad \& \quad \mathrm{U}_6 = \mathrm{U}_2 + \mathrm{U}_4 \quad \& \quad \mathrm{E}_6 = \mathrm{E}_2 + \mathrm{E}_4$$

The Principles

The linear momentum $[\,\mathbf{P}_1\,]$ of an isolated system of N particles remains constant if the internal non-kinetic forces obey Newton's third law in its weak form.

$$\mathbf{P}_1 \;=\; \text{constant} \qquad \left[\, d(\mathbf{P}_1)/dt \;=\; \textstyle\sum_i^{\rm N} m_i\, \mathbf{a}_i \;=\; \textstyle\sum_i^{\rm N} \mathbf{F}_i \;=\; 0 \,\right]$$

The angular momentum $[\,\mathbf{L}_1\,]$ of an isolated system of N particles remains constant if the internal non-kinetic forces obey Newton's third law in its strong form.

$$\mathbf{L}_1 \;=\; \text{constant} \qquad \left[\, d(\mathbf{L}_1)/dt \;=\; \textstyle\sum_i^{\rm N} m_i \left[\, \mathbf{r}_i \times \mathbf{a}_i \,\right] \;=\; \textstyle\sum_i^{\rm N} \mathbf{r}_i \times \mathbf{F}_i \;=\; 0 \,\right]$$

The angular momentum $[\,\mathbf{L}_2\,]$ of an isolated system of N particles remains constant if the internal non-kinetic forces obey Newton's third law in its strong form.

$$\mathbf{L}_2 \;=\; \text{constant} \qquad \left[\, d(\mathbf{L}_2)/dt \;=\; \textstyle\sum_i^{\rm N} m_i \left[\, (\mathbf{r}_i - \mathbf{R}_{cm}) \times (\mathbf{a}_i - \mathbf{A}_{cm}) \,\right] \;=\; \right.$$

$$\left. \textstyle\sum_i^{\rm N} m_i \left[\, (\mathbf{r}_i - \mathbf{R}_{cm}) \times \mathbf{a}_i \,\right] \;=\; \textstyle\sum_i^{\rm N} (\mathbf{r}_i - \mathbf{R}_{cm}) \times \mathbf{F}_i \;=\; 0 \,\right]$$

The mechanical energy $[\,\mathrm{E}_1\,]$ and the mechanical energy $[\,\mathrm{E}_2\,]$ of a system of N particles remain constant if the system is only subject to kinetic forces and to conservative non-kinetic forces.

$$\mathrm{E}_1 \;=\; \text{constant} \qquad \left[\, \Delta\, \mathrm{E}_1 \;=\; \Delta\, \mathrm{K}_1 + \Delta\, \mathrm{U}_1 \;=\; 0 \,\right]$$

$$\mathrm{E}_2 \;=\; \text{constant} \qquad \left[\, \Delta\, \mathrm{E}_2 \;=\; \Delta\, \mathrm{K}_2 + \Delta\, \mathrm{U}_2 \;=\; 0 \,\right]$$

The mechanical energy $[\,\mathrm{E}_3\,]$ and the mechanical energy $[\,\mathrm{E}_4\,]$ of a system of N particles are always zero (and therefore they always remain constant)

$$\mathrm{E}_3 \;=\; \text{constant} \qquad \left[\, \mathrm{E}_3 \;=\; \textstyle\sum_i^{\rm N} {}^1\!/_2 \left[\, m_i\, \mathbf{a}_i \cdot \mathbf{r}_i - \mathbf{F}_i \cdot \mathbf{r}_i \,\right] \;=\; 0 \,\right]$$

$$\mathrm{E}_4 \;=\; \text{constant} \qquad \left[\, \mathrm{E}_4 \;=\; \textstyle\sum_i^{\rm N} {}^1\!/_2 \left[\, m_i\, \mathbf{a}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) - \mathbf{F}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) \,\right] \;=\; 0 \,\right]$$

$$\textstyle\sum_i^{\rm N} {}^1\!/_2\, m_i \left[\, (\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) \,\right] \;=\; \textstyle\sum_i^{\rm N} {}^1\!/_2\, m_i\, \mathbf{a}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm})$$

The mechanical energy $[\,\mathrm{E}_5\,]$ and the mechanical energy $[\,\mathrm{E}_6\,]$ of a system of N particles remain constant if the system is only subject to kinetic forces and to conservative non-kinetic forces.

$$\mathrm{E}_5 \;=\; \text{constant} \qquad \left[\, \Delta\, \mathrm{E}_5 \;=\; \Delta\, \mathrm{K}_5 + \Delta\, \mathrm{U}_5 \;=\; 0 \,\right]$$

$$\mathrm{E}_6 \;=\; \text{constant} \qquad \left[\, \Delta\, \mathrm{E}_6 \;=\; \Delta\, \mathrm{K}_6 + \Delta\, \mathrm{U}_6 \;=\; 0 \,\right]$$

Observations

All equations of this paper can be applied in any inertial reference frame and also in any non-inertial reference frame.

Additionally, inertial reference frames and non-inertial reference frames must not introduce fictitious forces into $\mathbf{F}_i$.

In this paper, the magnitudes $[\, m, \mathbf{r}, \mathbf{v}, \mathbf{a}, \mathrm{M}, \mathbf{R}, \mathbf{V}, \mathbf{A}, \mathbf{T}, \mathbf{K}, \mathbf{F}, \mathbf{P}_1, \mathbf{L}_1, \mathbf{L}_2, \mathrm{W}_1, \mathrm{K}_1, \mathrm{U}_1, \mathrm{E}_1, \mathrm{L}_1$ W_2, K_2, U_2, E_2, L_2, W_3, K_3, U_3, E_3, W_4, K_4, U_4, E_4, W_5, K_5, U_5, E_5, W_6, K_6, U_6 and $\mathrm{E}_6\,]$ are invariant under transformations between inertial and non-inertial reference frames.

The mechanical energy E_3 of a system of particles is always zero $[\, \mathrm{E}_3 = \mathrm{K}_3 + \mathrm{U}_3 = 0\,]$

Therefore, the mechanical energy E_5 of a system of particles is always equal to the mechanical energy E_1 of the system of particles $[\, \mathrm{E}_5 = \mathrm{E}_1\,]$

The mechanical energy E_4 of a system of particles is always zero $[\, \mathrm{E}_4 = \mathrm{K}_4 + \mathrm{U}_4 = 0\,]$

Therefore, the mechanical energy E_6 of a system of particles is always equal to the mechanical energy E_2 of the system of particles $[\, \mathrm{E}_6 = \mathrm{E}_2\,]$

If the potential energy U_1 of a system of particles is a homogeneous function of degree k then the potential energy U_3 and the potential energy U_5 of the system of particles are given by: $[\, \mathrm{U}_3 = (\frac{k}{2})\, \mathrm{U}_1\,]$ and $[\, \mathrm{U}_5 = (1+\frac{k}{2})\, \mathrm{U}_1\,]$

If the potential energy U_2 of a system of particles is a homogeneous function of degree k then the potential energy U_4 and the potential energy U_6 of the system of particles are given by: $[\, \mathrm{U}_4 = (\frac{k}{2})\, \mathrm{U}_2\,]$ and $[\, \mathrm{U}_6 = (1+\frac{k}{2})\, \mathrm{U}_2\,]$

If the potential energy U_1 of a system of particles is a homogeneous function of degree k and if the kinetic energy K_5 of the system of particles is equal to zero, then we obtain: $[\, \mathrm{K}_1 = -\, \mathrm{K}_3 = \mathrm{U}_3 = (\frac{k}{2})\, \mathrm{U}_1 = (\frac{k}{2+k})\, \mathrm{E}_1\,]$

If the potential energy U_2 of a system of particles is a homogeneous function of degree k and if the kinetic energy K_6 of the system of particles is equal to zero, then we obtain: $[\, \mathrm{K}_2 = -\, \mathrm{K}_4 = \mathrm{U}_4 = (\frac{k}{2})\, \mathrm{U}_2 = (\frac{k}{2+k})\, \mathrm{E}_2\,]$

If the potential energy U_1 of a system of particles is a homogeneous function of degree k and if the average kinetic energy $\langle \mathrm{K}_5 \rangle$ of the system of particles is equal to zero, then we obtain: $[\, \langle \mathrm{K}_1 \rangle = -\, \langle \mathrm{K}_3 \rangle = \langle \mathrm{U}_3 \rangle = (\frac{k}{2})\, \langle \mathrm{U}_1 \rangle = (\frac{k}{2+k})\, \langle \mathrm{E}_1 \rangle\,]$

If the potential energy U_2 of a system of particles is a homogeneous function of degree k and if the average kinetic energy $\langle \mathrm{K}_6 \rangle$ of the system of particles is equal to zero, then we obtain: $[\, \langle \mathrm{K}_2 \rangle = -\, \langle \mathrm{K}_4 \rangle = \langle \mathrm{U}_4 \rangle = (\frac{k}{2})\, \langle \mathrm{U}_2 \rangle = (\frac{k}{2+k})\, \langle \mathrm{E}_2 \rangle\,]$

The average kinetic energy $\langle K_5 \rangle$ and the average kinetic energy $\langle K_6 \rangle$ of a system of particles with bounded motion (in $\langle K_5 \rangle$ relative to $\vec{R}$ and in $\langle K_6 \rangle$ relative to $\vec{R}_{cm}$) are always zero.

The kinetic energy K_5 and the kinetic energy K_6 of a system of N particles can also be expressed as follows : $[\ K_5 = \sum_i^N 1/2\ m_i\,(\,\dot{r}_i\,\dot{r}_i + \ddot{r}_i\,r_i\,)\ ]$ where $r_i \doteq |\,\vec{r}_i - \vec{R}\,|$ and $[\ K_6 = \sum_{j>i}^N 1/2\, m_i\, m_j\, M^{-1}(\,\dot{r}_{ij}\,\dot{r}_{ij} + \ddot{r}_{ij}\, r_{ij}\,)\ ]$ where $r_{ij} \doteq |\,\vec{r}_i - \vec{r}_j\,|$

The kinetic energy K_5 and the kinetic energy K_6 of a system of N particles can also be expressed as follows : $[\ K_5 = \sum_i^N 1/2\ m_i\,(\,\ddot{\tau}_i\,)\ ]$ where $\tau_i \doteq 1/2\ (\vec{r}_i - \vec{R}) \cdot (\vec{r}_i - \vec{R})$ and $[\ K_6 = \sum_{j>i}^N 1/2\, m_i\, m_j\, M^{-1}(\,\ddot{\tau}_{ij}\,)\ ]$ where $\tau_{ij} \doteq 1/2\ (\vec{r}_i - \vec{r}_j) \cdot (\vec{r}_i - \vec{r}_j)$

The kinetic energy K_6 is the only kinetic energy that can be expressed without the necessity of introducing any magnitude that is related to the Universe [such as: $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$, $\vec{\omega}$, $\vec{R}$, etc.]

In an isolated system of particles, the potential energy U_2 is equal to the potential energy U_1 if the internal non-kinetic forces obey Newton's third law in its weak form $[\,U_2 = U_1\,]$

In an isolated system of particles, the potential energy U_4 is equal to the potential energy U_3 if the internal non-kinetic forces obey Newton's third law in its weak form $[\,U_4 = U_3\,]$

In an isolated system of particles, the potential energy U_6 is equal to the potential energy U_5 if the internal non-kinetic forces obey Newton's third law in its weak form $[\,U_6 = U_5\,]$

A reference frame S is non-rotating if the angular velocity $\vec{\omega}$ of the Universe relative to S is equal to zero, and the reference frame S is also inertial if the acceleration $\vec{A}$ of the center of mass of the Universe relative to S is equal to zero.

If the origin of a non-rotating reference frame S $[\,\vec{\omega} = 0\,]$ always coincides with the center of mass of the Universe $[\,\vec{R} = \vec{V} = \vec{A} = 0\,]$ then relative to S: $[\,\mathbf{r}_i = \vec{r}_i,\ \mathbf{v}_i = \vec{v}_i$ and $\mathbf{a}_i = \vec{a}_i\,]$ Therefore, it is easy to see that always: $[\,\mathbf{v}_i = d(\mathbf{r}_i)/dt$ and $\mathbf{a}_i = d^2(\mathbf{r}_i)/dt^2\,]$

If kinetic forces are excluded, then this paper does not contradict Newton's first and second laws since they are valid in all inertial reference frames. The equation $[\,\mathbf{F}_i = m_i\,\mathbf{a}_i\,]$ is a simple reformulation of Newton's second law.

In this paper, the equation $[\,\mathbf{F}_i = m_i\,\mathbf{a}_i\,]$ would be false in all reference frames (inertial or non-inertial) if a new non-kinetic force were always disobeyed Newton's third law in its strong form or in its weak form.

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Appendix I

The Universe

The Universe is a system that contains all particles, that is always free of external forces, and that all internal non-kinetic forces always obey Newton's third law in its weak form and in its strong form.

The position $\vec{R}$, the velocity $\vec{V}$ and the acceleration $\vec{A}$ of the center of mass of the Universe relative to a reference frame S (and the angular velocity $\vec{\omega}$ and the angular acceleration $\vec{\alpha}$ of the Universe relative to the reference frame S) are given by:

$$\mathrm{M} \doteq \textstyle\sum_i^{All} m_i$$

$$\vec{R} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{All} m_i\, \vec{r}_i$$

$$\vec{V} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{All} m_i\, \vec{v}_i$$

$$\vec{A} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{All} m_i\, \vec{a}_i$$

$$\vec{\omega} \doteq \overleftrightarrow{I}^{-1} \cdot \vec{L}$$

$$\vec{\alpha} \doteq d(\vec{\omega})/dt$$

$$\overleftrightarrow{I} \doteq \textstyle\sum_i^{All} m_i \,[\, |\vec{r}_i - \vec{R}\,|^2\, \overleftrightarrow{1} - (\vec{r}_i - \vec{R}) \otimes (\vec{r}_i - \vec{R})\,]$$

$$\vec{L} \doteq \textstyle\sum_i^{All} m_i\, (\vec{r}_i - \vec{R}) \times (\vec{v}_i - \vec{V})$$

where M is the mass of the Universe, $\overleftrightarrow{I}$ is the inertia tensor of the Universe (relative to $\vec{R}$) and $\vec{L}$ is the angular momentum of the Universe relative to the reference frame S.

The Transformations

$$(\vec{r}_i - \vec{R}) \doteq \mathbf{r}_i = \mathbf{r}'_i$$

$$(\vec{r}'_i - \vec{R}') \doteq \mathbf{r}'_i = \mathbf{r}_i$$

$$(\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R}) \doteq \mathbf{v}_i = \mathbf{v}'_i$$

$$(\vec{v}'_i - \vec{V}') - \vec{\omega}' \times (\vec{r}'_i - \vec{R}') \doteq \mathbf{v}'_i = \mathbf{v}_i$$

$$(\vec{a}_i - \vec{A}) - 2\, \vec{\omega} \times (\vec{v}_i - \vec{V}) + \vec{\omega} \times [\, \vec{\omega} \times (\vec{r}_i - \vec{R})\,] - \vec{\alpha} \times (\vec{r}_i - \vec{R}) \doteq \mathbf{a}_i = \mathbf{a}'_i$$

$$(\vec{a}'_i - \vec{A}') - 2\, \vec{\omega}' \times (\vec{v}'_i - \vec{V}') + \vec{\omega}' \times [\, \vec{\omega}' \times (\vec{r}'_i - \vec{R}')\,] - \vec{\alpha}' \times (\vec{r}'_i - \vec{R}') \doteq \mathbf{a}'_i = \mathbf{a}_i$$

Appendix II

The Relations

In a system of particles, these relations can be obtained (The magnitudes $\mathbf{R}_{cm}$, $\mathbf{V}_{cm}$, $\mathbf{A}_{cm}$, $\vec{R}_{cm}$, $\vec{V}_{cm}$ and $\vec{A}_{cm}$ can be replaced by the magnitudes $\mathbf{R}$, $\mathbf{V}$, $\mathbf{A}$, $\vec{R}$, $\vec{V}$ and $\vec{A}$, or by the magnitudes $\mathbf{r}_j$, $\mathbf{v}_j$, $\mathbf{a}_j$, $\vec{r}_j$, $\vec{v}_j$ and $\vec{a}_j$, respectively. On the other hand, $\mathbf{R} = \mathbf{V} = \mathbf{A} = 0$)

$$\mathbf{r}_i \doteq (\vec{r}_i - \vec{R})$$

$$\mathbf{R}_{cm} \doteq (\vec{R}_{cm} - \vec{R})$$

$$\longrightarrow \quad (\mathbf{r}_i - \mathbf{R}_{cm}) = (\vec{r}_i - \vec{R}_{cm})$$

$$\mathbf{v}_i \doteq (\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R})$$

$$\mathbf{V}_{cm} \doteq (\vec{V}_{cm} - \vec{V}) - \vec{\omega} \times (\vec{R}_{cm} - \vec{R})$$

$$\longrightarrow \quad (\mathbf{v}_i - \mathbf{V}_{cm}) = (\vec{v}_i - \vec{V}_{cm}) - \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})$$

$$(\mathbf{v}_i - \mathbf{V}_{cm}) \cdot (\mathbf{v}_i - \mathbf{V}_{cm}) = \big[(\vec{v}_i - \vec{V}_{cm}) - \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] \cdot \big[(\vec{v}_i - \vec{V}_{cm}) - \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] =$$

$$(\vec{v}_i - \vec{V}_{cm}) \cdot (\vec{v}_i - \vec{V}_{cm}) - 2\,(\vec{v}_i - \vec{V}_{cm}) \cdot \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] + \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] \cdot \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] =$$

$$(\vec{v}_i - \vec{V}_{cm}) \cdot (\vec{v}_i - \vec{V}_{cm}) + 2\,(\vec{r}_i - \vec{R}_{cm}) \cdot \big[\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] + \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] \cdot \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] =$$

$$(\vec{v}_i - \vec{V}_{cm}) \cdot (\vec{v}_i - \vec{V}_{cm}) + \big[2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] \cdot \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] =$$

$$(\vec{v}_i - \vec{V}_{cm})^2 + \big[2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big]^2$$

$$(\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) = \big\{ (\vec{a}_i - \vec{A}_{cm}) - 2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) + \vec{\omega} \times [\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})] -$$

$$\vec{\alpha} \times (\vec{r}_i - \vec{R}_{cm}) \big\} \cdot (\vec{r}_i - \vec{R}_{cm}) = (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) - \big[2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) +$$

$$\big\{ \vec{\omega} \times [\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})] \big\} \cdot (\vec{r}_i - \vec{R}_{cm}) - \big[\vec{\alpha} \times (\vec{r}_i - \vec{R}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) = (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) -$$

$$\big[2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big\{ \big[\vec{\omega} \cdot (\vec{r}_i - \vec{R}_{cm}) \big] \vec{\omega} - (\vec{\omega} \cdot \vec{\omega})\, (\vec{r}_i - \vec{R}_{cm}) \big\} \cdot (\vec{r}_i - \vec{R}_{cm}) =$$

$$(\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) - \big[2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big[\vec{\omega} \cdot (\vec{r}_i - \vec{R}_{cm}) \big]^2 - (\vec{\omega})^2\, (\vec{r}_i - \vec{R}_{cm})^2$$

$$\longrightarrow \quad (\mathbf{v}_i - \mathbf{V}_{cm})^2 + (\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) = (\vec{v}_i - \vec{V}_{cm})^2 + (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm})$$

Appendix III

The Magnitudes

The magnitudes $\mathbf{L}_2$, W_2, K_2, U_2, W_4, K_4, U_4, W_6, K_6 and U_6 of a system of N particles can also be expressed as follows:

$$\mathbf{L}_2 = \sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[(\mathbf{r}_i - \mathbf{r}_j) \times (\mathbf{v}_i - \mathbf{v}_j) \big]$$

$$W_2 = \sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\mathbf{r}_i - \mathbf{r}_j) \big]$$

$$\Delta K_2 = \sum_{j>i}^{N} \Delta\, 1/2\, m_i\, m_j\, M^{-1} (\mathbf{v}_i - \mathbf{v}_j)^2 = W_2$$

$$\Delta U_2 = - \sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\mathbf{r}_i - \mathbf{r}_j) \big]$$

$$W_4 = \sum_{j>i}^{N} \Delta\, 1/2\, m_i\, m_j\, M^{-1} \big[(\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\mathbf{r}_i - \mathbf{r}_j) \big]$$

$$\Delta K_4 = \sum_{j>i}^{N} \Delta\, 1/2\, m_i\, m_j\, M^{-1} \big[(\mathbf{a}_i - \mathbf{a}_j) \cdot (\mathbf{r}_i - \mathbf{r}_j) \big] = W_4$$

$$\Delta U_4 = - \sum_{j>i}^{N} \Delta\, 1/2\, m_i\, m_j\, M^{-1} \big[(\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\mathbf{r}_i - \mathbf{r}_j) \big]$$

$$W_6 = \sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\vec{r}_i - \vec{r}_j) + \Delta\, 1/2\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\vec{r}_i - \vec{r}_j) \big]$$

$$\Delta K_6 = \sum_{j>i}^{N} \Delta\, 1/2\, m_i\, m_j\, M^{-1} \big[(\vec{v}_i - \vec{v}_j)^2 + (\vec{a}_i - \vec{a}_j) \cdot (\vec{r}_i - \vec{r}_j) \big] = W_6$$

$$\Delta U_6 = - \sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\vec{r}_i - \vec{r}_j) + \Delta\, 1/2\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\vec{r}_i - \vec{r}_j) \big]$$

The magnitudes $W_{(1\ \text{to}\ 6)}$ and $U_{(1\ \text{to}\ 6)}$ of an isolated system of N particles, whose internal non-kinetic forces obey Newton's third law in its weak form, can be reduced to:

$$W_1 = W_2 = \sum_i^N \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i$$

$$\Delta U_1 = \Delta U_2 = - \sum_i^N \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i$$

$$W_3 = W_4 = \sum_i^N \Delta\, 1/2\, \mathbf{F}_i \cdot \vec{r}_i$$

$$\Delta U_3 = \Delta U_4 = - \sum_i^N \Delta\, 1/2\, \mathbf{F}_i \cdot \vec{r}_i$$

$$W_5 = W_6 = \sum_i^N \big[\int_1^2 \mathbf{F}_i \cdot d\vec{r}_i + \Delta\, 1/2\, \mathbf{F}_i \cdot \vec{r}_i \big]$$

$$\Delta U_5 = \Delta U_6 = - \sum_i^N \big[\int_1^2 \mathbf{F}_i \cdot d\vec{r}_i + \Delta\, 1/2\, \mathbf{F}_i \cdot \vec{r}_i \big]$$

A New Theory in Relational Mechanics

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(Paper III)

In relational mechanics, a new theory is presented, which is invariant under transformations between inertial and non-inertial reference frames and which can be applied in any reference frame without introducing fictitious forces. Additionally, in this paper, we assume that all forces can obey or disobey Newton's third law.

Introduction

The new theory in relational mechanics presented in this paper is obtained starting from an auxiliary system of particles (called free-system) that is used to obtain kinematic magnitudes (such as inertial position, inertial velocity, etc.) that are invariant under transformations between inertial and non-inertial reference frames.

The inertial position $\mathbf{r}_i$, the inertial velocity $\mathbf{v}_i$ and the inertial acceleration $\mathbf{a}_i$ of a particle i are given by:

$$\mathbf{r}_i \doteq (\vec{r}_i - \vec{R})$$

$$\mathbf{v}_i \doteq (\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R})$$

$$\mathbf{a}_i \doteq (\vec{a}_i - \vec{A}) - 2\,\vec{\omega} \times (\vec{v}_i - \vec{V}) + \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,] - \vec{\alpha} \times (\vec{r}_i - \vec{R})$$

($\mathbf{v}_i \doteq d(\mathbf{r}_i)/dt$) and ($\mathbf{a}_i \doteq d^2(\mathbf{r}_i)/dt^2$) where $\vec{r}_i$ is the position vector of particle i, $\vec{R}$ is the position vector of the center of mass of the free-system, and $\vec{\omega}$ is the angular velocity vector of the free-system (see Appendix I)

A reference frame S is non-rotating if the angular velocity $\vec{\omega}$ of the free-system relative to S is equal to zero, and the reference frame S is also inertial if the acceleration $\vec{A}$ of the center of mass of the free-system relative to S is equal to zero.

The New Dynamics

[1] A force is always caused by the interaction between two or more particles.

[2] The net force $\mathbf{F}_i$ acting on a particle i of mass m_i produces an inertial acceleration $\mathbf{a}_i$ according to the following equation: [$\mathbf{F}_i = m_i\,\mathbf{a}_i$]

[3] In this paper, we assume that all forces can obey or disobey Newton's third law in its weak form or in its strong form.

The Definitions

For a system of N particles, the following definitions are applicable:

Mass	$\mathrm{M} \doteq \sum_i^{\mathrm{N}} m_i$
Position CM 1	$\vec{R}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \vec{r}_i$
Velocity CM 1	$\vec{V}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \vec{v}_i$
Acceleration CM 1	$\vec{A}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \vec{a}_i$
Position CM 2	$\mathbf{R}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \mathbf{r}_i$
Velocity CM 2	$\mathbf{V}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \mathbf{v}_i$
Acceleration CM 2	$\mathbf{A}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \mathbf{a}_i$
Linear Momentum 1	$\mathbf{P}_1 \doteq \sum_i^{\mathrm{N}} m_i \, \mathbf{v}_i$
Angular Momentum 1	$\mathbf{L}_1 \doteq \sum_i^{\mathrm{N}} m_i \left[\mathbf{r}_i \times \mathbf{v}_i \right]$
Angular Momentum 2	$\mathbf{L}_2 \doteq \sum_i^{\mathrm{N}} m_i \left[(\mathbf{r}_i - \mathbf{R}_{cm}) \times (\mathbf{v}_i - \mathbf{V}_{cm}) \right]$
Work 1	$\mathrm{W}_1 \doteq \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i = \Delta \mathrm{K}_1$
Kinetic Energy 1	$\Delta \mathrm{K}_1 \doteq \sum_i^{\mathrm{N}} \Delta \, {}^1\!/_2 \, m_i \, (\mathbf{v}_i)^2$
Potential Energy 1	$\Delta \mathrm{U}_1 \doteq - \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i$
Mechanical Energy 1	$\mathrm{E}_1 \doteq \mathrm{K}_1 + \mathrm{U}_1$
Lagrangian 1	$\mathrm{L}_1 \doteq \mathrm{K}_1 - \mathrm{U}_1$
Work 2	$\mathrm{W}_2 \doteq \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d(\mathbf{r}_i - \mathbf{R}_{cm}) = \Delta \mathrm{K}_2$
Kinetic Energy 2	$\Delta \mathrm{K}_2 \doteq \sum_i^{\mathrm{N}} \Delta \, {}^1\!/_2 \, m_i \, (\mathbf{v}_i - \mathbf{V}_{cm})^2$
Potential Energy 2	$\Delta \mathrm{U}_2 \doteq - \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d(\mathbf{r}_i - \mathbf{R}_{cm})$
Mechanical Energy 2	$\mathrm{E}_2 \doteq \mathrm{K}_2 + \mathrm{U}_2$
Lagrangian 2	$\mathrm{L}_2 \doteq \mathrm{K}_2 - \mathrm{U}_2$

Work 3 $\quad W_3 \doteq \sum_i^N \Delta\, ^1\!/_2\, \mathbf{F}_i \cdot \mathbf{r}_i = \Delta K_3$

Kinetic Energy 3 $\quad \Delta K_3 \doteq \sum_i^N \Delta\, ^1\!/_2\, m_i\, \mathbf{a}_i \cdot \mathbf{r}_i$

Potential Energy 3 $\quad \Delta U_3 \doteq -\sum_i^N \Delta\, ^1\!/_2\, \mathbf{F}_i \cdot \mathbf{r}_i$

Mechanical Energy 3 $\quad E_3 \doteq K_3 + U_3$

Work 4 $\quad W_4 \doteq \sum_i^N \Delta\, ^1\!/_2\, \mathbf{F}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) = \Delta K_4$

Kinetic Energy 4 $\quad \Delta K_4 \doteq \sum_i^N \Delta\, ^1\!/_2\, m_i \left[(\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) \right]$

Potential Energy 4 $\quad \Delta U_4 \doteq -\sum_i^N \Delta\, ^1\!/_2\, \mathbf{F}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm})$

Mechanical Energy 4 $\quad E_4 \doteq K_4 + U_4$

Work 5 $\quad W_5 \doteq \sum_i^N \left[\int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}) + \Delta\, ^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}) \right] = \Delta K_5$

Kinetic Energy 5 $\quad \Delta K_5 \doteq \sum_i^N \Delta\, ^1\!/_2\, m_i \left[(\vec{v}_i - \vec{V})^2 + (\vec{a}_i - \vec{A}) \cdot (\vec{r}_i - \vec{R}) \right]$

Potential Energy 5 $\quad \Delta U_5 \doteq -\sum_i^N \left[\int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}) + \Delta\, ^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}) \right]$

Mechanical Energy 5 $\quad E_5 \doteq K_5 + U_5$

Work 6 $\quad W_6 \doteq \sum_i^N \left[\int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}_{cm}) + \Delta\, ^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}_{cm}) \right] = \Delta K_6$

Kinetic Energy 6 $\quad \Delta K_6 \doteq \sum_i^N \Delta\, ^1\!/_2\, m_i \left[(\vec{v}_i - \vec{V}_{cm})^2 + (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) \right]$

Potential Energy 6 $\quad \Delta U_6 \doteq -\sum_i^N \left[\int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}_{cm}) + \Delta\, ^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}_{cm}) \right]$

Mechanical Energy 6 $\quad E_6 \doteq K_6 + U_6$

The Relations

From the above definitions, the following relations can be obtained (see Appendix II)

$$K_1 = K_2 + {}^1\!/_2\, M\, \mathbf{V}_{cm}^2$$

$$K_3 = K_4 + {}^1\!/_2\, M\, \mathbf{A}_{cm} \cdot \mathbf{R}_{cm}$$

$$K_5 = K_6 + {}^1\!/_2\, M \left[(\vec{V}_{cm} - \vec{V})^2 + (\vec{A}_{cm} - \vec{A}) \cdot (\vec{R}_{cm} - \vec{R}) \right]$$

$$K_5 = K_1 + K_3 \quad \& \quad U_5 = U_1 + U_3 \quad \& \quad E_5 = E_1 + E_3$$

$$K_6 = K_2 + K_4 \quad \& \quad U_6 = U_2 + U_4 \quad \& \quad E_6 = E_2 + E_4$$

The Principles

The linear momentum $[\,\mathbf{P}_1\,]$ of an isolated system of N particles remains constant if the internal forces obey Newton's third law in its weak form.

$$\mathbf{P}_1 \;=\; \text{constant} \qquad \Big[\; d(\mathbf{P}_1)/dt \;=\; \textstyle\sum_i^{\text{N}} m_i\,\mathbf{a}_i \;=\; \sum_i^{\text{N}} \mathbf{F}_i \;=\; 0 \;\Big]$$

The angular momentum $[\,\mathbf{L}_1\,]$ of an isolated system of N particles remains constant if the internal forces obey Newton's third law in its strong form.

$$\mathbf{L}_1 \;=\; \text{constant} \qquad \Big[\; d(\mathbf{L}_1)/dt \;=\; \textstyle\sum_i^{\text{N}} m_i\,\big[\,\mathbf{r}_i \times \mathbf{a}_i\,\big] \;=\; \sum_i^{\text{N}} \mathbf{r}_i \times \mathbf{F}_i \;=\; 0 \;\Big]$$

The angular momentum $[\,\mathbf{L}_2\,]$ of an isolated system of N particles remains constant if the internal forces obey Newton's third law in its strong form.

$$\mathbf{L}_2 \;=\; \text{constant} \qquad \Big[\; d(\mathbf{L}_2)/dt \;=\; \textstyle\sum_i^{\text{N}} m_i\,\big[\,(\mathbf{r}_i - \mathbf{R}_{cm}) \times (\mathbf{a}_i - \mathbf{A}_{cm})\,\big] \;=$$

$$\textstyle\sum_i^{\text{N}} m_i\,\big[\,(\mathbf{r}_i - \mathbf{R}_{cm}) \times \mathbf{a}_i\,\big] \;=\; \sum_i^{\text{N}} (\mathbf{r}_i - \mathbf{R}_{cm}) \times \mathbf{F}_i \;=\; 0 \;\Big]$$

The mechanical energy $[\,\mathrm{E}_1\,]$ and the mechanical energy $[\,\mathrm{E}_2\,]$ of a system of N particles remain constant if the system is only subject to conservative forces.

$$\mathrm{E}_1 \;=\; \text{constant} \qquad \Big[\; \Delta\,\mathrm{E}_1 \;=\; \Delta\,\mathrm{K}_1 + \Delta\,\mathrm{U}_1 \;=\; 0 \;\Big]$$

$$\mathrm{E}_2 \;=\; \text{constant} \qquad \Big[\; \Delta\,\mathrm{E}_2 \;=\; \Delta\,\mathrm{K}_2 + \Delta\,\mathrm{U}_2 \;=\; 0 \;\Big]$$

The mechanical energy $[\,\mathrm{E}_3\,]$ and the mechanical energy $[\,\mathrm{E}_4\,]$ of a system of N particles are always zero (and therefore they always remain constant)

$$\mathrm{E}_3 \;=\; \text{constant} \qquad \Big[\; \mathrm{E}_3 \;=\; \textstyle\sum_i^{\text{N}} 1/2\,\big[\,m_i\,\mathbf{a}_i \cdot \mathbf{r}_i - \mathbf{F}_i \cdot \mathbf{r}_i\,\big] \;=\; 0 \;\Big]$$

$$\mathrm{E}_4 \;=\; \text{constant} \qquad \Big[\; \mathrm{E}_4 \;=\; \textstyle\sum_i^{\text{N}} 1/2\,\big[\,m_i\,\mathbf{a}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) - \mathbf{F}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm})\,\big] \;=\; 0 \;\Big]$$

$$\textstyle\sum_i^{\text{N}} 1/2\, m_i\,\big[\,(\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm})\,\big] \;=\; \sum_i^{\text{N}} 1/2\, m_i\,\mathbf{a}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm})$$

The mechanical energy $[\,\mathrm{E}_5\,]$ and the mechanical energy $[\,\mathrm{E}_6\,]$ of a system of N particles remain constant if the system is only subject to conservative forces.

$$\mathrm{E}_5 \;=\; \text{constant} \qquad \Big[\; \Delta\,\mathrm{E}_5 \;=\; \Delta\,\mathrm{K}_5 + \Delta\,\mathrm{U}_5 \;=\; 0 \;\Big]$$

$$\mathrm{E}_6 \;=\; \text{constant} \qquad \Big[\; \Delta\,\mathrm{E}_6 \;=\; \Delta\,\mathrm{K}_6 + \Delta\,\mathrm{U}_6 \;=\; 0 \;\Big]$$

Observations

All equations of this paper can be applied in any inertial reference frame and also in any non-inertial reference frame.

Additionally, inertial reference frames and non-inertial reference frames must not introduce fictitious forces into $\mathbf{F}_i$.

In this paper, the magnitudes [m, $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$, M, $\mathbf{R}$, $\mathbf{V}$, $\mathbf{A}$, $\mathbf{F}$, $\mathbf{P}_1$, $\mathbf{L}_1$, $\mathbf{L}_2$, W_1, K_1, U_1, E_1, L_1, W_2, K_2, U_2, E_2, L_2, W_3, K_3, U_3, E_3, W_4, K_4, U_4, E_4, W_5, K_5, U_5, E_5, W_6, K_6, U_6 and E_6] are invariant under transformations between inertial and non-inertial reference frames.

The mechanical energy E_3 of a system of particles is always zero [$E_3 = K_3 + U_3 = 0$]

Therefore, the mechanical energy E_5 of a system of particles is always equal to the mechanical energy E_1 of the system of particles [$E_5 = E_1$]

The mechanical energy E_4 of a system of particles is always zero [$E_4 = K_4 + U_4 = 0$]

Therefore, the mechanical energy E_6 of a system of particles is always equal to the mechanical energy E_2 of the system of particles [$E_6 = E_2$]

If the potential energy U_1 of a system of particles is a homogeneous function of degree k then the potential energy U_3 and the potential energy U_5 of the system of particles are given by: [$U_3 = (\frac{k}{2})\, U_1$] and [$U_5 = (1+\frac{k}{2})\, U_1$]

If the potential energy U_2 of a system of particles is a homogeneous function of degree k then the potential energy U_4 and the potential energy U_6 of the system of particles are given by: [$U_4 = (\frac{k}{2})\, U_2$] and [$U_6 = (1+\frac{k}{2})\, U_2$]

If the potential energy U_1 of a system of particles is a homogeneous function of degree k and if the kinetic energy K_5 of the system of particles is equal to zero, then we obtain: [$K_1 = -\,K_3 = U_3 = (\frac{k}{2})\, U_1 = (\frac{k}{2+k})\, E_1$]

If the potential energy U_2 of a system of particles is a homogeneous function of degree k and if the kinetic energy K_6 of the system of particles is equal to zero, then we obtain: [$K_2 = -\,K_4 = U_4 = (\frac{k}{2})\, U_2 = (\frac{k}{2+k})\, E_2$]

If the potential energy U_1 of a system of particles is a homogeneous function of degree k and if the average kinetic energy $\langle K_5 \rangle$ of the system of particles is equal to zero, then we obtain: [$\langle K_1 \rangle = -\,\langle K_3 \rangle = \langle U_3 \rangle = (\frac{k}{2})\, \langle U_1 \rangle = (\frac{k}{2+k})\, \langle E_1 \rangle$]

If the potential energy U_2 of a system of particles is a homogeneous function of degree k and if the average kinetic energy $\langle K_6 \rangle$ of the system of particles is equal to zero, then we obtain: [$\langle K_2 \rangle = -\,\langle K_4 \rangle = \langle U_4 \rangle = (\frac{k}{2})\, \langle U_2 \rangle = (\frac{k}{2+k})\, \langle E_2 \rangle$]

The average kinetic energy $\langle \mathrm{K}_5 \rangle$ and the average kinetic energy $\langle \mathrm{K}_6 \rangle$ of a system of particles with bounded motion (in $\langle \mathrm{K}_5 \rangle$ relative to $\vec{R}$ and in $\langle \mathrm{K}_6 \rangle$ relative to $\vec{R}_{cm}$) are always zero.

The kinetic energy K_5 and the kinetic energy K_6 of a system of N particles can also be expressed as follows : [$\mathrm{K}_5 = \sum_i^{\mathrm{N}} 1/2 \; m_i \, (\dot{r}_i \, \dot{r}_i + \ddot{r}_i \, r_i)$] where $r_i \doteq | \, \vec{r}_i - \vec{R} \, |$ and [$\mathrm{K}_6 = \sum_{j>i}^{\mathrm{N}} 1/2 \, m_i \, m_j \, \mathrm{M}^{-1} (\, \dot{r}_{ij} \, \dot{r}_{ij} + \ddot{r}_{ij} \, r_{ij} \,)$] where $r_{ij} \doteq | \, \vec{r}_i - \vec{r}_j \, |$

The kinetic energy K_5 and the kinetic energy K_6 of a system of N particles can also be expressed as follows : [$\mathrm{K}_5 = \sum_i^{\mathrm{N}} 1/2 \; m_i \, (\, \ddot{\tau}_i \,)$] where $\tau_i \doteq 1/2 \; (\vec{r}_i - \vec{R}) \cdot (\vec{r}_i - \vec{R})$ and [$\mathrm{K}_6 = \sum_{j>i}^{\mathrm{N}} 1/2 \, m_i \, m_j \, \mathrm{M}^{-1} (\, \ddot{\tau}_{ij} \,)$] where $\tau_{ij} \doteq 1/2 \; (\vec{r}_i - \vec{r}_j) \cdot (\vec{r}_i - \vec{r}_j)$

The kinetic energy K_6 is the only kinetic energy that can be expressed without the necessity of introducing any magnitude that is related to the free-system [such as: $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$, $\vec{\omega}$, $\vec{R}$, etc.]

In an isolated system of particles, the potential energy U_2 is equal to the potential energy U_1 if the internal forces obey Newton's third law in its weak form [$\mathrm{U}_2 = \mathrm{U}_1$]

In an isolated system of particles, the potential energy U_4 is equal to the potential energy U_3 if the internal forces obey Newton's third law in its weak form [$\mathrm{U}_4 = \mathrm{U}_3$]

In an isolated system of particles, the potential energy U_6 is equal to the potential energy U_5 if the internal forces obey Newton's third law in its weak form [$\mathrm{U}_6 = \mathrm{U}_5$]

A reference frame S is non-rotating if the angular velocity $\vec{\omega}$ of the free-system relative to S is equal to zero, and the reference frame S is also inertial if the acceleration $\vec{A}$ of the center of mass of the free-system relative to S is equal to zero.

If the origin of a non-rotating reference frame S [$\vec{\omega} = 0$] always coincides with the center of mass of the free-system [$\vec{R} = \vec{V} = \vec{A} = 0$] then relative to S: [$\mathbf{r}_i = \vec{r}_i$, $\mathbf{v}_i = \vec{v}_i$ and $\mathbf{a}_i = \vec{a}_i$] Therefore, it is easy to see that always: [$\mathbf{v}_i = d(\mathbf{r}_i)/dt$ and $\mathbf{a}_i = d^2(\mathbf{r}_i)/dt^2$]

This paper does not contradict Newton's first and second laws since these two laws are valid in all inertial reference frames. The equation [$\mathbf{F}_i = m_i \, \mathbf{a}_i$] is a simple reformulation of Newton's second law.

In this paper, the equation [$\mathbf{F}_i = m_i \, \mathbf{a}_i$] would be valid in all reference frames (inertial or non-inertial) even if all forces were always disobeyed Newton's third law in its strong form and in its weak form.

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Appendix I

The Free-System

The free-system is a system of N particles that must always be free of internal and external forces, that must be three-dimensional, and that the relative distances between the N particles must be constant.

The position $\vec{R}$, the velocity $\vec{V}$ and the acceleration $\vec{A}$ of the center of mass of the free-system relative to a reference frame S (and the angular velocity $\vec{\omega}$ and the angular acceleration $\vec{\alpha}$ of the free-system relative to the reference frame S) are given by:

$$\mathrm{M} \doteq \textstyle\sum_i^{\mathrm{N}} m_i$$

$$\vec{R} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{\mathrm{N}} m_i \, \vec{r}_i$$

$$\vec{V} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{\mathrm{N}} m_i \, \vec{v}_i$$

$$\vec{A} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{\mathrm{N}} m_i \, \vec{a}_i$$

$$\vec{\omega} \doteq \overleftrightarrow{I}^{-1} \cdot \vec{L}$$

$$\vec{\alpha} \doteq d(\vec{\omega})/dt$$

$$\overleftrightarrow{I} \doteq \textstyle\sum_i^{\mathrm{N}} m_i \, [\, |\, \vec{r}_i - \vec{R}\, |^2 \, \overleftrightarrow{1} - (\vec{r}_i - \vec{R}) \otimes (\vec{r}_i - \vec{R})\,]$$

$$\vec{L} \doteq \textstyle\sum_i^{\mathrm{N}} m_i \, (\vec{r}_i - \vec{R}) \times (\vec{v}_i - \vec{V})$$

where M is the mass of the free-system, $\overleftrightarrow{I}$ is the inertia tensor of the free-system (relative to $\vec{R}$) and $\vec{L}$ is the angular momentum of the free-system relative to the reference frame S.

The Transformations

$$(\vec{r}_i - \vec{R}) \doteq \mathbf{r}_i = \mathbf{r}'_i$$

$$(\vec{r}\,'_i - \vec{R}') \doteq \mathbf{r}'_i = \mathbf{r}_i$$

$$(\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R}) \doteq \mathbf{v}_i = \mathbf{v}'_i$$

$$(\vec{v}\,'_i - \vec{V}') - \vec{\omega}\,' \times (\vec{r}\,'_i - \vec{R}') \doteq \mathbf{v}'_i = \mathbf{v}_i$$

$$(\vec{a}_i - \vec{A}) - 2\, \vec{\omega} \times (\vec{v}_i - \vec{V}) + \vec{\omega} \times [\, \vec{\omega} \times (\vec{r}_i - \vec{R})\,] - \vec{\alpha} \times (\vec{r}_i - \vec{R}) \doteq \mathbf{a}_i = \mathbf{a}'_i$$

$$(\vec{a}\,'_i - \vec{A}') - 2\, \vec{\omega}\,' \times (\vec{v}\,'_i - \vec{V}') + \vec{\omega}\,' \times [\, \vec{\omega}\,' \times (\vec{r}\,'_i - \vec{R}')\,] - \vec{\alpha}\,' \times (\vec{r}\,'_i - \vec{R}') \doteq \mathbf{a}'_i = \mathbf{a}_i$$

Appendix II

The Relations

In a system of particles, these relations can be obtained (The magnitudes $\mathbf{R}_{cm}$, $\mathbf{V}_{cm}$, $\mathbf{A}_{cm}$, $\vec{R}_{cm}$, $\vec{V}_{cm}$ and $\vec{A}_{cm}$ can be replaced by the magnitudes $\mathbf{R}$, $\mathbf{V}$, $\mathbf{A}$, $\vec{R}$, $\vec{V}$ and $\vec{A}$, or by the magnitudes $\mathbf{r}_j$, $\mathbf{v}_j$, $\mathbf{a}_j$, $\vec{r}_j$, $\vec{v}_j$ and $\vec{a}_j$, respectively. On the other hand, $\mathbf{R} = \mathbf{V} = \mathbf{A} = 0$)

$$\mathbf{r}_i \doteq (\vec{r}_i - \vec{R})$$

$$\mathbf{R}_{cm} \doteq (\vec{R}_{cm} - \vec{R})$$

$$\longrightarrow \quad (\mathbf{r}_i - \mathbf{R}_{cm}) = (\vec{r}_i - \vec{R}_{cm})$$

$$\mathbf{v}_i \doteq (\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R})$$

$$\mathbf{V}_{cm} \doteq (\vec{V}_{cm} - \vec{V}) - \vec{\omega} \times (\vec{R}_{cm} - \vec{R})$$

$$\longrightarrow \quad (\mathbf{v}_i - \mathbf{V}_{cm}) = (\vec{v}_i - \vec{V}_{cm}) - \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})$$

$$(\mathbf{v}_i - \mathbf{V}_{cm}) \cdot (\mathbf{v}_i - \mathbf{V}_{cm}) = \big[\, (\vec{v}_i - \vec{V}_{cm}) - \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \,\big] \cdot \big[\, (\vec{v}_i - \vec{V}_{cm}) - \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \,\big] =$$

$$(\vec{v}_i - \vec{V}_{cm}) \cdot (\vec{v}_i - \vec{V}_{cm}) - 2\, (\vec{v}_i - \vec{V}_{cm}) \cdot \big[\, \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \,\big] + \big[\, \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \,\big] \cdot \big[\, \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \,\big] =$$

$$(\vec{v}_i - \vec{V}_{cm}) \cdot (\vec{v}_i - \vec{V}_{cm}) + 2\, (\vec{r}_i - \vec{R}_{cm}) \cdot \big[\, \vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \,\big] + \big[\, \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \,\big] \cdot \big[\, \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \,\big] =$$

$$(\vec{v}_i - \vec{V}_{cm}) \cdot (\vec{v}_i - \vec{V}_{cm}) + \big[\, 2\, \vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \,\big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big[\, \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \,\big] \cdot \big[\, \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \,\big] =$$

$$(\vec{v}_i - \vec{V}_{cm})^2 + \big[\, 2\, \vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \,\big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big[\, \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \,\big]^2$$

$$(\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) = \big\{\, (\vec{a}_i - \vec{A}_{cm}) - 2\, \vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) + \vec{\omega} \times [\, \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \,] \, -$$

$$\vec{\alpha} \times (\vec{r}_i - \vec{R}_{cm}) \,\big\} \cdot (\vec{r}_i - \vec{R}_{cm}) = (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) - \big[\, 2\, \vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \,\big] \cdot (\vec{r}_i - \vec{R}_{cm}) \, +$$

$$\big\{\, \vec{\omega} \times [\, \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \,] \,\big\} \cdot (\vec{r}_i - \vec{R}_{cm}) - \big[\, \vec{\alpha} \times (\vec{r}_i - \vec{R}_{cm}) \,\big] \cdot (\vec{r}_i - \vec{R}_{cm}) = (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) \, -$$

$$\big[\, 2\, \vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \,\big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big\{\, \big[\, \vec{\omega} \cdot (\vec{r}_i - \vec{R}_{cm}) \,\big]\, \vec{\omega} - (\, \vec{\omega} \cdot \vec{\omega} \,)\, (\vec{r}_i - \vec{R}_{cm}) \,\big\} \cdot (\vec{r}_i - \vec{R}_{cm}) =$$

$$(\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) - \big[\, 2\, \vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \,\big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big[\, \vec{\omega} \cdot (\vec{r}_i - \vec{R}_{cm}) \,\big]^2 - (\, \vec{\omega} \,)^2\, (\vec{r}_i - \vec{R}_{cm})^2$$

$$\longrightarrow \quad (\mathbf{v}_i - \mathbf{V}_{cm})^2 + (\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) = (\vec{v}_i - \vec{V}_{cm})^2 + (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm})$$

Appendix III

The Magnitudes

The magnitudes $\mathbf{L}_2$, W_2, K_2, U_2, W_4, K_4, U_4, W_6, K_6 and U_6 of a system of N particles can also be expressed as follows:

$$\mathbf{L}_2 \;=\; \textstyle\sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\, (\mathbf{r}_i - \mathbf{r}_j) \times (\mathbf{v}_i - \mathbf{v}_j)\, \big]$$

$$W_2 \;=\; \textstyle\sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\mathbf{r}_i - \mathbf{r}_j)\, \big]$$

$$\Delta\, K_2 \;=\; \textstyle\sum_{j>i}^{N} \Delta\, {}^1\!/_2\, m_i\, m_j\, M^{-1}\, (\mathbf{v}_i - \mathbf{v}_j)^2 \;=\; W_2$$

$$\Delta\, U_2 \;=\; - \textstyle\sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\mathbf{r}_i - \mathbf{r}_j)\, \big]$$

$$W_4 \;=\; \textstyle\sum_{j>i}^{N} \Delta\, {}^1\!/_2\, m_i\, m_j\, M^{-1} \big[\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\mathbf{r}_i - \mathbf{r}_j)\, \big]$$

$$\Delta\, K_4 \;=\; \textstyle\sum_{j>i}^{N} \Delta\, {}^1\!/_2\, m_i\, m_j\, M^{-1} \big[\, (\mathbf{a}_i - \mathbf{a}_j) \cdot (\mathbf{r}_i - \mathbf{r}_j)\, \big] \;=\; W_4$$

$$\Delta\, U_4 \;=\; - \textstyle\sum_{j>i}^{N} \Delta\, {}^1\!/_2\, m_i\, m_j\, M^{-1} \big[\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\mathbf{r}_i - \mathbf{r}_j)\, \big]$$

$$W_6 \;=\; \textstyle\sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\vec{r}_i - \vec{r}_j) + \Delta\, {}^1\!/_2\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\vec{r}_i - \vec{r}_j)\, \big]$$

$$\Delta\, K_6 \;=\; \textstyle\sum_{j>i}^{N} \Delta\, {}^1\!/_2\, m_i\, m_j\, M^{-1} \big[\, (\vec{v}_i - \vec{v}_j)^2 + (\vec{a}_i - \vec{a}_j) \cdot (\vec{r}_i - \vec{r}_j)\, \big] \;=\; W_6$$

$$\Delta\, U_6 \;=\; - \textstyle\sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\vec{r}_i - \vec{r}_j) + \Delta\, {}^1\!/_2\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\vec{r}_i - \vec{r}_j)\, \big]$$

The magnitudes $W_{(1 \text{ to } 6)}$ and $U_{(1 \text{ to } 6)}$ of an isolated system of N particles, whose internal forces obey Newton's third law in its weak form, can be reduced to:

$$W_1 \;=\; W_2 \;=\; \textstyle\sum_i^N \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i$$

$$\Delta\, U_1 \;=\; \Delta\, U_2 \;=\; - \textstyle\sum_i^N \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i$$

$$W_3 \;=\; W_4 \;=\; \textstyle\sum_i^N \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \vec{r}_i$$

$$\Delta\, U_3 \;=\; \Delta\, U_4 \;=\; - \textstyle\sum_i^N \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \vec{r}_i$$

$$W_5 \;=\; W_6 \;=\; \textstyle\sum_i^N \big[\, \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \vec{r}_i\, \big]$$

$$\Delta\, U_5 \;=\; \Delta\, U_6 \;=\; - \textstyle\sum_i^N \big[\, \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \vec{r}_i\, \big]$$

A New Theory in Relational Mechanics

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(Paper IV)

In relational mechanics, a new theory is presented, which is invariant under transformations between inertial and non-inertial reference frames, which can be applied in any reference frame without introducing fictitious forces and which establishes the existence of new universal forces of interaction, called kinetic forces.

Introduction

The new theory in relational mechanics presented in this paper is obtained starting from an auxiliary system of particles (called free-system) that is used to obtain kinematic magnitudes (such as inertial position, inertial velocity, etc.) that are invariant under transformations between inertial and non-inertial reference frames.

The inertial position $\mathbf{r}_i$, the inertial velocity $\mathbf{v}_i$ and the inertial acceleration $\mathbf{a}_i$ of a particle i are given by:

$$\mathbf{r}_i \doteq (\vec{r}_i - \vec{R})$$

$$\mathbf{v}_i \doteq (\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R})$$

$$\mathbf{a}_i \doteq (\vec{a}_i - \vec{A}) - 2\,\vec{\omega} \times (\vec{v}_i - \vec{V}) + \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,] - \vec{\alpha} \times (\vec{r}_i - \vec{R})$$

$(\,\mathbf{v}_i \doteq d(\mathbf{r}_i)/dt\,)$ and $(\,\mathbf{a}_i \doteq d^2(\mathbf{r}_i)/dt^2\,)$ where $\vec{r}_i$ is the position vector of particle i, $\vec{R}$ is the position vector of the center of mass of the free-system, and $\vec{\omega}$ is the angular velocity vector of the free-system (see Appendix I)

A reference frame S is non-rotating if the angular velocity $\vec{\omega}$ of the free-system relative to S is equal to zero, and the reference frame S is also inertial if the acceleration $\vec{A}$ of the center of mass of the free-system relative to S is equal to zero.

The New Dynamics

[1] A force is always caused by the interaction between two or more particles.

[2] The total force $\mathbf{T}_i$ acting on a particle i is always zero $[\,\mathbf{T}_i = 0\,]$

[3] In this paper, we assume that all non-kinetic forces can obey or disobey Newton's third law in its weak form or in its strong form.

The Kinetic Forces

The kinetic force $\mathbf{K}^a_{ij}$ exerted on a particle i of mass m_i by another particle j of mass m_j, caused by the interaction between particle i and particle j, is given by:

$$\mathbf{K}^a_{ij} \;=\; -\, m_i\, m_j\, \mathrm{M}^{-1}\, (\mathbf{a}_i - \mathbf{a}_j)$$

where $\mathbf{a}_i$ is the inertial acceleration of particle i, $\mathbf{a}_j$ is the inertial acceleration of particle j, and M is the mass of the Universe.

The kinetic force $\mathbf{K}^u_i$ exerted on a particle i of mass m_i by the center of mass of the Universe, caused by the interaction between particle i and the Universe, is given by:

$$\mathbf{K}^u_i \;=\; -\, m_i\, \mathbf{A}_{cm}$$

where $\mathbf{A}_{cm}$ is the inertial acceleration of the center of mass of the Universe.

From the above equations it follows that the net kinetic force $\mathbf{K}_i$ ($= \sum_j^{All} \mathbf{K}^a_{ij} + \mathbf{K}^u_i$) acting on a particle i of mass m_i is given by:

$$\mathbf{K}_i \;=\; -\, m_i\, \mathbf{a}_i$$

where $\mathbf{a}_i$ is the inertial acceleration of particle i.

The kinetic force $\mathbf{K}^a_{ij}$ always obey Newton's third law in its weak form.

If all non-kinetic forces always obey Newton's third law in its weak form then the inertial acceleration of the center of mass of the Universe $\mathbf{A}_{cm}$ is always zero.

The [2] Principle

The second principle of the new dynamics establishes that the total force $\mathbf{T}_i$ acting on a particle i is always zero.

$$\mathbf{T}_i \;=\; 0$$

If the total force $\mathbf{T}_i$ is divided into the following two parts: the net kinetic force $\mathbf{K}_i$ and the net non-kinetic force $\mathbf{F}_i$ ($\sum$ of gravitational forces, electrostatic forces, etc.) then we have:

$$\mathbf{K}_i + \mathbf{F}_i \;=\; 0$$

Now, substituting ($\mathbf{K}_i \;=\; -\, m_i\, \mathbf{a}_i$) and rearranging, we finally obtain:

$$\mathbf{F}_i \;=\; m_i\, \mathbf{a}_i$$

This equation (similar to Newton's second law) will be used throughout this paper.

On the other hand, in this paper a system of particles is isolated when the system is free of external non-kinetic forces.

The Definitions

For a system of N particles, the following definitions are applicable:

Mass $\mathrm{M} \doteq \sum_i^{\mathrm{N}} m_i$

Position CM 1 $\vec{R}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \vec{r}_i$

Velocity CM 1 $\vec{V}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \vec{v}_i$

Acceleration CM 1 $\vec{A}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \vec{a}_i$

Position CM 2 $\mathbf{R}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \mathbf{r}_i$

Velocity CM 2 $\mathbf{V}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \mathbf{v}_i$

Acceleration CM 2 $\mathbf{A}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \mathbf{a}_i$

Linear Momentum 1 $\mathbf{P}_1 \doteq \sum_i^{\mathrm{N}} m_i \, \mathbf{v}_i$

Angular Momentum 1 $\mathbf{L}_1 \doteq \sum_i^{\mathrm{N}} m_i \left[\mathbf{r}_i \times \mathbf{v}_i \right]$

Angular Momentum 2 $\mathbf{L}_2 \doteq \sum_i^{\mathrm{N}} m_i \left[(\mathbf{r}_i - \mathbf{R}_{cm}) \times (\mathbf{v}_i - \mathbf{V}_{cm}) \right]$

Work 1 $\mathrm{W}_1 \doteq \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i = \Delta \mathrm{K}_1$

Kinetic Energy 1 $\Delta \mathrm{K}_1 \doteq \sum_i^{\mathrm{N}} \Delta \, {}^1\!/_2 \, m_i \, (\mathbf{v}_i)^2$

Potential Energy 1 $\Delta \mathrm{U}_1 \doteq - \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i$

Mechanical Energy 1 $\mathrm{E}_1 \doteq \mathrm{K}_1 + \mathrm{U}_1$

Lagrangian 1 $\mathrm{L}_1 \doteq \mathrm{K}_1 - \mathrm{U}_1$

Work 2 $\mathrm{W}_2 \doteq \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d(\mathbf{r}_i - \mathbf{R}_{cm}) = \Delta \mathrm{K}_2$

Kinetic Energy 2 $\Delta \mathrm{K}_2 \doteq \sum_i^{\mathrm{N}} \Delta \, {}^1\!/_2 \, m_i \, (\mathbf{v}_i - \mathbf{V}_{cm})^2$

Potential Energy 2 $\Delta \mathrm{U}_2 \doteq - \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d(\mathbf{r}_i - \mathbf{R}_{cm})$

Mechanical Energy 2 $\mathrm{E}_2 \doteq \mathrm{K}_2 + \mathrm{U}_2$

Lagrangian 2 $\mathrm{L}_2 \doteq \mathrm{K}_2 - \mathrm{U}_2$

Work 3 $\quad \mathrm{W}_3 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \mathbf{r}_i = \Delta\, \mathrm{K}_3$

Kinetic Energy 3 $\quad \Delta\, \mathrm{K}_3 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i\, \mathbf{a}_i \cdot \mathbf{r}_i$

Potential Energy 3 $\quad \Delta\, \mathrm{U}_3 \doteq -\sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \mathbf{r}_i$

Mechanical Energy 3 $\quad \mathrm{E}_3 \doteq \mathrm{K}_3 + \mathrm{U}_3$

Work 4 $\quad \mathrm{W}_4 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) = \Delta\, \mathrm{K}_4$

Kinetic Energy 4 $\quad \Delta\, \mathrm{K}_4 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i \left[\, (\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm})\, \right]$

Potential Energy 4 $\quad \Delta\, \mathrm{U}_4 \doteq -\sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm})$

Mechanical Energy 4 $\quad \mathrm{E}_4 \doteq \mathrm{K}_4 + \mathrm{U}_4$

Work 5 $\quad \mathrm{W}_5 \doteq \sum_i^{\mathrm{N}} \left[\, \int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R})\, \right] = \Delta\, \mathrm{K}_5$

Kinetic Energy 5 $\quad \Delta\, \mathrm{K}_5 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i \left[\, (\vec{v}_i - \vec{V})^2 + (\vec{a}_i - \vec{A}) \cdot (\vec{r}_i - \vec{R})\, \right]$

Potential Energy 5 $\quad \Delta\, \mathrm{U}_5 \doteq -\sum_i^{\mathrm{N}} \left[\, \int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R})\, \right]$

Mechanical Energy 5 $\quad \mathrm{E}_5 \doteq \mathrm{K}_5 + \mathrm{U}_5$

Work 6 $\quad \mathrm{W}_6 \doteq \sum_i^{\mathrm{N}} \left[\, \int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}_{cm}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}_{cm})\, \right] = \Delta\, \mathrm{K}_6$

Kinetic Energy 6 $\quad \Delta\, \mathrm{K}_6 \doteq \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i \left[\, (\vec{v}_i - \vec{V}_{cm})^2 + (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm})\, \right]$

Potential Energy 6 $\quad \Delta\, \mathrm{U}_6 \doteq -\sum_i^{\mathrm{N}} \left[\, \int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}_{cm}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}_{cm})\, \right]$

Mechanical Energy 6 $\quad \mathrm{E}_6 \doteq \mathrm{K}_6 + \mathrm{U}_6$

The Relations

From the above definitions, the following relations can be obtained (see Appendix II)

$$\mathrm{K}_1 = \mathrm{K}_2 + {}^1\!/_2\, \mathrm{M}\, \mathbf{V}_{cm}^2$$

$$\mathrm{K}_3 = \mathrm{K}_4 + {}^1\!/_2\, \mathrm{M}\, \mathbf{A}_{cm} \cdot \mathbf{R}_{cm}$$

$$\mathrm{K}_5 = \mathrm{K}_6 + {}^1\!/_2\, \mathrm{M} \left[\, (\vec{V}_{cm} - \vec{V})^2 + (\vec{A}_{cm} - \vec{A}) \cdot (\vec{R}_{cm} - \vec{R})\, \right]$$

$$\mathrm{K}_5 = \mathrm{K}_1 + \mathrm{K}_3 \quad \& \quad \mathrm{U}_5 = \mathrm{U}_1 + \mathrm{U}_3 \quad \& \quad \mathrm{E}_5 = \mathrm{E}_1 + \mathrm{E}_3$$

$$\mathrm{K}_6 = \mathrm{K}_2 + \mathrm{K}_4 \quad \& \quad \mathrm{U}_6 = \mathrm{U}_2 + \mathrm{U}_4 \quad \& \quad \mathrm{E}_6 = \mathrm{E}_2 + \mathrm{E}_4$$

The Principles

The linear momentum [$\mathbf{P}_1$] of an isolated system of N particles remains constant if the internal non-kinetic forces obey Newton's third law in its weak form.

$$\mathbf{P}_1 \; = \; \text{constant} \qquad \left[\, d(\mathbf{P}_1)/dt \; = \; \textstyle\sum_i^{\mathrm{N}} m_i \, \mathbf{a}_i \; = \; \sum_i^{\mathrm{N}} \mathbf{F}_i \; = \; 0 \,\right]$$

The angular momentum [$\mathbf{L}_1$] of an isolated system of N particles remains constant if the internal non-kinetic forces obey Newton's third law in its strong form.

$$\mathbf{L}_1 \; = \; \text{constant} \qquad \left[\, d(\mathbf{L}_1)/dt \; = \; \textstyle\sum_i^{\mathrm{N}} m_i \left[\, \mathbf{r}_i \times \mathbf{a}_i \,\right] \; = \; \sum_i^{\mathrm{N}} \mathbf{r}_i \times \mathbf{F}_i \; = \; 0 \,\right]$$

The angular momentum [$\mathbf{L}_2$] of an isolated system of N particles remains constant if the internal non-kinetic forces obey Newton's third law in its strong form.

$$\mathbf{L}_2 \; = \; \text{constant} \qquad \left[\, d(\mathbf{L}_2)/dt \; = \; \textstyle\sum_i^{\mathrm{N}} m_i \left[\, (\mathbf{r}_i - \mathbf{R}_{cm}) \times (\mathbf{a}_i - \mathbf{A}_{cm}) \,\right] \; = \right.$$

$$\left. \textstyle\sum_i^{\mathrm{N}} m_i \left[\, (\mathbf{r}_i - \mathbf{R}_{cm}) \times \mathbf{a}_i \,\right] \; = \; \sum_i^{\mathrm{N}} (\mathbf{r}_i - \mathbf{R}_{cm}) \times \mathbf{F}_i \; = \; 0 \,\right]$$

The mechanical energy [E_1] and the mechanical energy [E_2] of a system of N particles remain constant if the system is only subject to kinetic forces and to conservative non-kinetic forces.

$$\mathrm{E}_1 \; = \; \text{constant} \qquad \left[\, \Delta\, \mathrm{E}_1 \; = \; \Delta\, \mathrm{K}_1 + \Delta\, \mathrm{U}_1 \; = \; 0 \,\right]$$

$$\mathrm{E}_2 \; = \; \text{constant} \qquad \left[\, \Delta\, \mathrm{E}_2 \; = \; \Delta\, \mathrm{K}_2 + \Delta\, \mathrm{U}_2 \; = \; 0 \,\right]$$

The mechanical energy [E_3] and the mechanical energy [E_4] of a system of N particles are always zero (and therefore they always remain constant)

$$\mathrm{E}_3 \; = \; \text{constant} \qquad \left[\, \mathrm{E}_3 \; = \; \textstyle\sum_i^{\mathrm{N}} 1/2 \left[\, m_i \, \mathbf{a}_i \cdot \mathbf{r}_i - \mathbf{F}_i \cdot \mathbf{r}_i \,\right] \; = \; 0 \,\right]$$

$$\mathrm{E}_4 \; = \; \text{constant} \qquad \left[\, \mathrm{E}_4 \; = \; \textstyle\sum_i^{\mathrm{N}} 1/2 \left[\, m_i \, \mathbf{a}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) - \mathbf{F}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) \,\right] \; = \; 0 \,\right]$$

$$\textstyle\sum_i^{\mathrm{N}} 1/2 \, m_i \left[\, (\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) \,\right] \; = \; \sum_i^{\mathrm{N}} 1/2 \, m_i \, \mathbf{a}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm})$$

The mechanical energy [E_5] and the mechanical energy [E_6] of a system of N particles remain constant if the system is only subject to kinetic forces and to conservative non-kinetic forces.

$$\mathrm{E}_5 \; = \; \text{constant} \qquad \left[\, \Delta\, \mathrm{E}_5 \; = \; \Delta\, \mathrm{K}_5 + \Delta\, \mathrm{U}_5 \; = \; 0 \,\right]$$

$$\mathrm{E}_6 \; = \; \text{constant} \qquad \left[\, \Delta\, \mathrm{E}_6 \; = \; \Delta\, \mathrm{K}_6 + \Delta\, \mathrm{U}_6 \; = \; 0 \,\right]$$

Observations

All equations of this paper can be applied in any inertial reference frame and also in any non-inertial reference frame.

Additionally, inertial reference frames and non-inertial reference frames must not introduce fictitious forces into $\mathbf{F}_i$.

In this paper, the magnitudes $[\, m, \mathbf{r}, \mathbf{v}, \mathbf{a}, \mathrm{M}, \mathbf{R}, \mathbf{V}, \mathbf{A}, \mathbf{T}, \mathbf{K}, \mathbf{F}, \mathbf{P}_1, \mathbf{L}_1, \mathbf{L}_2, \mathrm{W}_1, \mathrm{K}_1, \mathrm{U}_1, \mathrm{E}_1, \mathrm{L}_1$ W_2, K_2, U_2, E_2, L_2, W_3, K_3, U_3, E_3, W_4, K_4, U_4, E_4, W_5, K_5, U_5, E_5, W_6, K_6, U_6 and $\mathrm{E}_6\,]$ are invariant under transformations between inertial and non-inertial reference frames.

The mechanical energy E_3 of a system of particles is always zero $[\,\mathrm{E}_3 = \mathrm{K}_3 + \mathrm{U}_3 = 0\,]$

Therefore, the mechanical energy E_5 of a system of particles is always equal to the mechanical energy E_1 of the system of particles $[\,\mathrm{E}_5 = \mathrm{E}_1\,]$

The mechanical energy E_4 of a system of particles is always zero $[\,\mathrm{E}_4 = \mathrm{K}_4 + \mathrm{U}_4 = 0\,]$

Therefore, the mechanical energy E_6 of a system of particles is always equal to the mechanical energy E_2 of the system of particles $[\,\mathrm{E}_6 = \mathrm{E}_2\,]$

If the potential energy U_1 of a system of particles is a homogeneous function of degree k then the potential energy U_3 and the potential energy U_5 of the system of particles are given by: $[\,\mathrm{U}_3 = (\frac{k}{2})\,\mathrm{U}_1\,]$ and $[\,\mathrm{U}_5 = (1+\frac{k}{2})\,\mathrm{U}_1\,]$

If the potential energy U_2 of a system of particles is a homogeneous function of degree k then the potential energy U_4 and the potential energy U_6 of the system of particles are given by: $[\,\mathrm{U}_4 = (\frac{k}{2})\,\mathrm{U}_2\,]$ and $[\,\mathrm{U}_6 = (1+\frac{k}{2})\,\mathrm{U}_2\,]$

If the potential energy U_1 of a system of particles is a homogeneous function of degree k and if the kinetic energy K_5 of the system of particles is equal to zero, then we obtain: $[\,\mathrm{K}_1 = -\,\mathrm{K}_3 = \mathrm{U}_3 = (\frac{k}{2})\,\mathrm{U}_1 = (\frac{k}{2+k})\,\mathrm{E}_1\,]$

If the potential energy U_2 of a system of particles is a homogeneous function of degree k and if the kinetic energy K_6 of the system of particles is equal to zero, then we obtain: $[\,\mathrm{K}_2 = -\,\mathrm{K}_4 = \mathrm{U}_4 = (\frac{k}{2})\,\mathrm{U}_2 = (\frac{k}{2+k})\,\mathrm{E}_2\,]$

If the potential energy U_1 of a system of particles is a homogeneous function of degree k and if the average kinetic energy $\langle \mathrm{K}_5 \rangle$ of the system of particles is equal to zero, then we obtain: $[\,\langle \mathrm{K}_1 \rangle = -\,\langle \mathrm{K}_3 \rangle = \langle \mathrm{U}_3 \rangle = (\frac{k}{2})\,\langle \mathrm{U}_1 \rangle = (\frac{k}{2+k})\,\langle \mathrm{E}_1 \rangle\,]$

If the potential energy U_2 of a system of particles is a homogeneous function of degree k and if the average kinetic energy $\langle \mathrm{K}_6 \rangle$ of the system of particles is equal to zero, then we obtain: $[\,\langle \mathrm{K}_2 \rangle = -\,\langle \mathrm{K}_4 \rangle = \langle \mathrm{U}_4 \rangle = (\frac{k}{2})\,\langle \mathrm{U}_2 \rangle = (\frac{k}{2+k})\,\langle \mathrm{E}_2 \rangle\,]$

The average kinetic energy $\langle \mathrm{K}_5 \rangle$ and the average kinetic energy $\langle \mathrm{K}_6 \rangle$ of a system of particles with bounded motion (in $\langle \mathrm{K}_5 \rangle$ relative to $\vec{R}$ and in $\langle \mathrm{K}_6 \rangle$ relative to $\vec{R}_{cm}$) are always zero.

The kinetic energy K_5 and the kinetic energy K_6 of a system of N particles can also be expressed as follows : $[\ \mathrm{K}_5 = \sum_i^{\mathrm{N}} 1/2\ m_i\,(\,\dot{r}_i\,\dot{r}_i + \ddot{r}_i\,r_i\,)\ ]$ where $r_i \doteq |\,\vec{r}_i - \vec{R}\,|$ and $[\ \mathrm{K}_6 = \sum_{j>i}^{\mathrm{N}} 1/2\,m_i\,m_j\,\mathrm{M}^{-1}(\,\dot{r}_{ij}\,\dot{r}_{ij} + \ddot{r}_{ij}\,r_{ij}\,)\,]$ where $r_{ij} \doteq |\,\vec{r}_i - \vec{r}_j\,|$

The kinetic energy K_5 and the kinetic energy K_6 of a system of N particles can also be expressed as follows : $[\ \mathrm{K}_5 = \sum_i^{\mathrm{N}} 1/2\ m_i\,(\,\ddot{\tau}_i\,)\ ]$ where $\tau_i \doteq 1/2\ (\vec{r}_i - \vec{R}) \cdot (\vec{r}_i - \vec{R})$ and $[\ \mathrm{K}_6 = \sum_{j>i}^{\mathrm{N}} 1/2\,m_i\,m_j\,\mathrm{M}^{-1}(\,\ddot{\tau}_{ij}\,)\,]$ where $\tau_{ij} \doteq 1/2\ (\vec{r}_i - \vec{r}_j) \cdot (\vec{r}_i - \vec{r}_j)$

The kinetic energy K_6 is the only kinetic energy that can be expressed without the necessity of introducing any magnitude that is related to the free-system [such as: $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$, $\vec{\omega}$, $\vec{R}$, etc.]

In an isolated system of particles, the potential energy U_2 is equal to the potential energy U_1 if the internal non-kinetic forces obey Newton's third law in its weak form $[\,\mathrm{U}_2 = \mathrm{U}_1\,]$

In an isolated system of particles, the potential energy U_4 is equal to the potential energy U_3 if the internal non-kinetic forces obey Newton's third law in its weak form $[\,\mathrm{U}_4 = \mathrm{U}_3\,]$

In an isolated system of particles, the potential energy U_6 is equal to the potential energy U_5 if the internal non-kinetic forces obey Newton's third law in its weak form $[\,\mathrm{U}_6 = \mathrm{U}_5\,]$

A reference frame S is non-rotating if the angular velocity $\vec{\omega}$ of the free-system relative to S is equal to zero, and the reference frame S is also inertial if the acceleration $\vec{A}$ of the center of mass of the free-system relative to S is equal to zero.

If the origin of a non-rotating reference frame S $[\,\vec{\omega} = 0\,]$ always coincides with the center of mass of the free-system $[\,\vec{R} = \vec{V} = \vec{A} = 0\,]$ then relative to S: $[\,\mathbf{r}_i = \vec{r}_i,\ \mathbf{v}_i = \vec{v}_i$ and $\mathbf{a}_i = \vec{a}_i\,]$ Therefore, it is easy to see that always: $[\,\mathbf{v}_i = d(\mathbf{r}_i)/dt$ and $\mathbf{a}_i = d^2(\mathbf{r}_i)/dt^2\,]$

If kinetic forces are excluded, then this paper does not contradict Newton's first and second laws since they are valid in all inertial reference frames. The equation $[\,\mathbf{F}_i = m_i\,\mathbf{a}_i\,]$ is a simple reformulation of Newton's second law.

In this paper, the equation $[\,\mathbf{F}_i = m_i\,\mathbf{a}_i\,]$ would be valid in all reference frames (inertial or non-inertial) even if all non-kinetic forces were always disobeyed Newton's third law in its strong form and in its weak form.

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Appendix I

The Free-System

The free-system is a system of N particles that must always be free of internal and external non-kinetic forces, that must be three-dimensional, and that the relative distances between the N particles must be constant.

The position $\vec{R}$, the velocity $\vec{V}$ and the acceleration $\vec{A}$ of the center of mass of the free-system relative to a reference frame S (and the angular velocity $\vec{\omega}$ and the angular acceleration $\vec{\alpha}$ of the free-system relative to the reference frame S) are given by:

$$\mathrm{M} \doteq \textstyle\sum_i^{\mathrm{N}} m_i$$

$$\vec{R} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{\mathrm{N}} m_i \, \vec{r}_i$$

$$\vec{V} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{\mathrm{N}} m_i \, \vec{v}_i$$

$$\vec{A} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{\mathrm{N}} m_i \, \vec{a}_i$$

$$\vec{\omega} \doteq \overleftrightarrow{I}^{-1} \cdot \vec{L}$$

$$\vec{\alpha} \doteq d(\vec{\omega})/dt$$

$$\overleftrightarrow{I} \doteq \textstyle\sum_i^{\mathrm{N}} m_i \, [\, |\, \vec{r}_i - \vec{R}\,|^2 \, \overleftrightarrow{1} - (\vec{r}_i - \vec{R}) \otimes (\vec{r}_i - \vec{R})\,]$$

$$\vec{L} \doteq \textstyle\sum_i^{\mathrm{N}} m_i \, (\vec{r}_i - \vec{R}) \times (\vec{v}_i - \vec{V})$$

where M is the mass of the free-system, $\overleftrightarrow{I}$ is the inertia tensor of the free-system (relative to $\vec{R}$) and $\vec{L}$ is the angular momentum of the free-system relative to the reference frame S.

The Transformations

$$(\vec{r}_i - \vec{R}) \doteq \mathbf{r}_i = \mathbf{r}'_i$$

$$(\vec{r}'_i - \vec{R}') \doteq \mathbf{r}'_i = \mathbf{r}_i$$

$$(\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R}) \doteq \mathbf{v}_i = \mathbf{v}'_i$$

$$(\vec{v}'_i - \vec{V}') - \vec{\omega}' \times (\vec{r}'_i - \vec{R}') \doteq \mathbf{v}'_i = \mathbf{v}_i$$

$$(\vec{a}_i - \vec{A}) - 2\,\vec{\omega} \times (\vec{v}_i - \vec{V}) + \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,] - \vec{\alpha} \times (\vec{r}_i - \vec{R}) \doteq \mathbf{a}_i = \mathbf{a}'_i$$

$$(\vec{a}'_i - \vec{A}') - 2\,\vec{\omega}' \times (\vec{v}'_i - \vec{V}') + \vec{\omega}' \times [\,\vec{\omega}' \times (\vec{r}'_i - \vec{R}')\,] - \vec{\alpha}' \times (\vec{r}'_i - \vec{R}') \doteq \mathbf{a}'_i = \mathbf{a}_i$$

Appendix II

The Relations

In a system of particles, these relations can be obtained (The magnitudes $\mathbf{R}_{cm}$, $\mathbf{V}_{cm}$, $\mathbf{A}_{cm}$, $\vec{R}_{cm}$, $\vec{V}_{cm}$ and $\vec{A}_{cm}$ can be replaced by the magnitudes $\mathbf{R}$, $\mathbf{V}$, $\mathbf{A}$, $\vec{R}$, $\vec{V}$ and $\vec{A}$, or by the magnitudes $\mathbf{r}_j$, $\mathbf{v}_j$, $\mathbf{a}_j$, $\vec{r}_j$, $\vec{v}_j$ and $\vec{a}_j$, respectively. On the other hand, $\mathbf{R} = \mathbf{V} = \mathbf{A} = 0$)

$$\mathbf{r}_i \doteq (\vec{r}_i - \vec{R})$$

$$\mathbf{R}_{cm} \doteq (\vec{R}_{cm} - \vec{R})$$

$$\longrightarrow \quad (\mathbf{r}_i - \mathbf{R}_{cm}) = (\vec{r}_i - \vec{R}_{cm})$$

$$\mathbf{v}_i \doteq (\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R})$$

$$\mathbf{V}_{cm} \doteq (\vec{V}_{cm} - \vec{V}) - \vec{\omega} \times (\vec{R}_{cm} - \vec{R})$$

$$\longrightarrow \quad (\mathbf{v}_i - \mathbf{V}_{cm}) = (\vec{v}_i - \vec{V}_{cm}) - \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})$$

$$(\mathbf{v}_i - \mathbf{V}_{cm}) \cdot (\mathbf{v}_i - \mathbf{V}_{cm}) = \big[(\vec{v}_i - \vec{V}_{cm}) - \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] \cdot \big[(\vec{v}_i - \vec{V}_{cm}) - \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] =$$

$$(\vec{v}_i - \vec{V}_{cm}) \cdot (\vec{v}_i - \vec{V}_{cm}) - 2\,(\vec{v}_i - \vec{V}_{cm}) \cdot \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] + \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] \cdot \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] =$$

$$(\vec{v}_i - \vec{V}_{cm}) \cdot (\vec{v}_i - \vec{V}_{cm}) + 2\,(\vec{r}_i - \vec{R}_{cm}) \cdot \big[\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] + \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] \cdot \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] =$$

$$(\vec{v}_i - \vec{V}_{cm}) \cdot (\vec{v}_i - \vec{V}_{cm}) + \big[2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] \cdot \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] =$$

$$(\vec{v}_i - \vec{V}_{cm})^2 + \big[2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big]^2$$

$$(\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) = \big\{ (\vec{a}_i - \vec{A}_{cm}) - 2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) + \vec{\omega} \times [\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})] -$$

$$\vec{\alpha} \times (\vec{r}_i - \vec{R}_{cm}) \big\} \cdot (\vec{r}_i - \vec{R}_{cm}) = (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) - \big[2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) +$$

$$\big\{ \vec{\omega} \times [\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})] \big\} \cdot (\vec{r}_i - \vec{R}_{cm}) - \big[\vec{\alpha} \times (\vec{r}_i - \vec{R}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) = (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) -$$

$$\big[2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big\{ \big[\vec{\omega} \cdot (\vec{r}_i - \vec{R}_{cm}) \big] \vec{\omega} - (\vec{\omega} \cdot \vec{\omega})\,(\vec{r}_i - \vec{R}_{cm}) \big\} \cdot (\vec{r}_i - \vec{R}_{cm}) =$$

$$(\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) - \big[2\,\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big[\vec{\omega} \cdot (\vec{r}_i - \vec{R}_{cm}) \big]^2 - (\vec{\omega})^2 (\vec{r}_i - \vec{R}_{cm})^2$$

$$\longrightarrow \quad (\mathbf{v}_i - \mathbf{V}_{cm})^2 + (\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) = (\vec{v}_i - \vec{V}_{cm})^2 + (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm})$$

Appendix III

The Magnitudes

The magnitudes $\mathbf{L}_2$, W_2, K_2, U_2, W_4, K_4, U_4, W_6, K_6 and U_6 of a system of N particles can also be expressed as follows:

$$\mathbf{L}_2 \;=\; \textstyle\sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\, (\mathbf{r}_i - \mathbf{r}_j) \times (\mathbf{v}_i - \mathbf{v}_j) \,\big]$$

$$W_2 \;=\; \textstyle\sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\mathbf{r}_i - \mathbf{r}_j) \,\big]$$

$$\Delta K_2 \;=\; \textstyle\sum_{j>i}^{N} \Delta\, {}^1\!/_2\, m_i\, m_j\, M^{-1} (\mathbf{v}_i - \mathbf{v}_j)^2 \;=\; W_2$$

$$\Delta U_2 \;=\; -\textstyle\sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\mathbf{r}_i - \mathbf{r}_j) \,\big]$$

$$W_4 \;=\; \textstyle\sum_{j>i}^{N} \Delta\, {}^1\!/_2\, m_i\, m_j\, M^{-1} \big[\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\mathbf{r}_i - \mathbf{r}_j) \,\big]$$

$$\Delta K_4 \;=\; \textstyle\sum_{j>i}^{N} \Delta\, {}^1\!/_2\, m_i\, m_j\, M^{-1} \big[\, (\mathbf{a}_i - \mathbf{a}_j) \cdot (\mathbf{r}_i - \mathbf{r}_j) \,\big] \;=\; W_4$$

$$\Delta U_4 \;=\; -\textstyle\sum_{j>i}^{N} \Delta\, {}^1\!/_2\, m_i\, m_j\, M^{-1} \big[\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\mathbf{r}_i - \mathbf{r}_j) \,\big]$$

$$W_6 \;=\; \textstyle\sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\vec{r}_i - \vec{r}_j) + \Delta\, {}^1\!/_2\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\vec{r}_i - \vec{r}_j) \,\big]$$

$$\Delta K_6 \;=\; \textstyle\sum_{j>i}^{N} \Delta\, {}^1\!/_2\, m_i\, m_j\, M^{-1} \big[\, (\vec{v}_i - \vec{v}_j)^2 + (\vec{a}_i - \vec{a}_j) \cdot (\vec{r}_i - \vec{r}_j) \,\big] \;=\; W_6$$

$$\Delta U_6 \;=\; -\textstyle\sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\vec{r}_i - \vec{r}_j) + \Delta\, {}^1\!/_2\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\vec{r}_i - \vec{r}_j) \,\big]$$

The magnitudes $W_{(1 \text{ to } 6)}$ and $U_{(1 \text{ to } 6)}$ of an isolated system of N particles, whose internal non-kinetic forces obey Newton's third law in its weak form, can be reduced to:

$$W_1 \;=\; W_2 \;=\; \textstyle\sum_{i}^{N} \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i$$

$$\Delta U_1 \;=\; \Delta U_2 \;=\; -\textstyle\sum_{i}^{N} \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i$$

$$W_3 \;=\; W_4 \;=\; \textstyle\sum_{i}^{N} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \vec{r}_i$$

$$\Delta U_3 \;=\; \Delta U_4 \;=\; -\textstyle\sum_{i}^{N} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \vec{r}_i$$

$$W_5 \;=\; W_6 \;=\; \textstyle\sum_{i}^{N} \big[\, \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \vec{r}_i \,\big]$$

$$\Delta U_5 \;=\; \Delta U_6 \;=\; -\textstyle\sum_{i}^{N} \big[\, \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \vec{r}_i \,\big]$$