LAWSON HOMOLOGY FOR ABELIAN VARIETIES

WENCHUAN HU

Abstract. In this paper we introduce the Fourier-Mukai transform for Lawson homology of abelian varieties and prove an inversion theorem for the Lawson homology as well as the morphic cohomology of abelian varieties. As applications, we obtain the direct sum decomposition of the Lawson homology and the morphic cohomology groups with rational coefficients, inspired by Beauville's works on the Chow theory. An analogue of the Beauville conjecture for Chow groups is proposed and is shown to be equivalent to the (weak) Suslin conjecture for Lawson homology. A filtration on Lawson homology is proposed and conjecturally it coincides to the filtration given by the direct sum decomposition of Lawson homology for abelian varieties. Moreover, a refined Friedlander-Lawson duality theorem is obtained for abelian varieties. We summarize several related conjectures in Lawson homology theory in the appendix for convenience.

Contents

1. Notations

In this paper, all varieties are defined over the complex number field $\mathbb{C}$. Let X be a projective variety of dimension n. Denoted by $\mathcal{Z}_p(X)$ the space of algebraic p-cycles on X. Let $\mathrm{Ch}_p(X)$ be the Chow group of p-cycles on X, i.e. $\mathrm{Ch}_p(X) = \mathcal{Z}_p(X)/\{\text{rational equivalence}\}$. Set $\mathrm{Ch}_p(X)_{\mathbb{Q}} := \mathrm{Ch}_p(X)\otimes\mathbb{Q}$, $\mathrm{Ch}_p(X) = \bigoplus_{p\geq 0}\mathrm{Ch}_p(X)$ and $\mathrm{Ch}_*(X)_{\mathbb{Q}} = \bigoplus_{p\geq 0}\mathrm{Ch}_p(X)_{\mathbb{Q}}$. Let $A_p(X)$ be the space of p-cycles

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on X modulo the algebraic equivalence, i.e. $A_p(X) = \mathcal{Z}_p(X)/\sim_{alg}$, where $\sim_{alg}$ denotes the algebraic equivalence. Set $A_p(X)_{\mathbb{Q}} := A_p(X) \otimes \mathbb{Q}$, $A(X) = \bigoplus_{p\geq 0} A_p(X)$ and $A_*(X)_{\mathbb{Q}} = \bigoplus_{p\geq 0} A_p(X)_{\mathbb{Q}}$.

2. Lawson homology

The *Lawson homology* $L_pH_k(X)$ of p-cycles for a projective variety is defined by

$$L_pH_k(X) := \pi_{k-2p}(\mathcal{Z}_p(X)) \quad for \quad k \geq 2p \geq 0,$$

where $\mathcal{Z}_p(X)$ is provided with a natural topology (cf. [F1], [L1]). It has been extended to define for a quasi-projective variety by Lima-Filho (cf. [LF]) and Chow motives (cf. [HL]). For general background, the reader is referred to Lawson' survey paper [L2]. The definition of Lawson homology has been extended to negative integer p. Formally for $p < 0$, we have $L_pH_k(X) = \pi_{k-2p}(\mathcal{Z}_0(X \times \mathbb{C}^{-p})) = H^{BM}_{k-2p}(X \times \mathbb{C}^{-p}) = H^{BM}_k(X) = L_0H_k(X)$ (cf. [FHW]), where $H^{BM}_*(-)$ denotes the Borel-Moore homology.

In [FM], Friedlander and Mazur showed that there are natural transformations, called *Friedlander-Mazur cycle class maps*

$$\Phi_{p,k} : L_pH_k(X) \to H_k(X) \tag{1}$$

for all $k \geq 2p \geq 0$.

Recall that Friedlander and Mazur constructed a map called the s-map $s : L_pH_k(X) \to L_{p-1}H_k(X)$ such that the cycle class map $\Phi_{p,k} = s^p$ ([FM]). Explicitly, if $\alpha \in L_pH_k(X)$ is represented by the homotopy class of a contiunous map $f : S^{k-2p} \to \mathcal{Z}_p(X)$, then $\Phi_{p,k}(\alpha) = [f \wedge S^{2p}]$, where $S^{2p} = S^2 \wedge \cdots \wedge S^2$ denotes the $2p$-dimensional topological sphere.

Set

$$\begin{array}{lcl} L_pH_k(X)_{hom} & := & \ker\{\Phi_{p,k} : L_pH_k(X) \to H_k(X)\}; \\ L_pH_k(X)_{\mathbb{Q}} & := & L_pH_k(X) \otimes \mathbb{Q}; \\ T_pH_k(X) & := & \mathrm{Image}\{\Phi_{p,k} : L_pH_k(X) \to H_k(X)\}; \\ T_pH_k(X,\mathbb{Q}) & := & T_pH_k(X) \otimes \mathbb{Q}. \end{array}$$

Denoted by $\Phi_{p,k,\mathbb{Q}}$ the map $\Phi_{p,k} \otimes \mathbb{Q} : L_pH_k(X)_{\mathbb{Q}} \to H_k(X,\mathbb{Q})$. The *Griffiths group* of dimension p-cycles is defined to be

$$\mathrm{Griff}_p(X) := \mathcal{Z}_p(X)_{hom}/\mathcal{Z}_p(X)_{alg}.$$

Set

$$\begin{array}{lcl} \mathrm{Griff}_p(X)_{\mathbb{Q}} & := & \mathrm{Griff}_p(X) \otimes \mathbb{Q}; \\ \mathrm{Griff}^q(X) & := & \mathrm{Griff}_{n-q}(X); \\ \mathrm{Griff}^q(X)_{\mathbb{Q}} & := & \mathrm{Griff}_{n-q}(X)_{\mathbb{Q}}. \end{array}$$

It was proved by Friedlander [F1] that, for any smooth projective variety X,

$$L_pH_{2p}(X) \cong \mathcal{Z}_p(X)/\mathcal{Z}_p(X)_{alg} = A_p(X).$$

Therefore

$$L_pH_{2p}(X)_{hom} \cong \mathrm{Griff}_p(X).$$

For any smooth quasi-projective variety X, there is an intersection pairing (cf. [FG])

$$L_pH_k(X) \otimes L_qH_l(X) \to L_{p+q-n}H_{k+l-2n}(X),$$

induced by the diagonal map $\Delta : X \to X \times X$. More precisely, the composition $\mathcal{Z}_p(X) \times \mathcal{Z}_q(X) \stackrel{\times}{\to} \mathcal{Z}_{p+q}(X \times X) \stackrel{\Delta^!}{\to} Z_{p+q-n}(X)$, where $\times$ is the Cartesian product of cycles and $\Delta^!$ is the Gysin map, factors through $\mathcal{Z}_p(X) \wedge \mathcal{Z}_q(X)$. On the level of homotopy groups we have intersection pairing

$$\pi_{k-2p}(\mathcal{Z}_p(X)) \otimes \pi_{l-2q}(\mathcal{Z}_q(X)) \xrightarrow{\bullet} \pi_{k+l-2(p+q)}(\mathcal{Z}_{p+q-n}(X)),$$

that is,

$$L_pH_k(X) \otimes L_qH_l(X) \xrightarrow{\bullet} L_{p+q-n}H_{k+l-2n}(X).$$

3. Correspondences

In this section we recall basic materials of correspondences and their actions on Lawson homology (cf. [FM], [Pe], [HL]). For closely related materials on Chow correspondences we refer to Manin [Ma] and Fulton [Fu].

A **correspondence** Γ from X to Y is an algebraic cycle (or an equivalent class of cycles depending on the context) on $X \times Y$. We denote the group of correspondences of rational equivalence classes between varieties X and Y by

$$Corr_d(X,Y) := A_{\dim X+d}(X \times Y).$$

Let X, Y be smooth projective varieties and let $\Gamma \in Corr_d(X,Y)$ for $d \in \mathbb{Z}$. Then for any element $\alpha \in L_pH_k(X)$, the push-forward morphism is defined by

$$\Gamma_* : L_pH_k(X) \to L_{p+d}H_{k+2d}(Y)$$
$$\Gamma_*(\alpha) = p_{2*}(p_1^*\alpha \bullet \Gamma),$$

where p_1 (resp. p_2) denotes the projection from $X \times Y$ onto X (resp. Y) and "$\bullet$" is the intersection product on the group $A(X \times Y)$.

Let X, Y, Z be smooth projective varieties. The composition of two correspondences $\Gamma_1 \in Corr_{d_1}(X,Y)$ and $\Gamma_2 \in Corr_{d_2}(Y,Z)$ is given by the formula

$$\Gamma_2 \circ \Gamma_1 = p_{13*}(p_{12}^*\Gamma_1 \cdot p_{23}^*\Gamma_2) \in Corr_{d_1+d_2}(X,Z)$$

where p_{ij}, $i,j = 1,2,3$ are the projection of $X \times Y \times Z$ on the product of its ith and jth factors.

Lemma 3.1. *Let X, Y, Z be smooth projective varieties, $\Gamma_1 \in Corr_d(X,Y)$ and $\Gamma_2 \in Corr_e(Y,Z)$. Then for any $u \in L_pH_k(X)$, we have*

$$(\Gamma_2 \circ \Gamma_1)_*u = \Gamma_{2*}\Gamma_{1*}u \in L_{p+d+e}H_{k+2d+2e}(Z).$$

Proof. cf. [HL, Prop. 4.2]. □

For a projective morphism $f : X_1 \to X_2$ the *graph* of f is defined to be the correspondence

$$\Gamma_f := (\mathrm{id}_{X_1}, f)_*(X_1) \in A(X_1 \times X_2).$$

Lemma 3.2. (1) *$(\Gamma_f)_*(\alpha) = f_*(\alpha)$ for $\alpha \in L_pH_k(X_1)$.*
(2) *$({}^t\Gamma_f)_*(\beta) = f^*(\beta)$ for $\beta \in L_qH_l(X_2)$.*

Proof. From the projection formula (cf. [Pe, Lemma 11. c)]), we have

$$\begin{aligned} (\Gamma_f)_*(\alpha) &= p_{2*}\big((\mathrm{id}_{X_1}, f)_*(X_1) \bullet p_1^*\alpha\big) \\ &= p_{2*}(\mathrm{id}_{X_1}, f)_*\big((\mathrm{id}_{X_1}, f)^* p_1^*\alpha\big) \\ &= f_*(\alpha), \end{aligned}$$

since $p_1 \circ (\mathrm{id}_{X_1}, f) = \mathrm{id}_{X_1}$ and $p_2 \circ (\mathrm{id}_{X_1}, f) = f$.

The proof of (2) is similar. □

Lemma 3.3. *Let $f_i : Y_i \to X_i$, for $i = 1, 2$ be projective morphisms of smooth projective varieties. Then*

(1) $(f_1 \times f_2)^* Z =^t \Gamma_{f_2} \circ Z \circ \Gamma_{f_1}$ *for all* $Z \in A(X_1 \times X_2)$*;*
(2) $(f_1 \times f_2)_* \tilde{Z} = \Gamma_{f_2} \circ \tilde{Z} \circ^t \Gamma_{f_1}$ *for all* $\tilde{Z} \in A(Y_1 \times Y_2)$*.*

Proof. We give a proof for 1), the proof of 2) is similar. To prove 1) it is enough to show that $(f_1 \times id)^* Z = Z \circ \Gamma_{f_1}$ and $(id \times f_2)^* Z =^t \Gamma_{f_2} \circ Z$. Denote by q_1 the projection $Y_1 \times X_2 \to Y_1$ and p_{ij} the projections of $Y_1 \times X_1 \times X_2$, e.g. $p_{12} : Y_1 \times X_1 \times X_2 \to Y_1 \times X_1$ is the projection onto the Cartesian product first two varieties. Then by applying the base change formula for Lawson homology (cf. [Pe, Lemma 11 a)]) to $(id_{Y_1}, f_1) \circ q_1 = p_{12} \circ ((id_{Y_1}, f_1) \times id_{X_2})$ and the projection formula (cf. [Pe, Lemma 11 c)]), we have

$$\begin{aligned} Z \circ \Gamma_{f_1} &= p_{13*}(p_{23}^* Z \cdot p_{12}^* (id_{Y_1}, f_1)_*(Y_1)) \\ &= p_{13*}(p_{23}^* Z \cdot ((id_{Y_1}, f_1) \times id_{X_2})_* q_1^*(Y_1)) \\ &= p_{13*}((id_{Y_1}, f_1) \times id_{X_2})_*((id_{Y_1}, f_1) \times id_{X_2})^* p_{23}^* Z \cdot q_1^*(Y_1)) \\ &= p_{13*}((id_{Y_1}, f_1) \times id_{X_2})_*((id_{Y_1}, f_1) \times id_{X_2})^* p_{23}^* Z) \\ &= (f_1 \times id_{X_2})^* Z, \end{aligned}$$

where we used $p_{13} \circ ((id_{Y_1}, f_1) \times id_{X_2}) = id_{Y_1 \times X_2}$ and $p_{23} \circ ((id_{Y_1}, f_1) \times id_{X_2}) = f_1 \times id_{X_2}$. From this we get

$$(id_{X_1} \times f_2)^* Z = {}^t((f_2 \times id_{X_1})^{*t} Z) = {}^t({}^t Z \circ \Gamma_{f_2}) = {}^t\Gamma_{f_2} \circ Z.$$

□

From Lemma 3.1-3.3, we have

$$(f_1 \times f_2)^* Z_*(\alpha) = f_2^*(Z_* f_{1*}(\alpha)), \forall \alpha \in L_* H_*(Y_1) \tag{2}$$

and

$$(f_1 \times f_2)_* Z_*(\beta) = f_{2*}(Z_* f_1^*(\beta)), \forall \beta \in L_* H_*(X_1). \tag{3}$$

4. Fourier-Mukai transform

Let X be an abelian variety of dimension n and let $\widehat{X}$ be the dual abelian variety of X, i.e., $\widehat{X} = Pic^0(X)$. Consider the Poincaré bundle $\mathcal{P} = \mathcal{P}_X \in Pic(X \times \widehat{X}) = \mathrm{Ch}^1(X \times \widehat{X})$. The correspondence

$$e^{\mathcal{P}} := \sum_{i \geq 0} \frac{1}{i!} \mathcal{P}^i \in \mathrm{Ch}_*(X)_{\mathbb{Q}}$$

is well-defined since the sum is finite, where $\mathcal{P}^i$ denotes the i-th intersection product.

The Fourier-Mukai transform on Chow group $\mathrm{Ch}(X)_{\mathbb{Q}}$ with rational coefficients is defined to be the homomorphism of groups $F = F_X : \mathrm{Ch}(X)_{\mathbb{Q}} \to \mathrm{Ch}(\widehat{X})_{\mathbb{Q}}$, $\alpha \mapsto p_{2*}(e^{\mathcal{P}} \cdot p_1^* \alpha)$.

Similarly, the **Fourier-Mukai transform** on Lawson homology $L_* H_*(X)_{\mathbb{Q}}$ with rational coefficients is defined to be the homomorphism of groups $F = F_X : L_* H_*(X)_{\mathbb{Q}} \to L_* H_*(\widehat{X})_{\mathbb{Q}}$, $\alpha \mapsto p_{2*}(e^{\mathcal{P}} \cdot p_1^* \alpha)$. In particular, the Fourier-Mukai transform on homology $H_*(X, \mathbb{Q})$ with rational coefficients is defined to be the homomorphism of groups $F = F_X : H_*(X, \mathbb{Q}) \to H_*(\widehat{X}, \mathbb{Q})$, $\alpha \mapsto p_{2*}(e^{\mathcal{P}} \cdot p_1^* \alpha)$.

Theorem 4.1 (Inversion Theorem)**.** *Let X be an abelian variety of dimension n. Then we have*

$$F_{\widehat{X}} \circ F_X = (-1)^d(-1)^*_X : L_*H_*(X)_{\mathbb{Q}} \to L_*H_*(X)_{\mathbb{Q}},$$

where $-1 : X \to X$ is the multiplication by -1 on X.

Proof. By definition, we need to show that

$$e^{\mathcal{P}_{\widehat{X}}} \circ e^{\mathcal{P}_X} = (-1)^d\Gamma_{-1} \in L_*H_*(X)_{\mathbb{Q}}.$$

Since (cf. [Mk])

$$e^{\mathcal{P}_{\widehat{X}}} \circ e^{\mathcal{P}_X} = (-1)^d\Gamma_{-1} \in \mathrm{Ch}_*(X \times X)_{\mathbb{Q}},$$

we get

$$e^{\mathcal{P}_{\widehat{X}}} \circ e^{\mathcal{P}_X} = (-1)^d\Gamma_{-1} \in A_*(X \times X)_{\mathbb{Q}}$$

Since by construction the action of correspondence $e^{\mathcal{P}_{\widehat{X}}} \circ e^{\mathcal{P}_X}$ on Lawson homology depends only on its class in $A_*(X \times X)$ (cf. [Pe]), it implies that $e^{\mathcal{P}_{\widehat{X}}} \circ e^{\mathcal{P}_X} = (-1)^d\Gamma_{-1} : L_*H_*(X)_{\mathbb{Q}} \to L_*H_*(X)_{\mathbb{Q}}$.

□

Proposition 4.2. *Let $f : Y \to X$ be an isogeny of abelian varieties. Then for all $\alpha \in L_*H_*(Y)_{\mathbb{Q}}$ and $\beta \in L_*H_*(X)_{\mathbb{Q}}$*

1. *$F_X f_*(\alpha) = \hat{f}^* F_Y(\alpha)$;*
2. *$F_Y f^*(\beta) = \hat{f}_* F_X(\beta)$.*

Proof. (1) The universal property of the Poincaré bundle implies that

$$(f \times \mathrm{id}_{\widehat{X}})\mathcal{P}_X = (\mathrm{id}_Y \times \hat{f})^*\mathcal{P}_Y.$$

By Equation (2), we get

$$\begin{aligned} F_X f_*(\alpha) &= (e^{\mathcal{P}_X})_* f_*\alpha \\ &= (f \times id_{\widehat{X}})^*(e^{\mathcal{P}_X})_*\alpha \\ &= (id_Y \times \hat{f})^*(e^{\mathcal{P}_Y})_*\alpha \\ &= \hat{f}^*(e^{\mathcal{P}_Y})_*\alpha \\ &= \hat{f}^* F_Y(\alpha). \end{aligned}$$

(2) By applying (1) to $\hat{f} : \widehat{X} \to \widehat{Y}$ we get

$$\begin{aligned} F_Y f^* &= (-1)^g(-1)^*_{\widehat{Y}} F_Y f^* F_{\widehat{X}} F_X && \text{(By Theorem 4.1)} \\ &= (-1)^g(-1)^*_{\widehat{Y}} F_Y F_{\widehat{Y}} \hat{f}^* F_X && \text{(By Part (1))} \\ &= \hat{f}^* F_X. && \text{(By Theorem 4.1)} \end{aligned}$$

□

Now we show that the Fourier-Mukai transform $F : L_*H_*(X)_{\mathbb{Q}} \to L_*H_*(X)_{\mathbb{Q}}$ on Lawson homology groups is compatible to that on the singular homology with rational coefficients. That is, we have the following result.

Proposition 4.3. *There is a commutative diagram*

$$\begin{array}{ccc} L_pH_k(X)_{\mathbb{Q}} & \xrightarrow{F_X} & L_pH_k(\widehat{X})_{\mathbb{Q}} \\ \downarrow{\scriptstyle \Phi_{p,k,\mathbb{Q}}} & & \downarrow{\scriptstyle \Phi_{p,k,\mathbb{Q}}} \\ H_k(X, \mathbb{Q}) & \xrightarrow{F_X} & H_k(\widehat{X}, \mathbb{Q}) \end{array}$$

Proof. It follows from the definitions of the Fourier-Mukai transform on Lawson homology and the singular homology. □

5. Pontryagin Product

In this section, X denotes an abelian variety of dimension n. Let $\mu : X \times X \to X$ denote the sum map $\mu(z, z') = z + z'$. The morphism μ induces a continuous map $\mu_* : \mathcal{Z}_p(X \times X) \to \mathcal{Z}_p(X)$ between the space of algebraic cycles.

For $Z = \sum n_i V_i \in \mathcal{Z}_p(X)$ and $Z' = \sum m_j W_j \in \mathcal{Z}_q(X)$, we set $Z \times Z' = \sum n_i m_j V_i \times W_j \in \mathcal{Z}_{p+q}(X \times X)$. So we get a bilinear continuous map $\times : \mathcal{Z}_p(X) \times \mathcal{Z}_q(X) \to \mathcal{Z}_{p+q}(X \times X)$.

Therefore we get a continuous composed map $\mu_* \circ \times : \mathcal{Z}_p(X) \times \mathcal{Z}_q(X) \to \mathcal{Z}_{p+q}(X)$. We choose the "empty cycle" $\emptyset_p$ (resp. $\emptyset_q$, $\emptyset_{p+q}$) as the base point in $\mathcal{Z}_p(X)$ (resp. $\mathcal{Z}_q(X)$, $\mathcal{Z}_{p+q}(X)$) so that each of $\mathcal{Z}_p(X)$, $\mathcal{Z}_q(X)$ and $\mathcal{Z}_{p+q}(X)$ is a point topological abelian group. Note that we have both $\mu_* \circ \times (Z \times \emptyset_q) = \emptyset_{p+q}$ and $\mu_* \circ \times (\emptyset_p \times Z') = \emptyset_{p+q}$ This implies that the map $\times$ factors through $\mathcal{Z}_p(X) \wedge \mathcal{Z}_q(X)$, i.e., there is a commutative diagram of continuous maps

$$\begin{array}{ccccc} \mathcal{Z}_p(X) \times \mathcal{Z}_q(X) & & & & \\ \downarrow & \searrow^{\times} & & & \\ \mathcal{Z}_p(X) \wedge \mathcal{Z}_q(X) & \dashrightarrow & \mathcal{Z}_{p+q}(X \times X) & \xrightarrow{\mu_*} & \mathcal{Z}_{p+q}(X). \end{array}$$

For $\alpha \in L_pH_k(X)$ and $\beta \in L_qH_l(X)$, the **Pontryagin Product**

$$* : L_pH_k(X) \otimes L_qH_l(X) \to L_{p+q}H_{k+l}(X),\ (\alpha, \beta) \mapsto \alpha * \beta$$

is defined to be the image of the homotopy class of $f \wedge g : S^{k+l-2(p+q)} \to \mathcal{Z}_{p+q}(X)$ under μ_*, where $f : S^{k-2p} \to \mathcal{Z}_p(X)$ (resp. $g : S^{l-2q} \to \mathcal{Z}_q(X)$) is a representative element of α (resp. β) and $f \wedge g$ is the composed map $S^{k+l-2(p+q)} = S^{k-2p} \wedge S^{l-2q} \to \mathcal{Z}_p(X) \wedge \mathcal{Z}_q(X) \to \mathcal{Z}_{p+q}(X \times X)$. The homotopy class of $f \wedge g$ defines an element in $L_{p+q}H_{k+l}(X \times X)$ and so $\mu_*([f \wedge g])$ gives us an element in $L_{p+q}H_{k+l}(X)$. By comparing the intersection of Lawson homology defined by Friedlander and Gabber ([FG]), this product $\alpha * \beta := \mu_*([f \wedge g])$ is equal to $\mu_*(p_1^*\alpha \bullet p_2^*\beta)$.

Lemma 5.1. *The Pontryagin product is bilinear, associative and anti-commutative for the second index on* $L_*H_*(X)$.

Proof. The bilinear property follows exactly from the above definition of $*$. Note that $f \wedge g = (-1)^{k+l-2(p+q)} g \wedge f = (-1)^{k+l} g \wedge f$, where f, g are given as above. Hence we get the anti-commutativity for the second index. The associativity follows from the associativity of $\wedge$. □

Proposition 5.2. *For all* $\alpha, \beta \in L_*H_*(X)$, *we have*

(1) $F(\alpha * \beta) = F(\alpha) \cdot F(\beta)$;
(2) $F(\alpha \cdot \beta) = (-1)^n F(\alpha) * F(\beta)$.

Proof. (1) We denote by q_i and q_{ij} the projections of $X \times X \times \widehat{X}$. Note first that $(\mu \times 1_{\widehat{X}})^* \mathcal{P}_X = q_{13}^* \mathcal{P}_X \cdot q_{23}^* \mathcal{P}_X$ holds in $\mathrm{Ch}_*(X \times X \times \widehat{X})$ (cf. [BL, Lemma 14.1.7]) implies that $(\mu \times 1_{\widehat{X}})^* e^{\mathcal{P}_X} = q_{13}^* e^{\mathcal{P}_X} \cdot q_{23}^* e^{\mathcal{P}_X}$ holds in

$\mathrm{Ch}_*(X \times X \times \widehat{X})$ and so in $A(X \times X \times \widehat{X})$. Then

$$
\begin{array}{rcl}
F(\alpha * \beta) & = & p_{2*}\big(e^{\mathcal{P}_X} \cdot p_1^*(\mu(\alpha \times \beta))\big) \\
& = & p_{2*}(e^{\mathcal{P}_X} \cdot (\mu \times 1_{\widehat{X}})_* q_{12}^*(\alpha \times \beta)) \\
& = & p_{2*}(e^{\mathcal{P}_X} \cdot (\mu \times 1_{\widehat{X}})_* q_1^*\alpha \cdot q_2^*\beta)) \\
& = & p_{2*}(\mu \times 1_{\widehat{X}})_*\big((\mu \times 1_{\widehat{X}})^* e^{\mathcal{P}_X} \cdot q_1^*\alpha \cdot q_2^*\beta)\big) \\
& = & p_{2*}(\mu \times 1_{\widehat{X}})_*\big(q_{13}^* e^{\mathcal{P}_X} \cdot q_{23}^* e^{\mathcal{P}_X} \cdot q_1^*\alpha \cdot q_2^*\beta)\big) \\
& = & q_{3*}(q_{13}^* e^{\mathcal{P}_X} \cdot q_1^* \cdot q_{23}^* e^{\mathcal{P}_X} \cdot q_2^*\beta) \\
& & \quad (\text{since } q_3 = \mu \times id_{\widehat{X}}) \\
& = & p_{2*}q_{13*}(q_{13}^* e^{\mathcal{P}_X} \cdot q_1^*\alpha \cdot q_{23}^* e^{\mathcal{P}_X} \cdot q_2^*\beta) \\
& & \quad (\text{since } q_3 = p_2 \circ q_{13}) \\
& = & p_{2*}q_{13*}\big(q_{13}^*(e^{\mathcal{P}_X} \cdot p_1^*\alpha) \cdot q_{23}^*(e^{\mathcal{P}_X} \cdot p_1^*\beta)\big) \\
& & \quad (\text{since } q_1 = p_1 \circ q_{13}, q_2 = p_1 \circ q_{23}) \\
& = & p_{2*}\big(e^{\mathcal{P}_X} \cdot p_1^*\alpha \cdot q_{13*}q_{23}^*(e^{\mathcal{P}_X} \cdot p_1^*\beta)\big) \\
& & \quad (\text{since } q_3 = p_2 \circ q_{13} \text{ and } q_3 = p_2 \circ q_{23}) \\
& = & p_{2*}\big(e^{\mathcal{P}_X} \cdot p_1^*\alpha \cdot p_2^* p_{2*}(e^{\mathcal{P}_X} \cdot p_1^*\beta)\big) \\
& = & p_{2*}(e^{\mathcal{P}_X} \cdot p_1^*\alpha) \cdot p_{2*}(e^{\mathcal{P}_X} \cdot p_1^*\beta)) \\
& = & F(\alpha) \cdot F(\beta).
\end{array}
$$

(2) The statement (2) follows from Part (1) by the Inversion Theorem 4.1.

□

Proposition 5.3. *The Pontryagin product is compatible with the natural transformation* $\Phi_{p,k} : L_pH_k(X) \to H_k(X)$. *More precisely, we have the following commutative diagram:*

$$
\begin{array}{ccc}
L_pH_k(X) \otimes L_qH_l(X) & \xrightarrow{\ *\ } & L_{p+q}H_{k+l}(X) \\
\Big\downarrow{\scriptstyle \Phi_{p,k}\otimes\Phi_{q,l}} & & \Big\downarrow{\scriptstyle \Phi_{p+q,k+l}} \\
H_k(X) \otimes H_l(X) & \xrightarrow{\ *\ } & H_{k+l}(X).
\end{array}
$$

Proof. Let $\alpha \in L_pH_k(X)$ (resp. $\beta \in L_qH_l(X)$) be represented by the homotopy class of the map $f : S^{k-2p} \to \mathcal{Z}_p(X)$ (resp. $g : S^{l-2q} \to \mathcal{Z}_q(X)$). Then by definition we have $\alpha * \beta = \mu_*([f \wedge g])$.

Recall that from the property of s-map (cf. [FM, Chapter 6]), one has the explicitly formulas

$$\Phi_{p,k}([f]) = [f \wedge S^{2p}]; \Phi_{q,l}([g]) = [g \wedge S^{2q}].$$

Hence

$$
\begin{array}{rcl}
\Phi_{p,k}(\alpha) * \Phi_{q,l}(\beta) & = & \Phi_{p,k}([f]) * \Phi_{q,l}([g]) \\
& = & \mu_*[f \wedge S^{2p} \wedge g \wedge S^{2q}] \\
& = & \mu_*[f \wedge g \wedge S^{2p} \wedge S^{2q}] \\
& = & \mu_*[f \wedge g \wedge S^{2(p+q)}] \\
& = & \mu_*\big(\Phi_{p+q,k+l}([f \wedge g])\big) \\
& = & \Phi_{p+q,k+l}\big(\mu_*([f \wedge g])\big) \\
& = & \Phi_{p+q,k+l}\big(\alpha * \beta\big).
\end{array}
$$

The penultimate equality holds since $\Phi_{p+q,k+l} : L_{p+q}H_{k+l}(X) \to H_{k+l}(X)$ is a natural transformation from Lawson homology to the singular homology. This completes the proof of the commutative diagram. □

Remark 5.4. *From the proof of the above proposition, we observe that the Pontryagin product $*$ is compatible with the s-map.*

6. Decompositions of Lawson homology groups for abelian varieties

Let X be an abelian variety of dimension n. For each integer m, there is a homomorphism $m_X : X \to X$ defined by $x \mapsto m \cdot x$. Recall that we have cycle class map $\Phi_{p,k} \otimes \mathbb{Q} : L_pH_k(X)_{\mathbb{Q}} \to H_k(X, \mathbb{Q})$ for all $k \geq 2p \geq 0$. By considering elements in $H_k(X, \mathbb{Q}) \cong H^k(X, \mathbb{Q})$ as the dual of differential forms, it is easy to see that the induced map $m_{X*} : H_k(X, \mathbb{Q}) \to H_k(X, \mathbb{Q})$ is multiplication by m^k.

There is an eigenspace decomposition of $L_pH_k(X)_{\mathbb{Q}}$ for each pair of p, k such that $k \geq 2p \geq 0$. Set

$$L_pH_k(X)^s_{\mathbb{Q}} := \{\alpha \in L_pH_k(X)_{\mathbb{Q}} | m_{X*}\alpha = m^{k+s}\alpha, \forall\ m \in \mathbb{Z}\}$$

Theorem 6.1. *Let X be an abelian variety of dimension n. Then we have the following decomposition*

$$L_pH_k(X)_{\mathbb{Q}} = \bigoplus_{s=p-k}^{n-[\frac{k+1}{2}]} L_pH_k(X)^s_{\mathbb{Q}}, \tag{4}$$

where $[a]$ denotes the largest integer less than or equal to a.

We have a direct corollary from Theorem 6.1.

Corollary 6.2. *Let X be an abelian variety of dimension n. Then $L_pH_k(X)^s_{\mathbb{Q}} = 0$ for $s > n - [\frac{k+1}{2}]$ or $s < p - k$.*

Lemma 6.3. *Suppose $\alpha \in L_pH_k(X)_{\mathbb{Q}}$ and*

$$F(\alpha) = \sum_{q=p-[\frac{k}{2}]}^{n} \beta_q$$

with $\beta_q \in L_qH_l(\widehat{X})_{\mathbb{Q}}$, where $l = k + 2(q - p)$. Then for all $m \in \mathbb{Z}$, we have

$$m^*_{\widehat{X}}\beta_q = m^{n-q+p}\beta_q.$$

Proof. By the definition of F we have

$$\beta_q = \frac{1}{(n-q+p)!} p_{2*}(\mathcal{P}^{(n-q+p)} \cdot p_1^*\alpha) \in L_qH_l(\widehat{X})_{\mathbb{Q}}.$$

Hence using flat base change with $m_{\widehat{X}} \circ p_2 = p_2 \circ (1_X \times m_{\widehat{X}})$ (cf. [FG, §3]) and the fact that $(1_X \times m_{\widehat{X}})^*\mathcal{P} = m\mathcal{P}$, we get

$$\begin{aligned} m^*_{\widehat{X}}\beta_q &= \tfrac{1}{(n-q+p)!} m^*_{\widehat{X}} p_{2*}(\mathcal{P}^{(n-q+p)} \cdot p_1^*\alpha) \\ &= \tfrac{1}{(n-q+p)!} p_{2*}((1_X \times m_{\widehat{X}})^*\mathcal{P}^{(n-q+p)} \cdot p_1^*\alpha) \\ &= \tfrac{m^{n-q+p}}{(n-q+p)!} p_{2*}(\mathcal{P}^{(n-q+p)} \cdot p_1^*\alpha) \\ &= m^{n-q+p}\beta_q. \end{aligned}$$

□

Proposition 6.4. *For $\alpha \in L_pH_k(X)_{\mathbb{Q}}$ and $m \in \mathbb{Z} - \{-1, 0, 1\}$ the following statements are equivalent:*

1. *$\alpha \in L_pH_k(X)^s_{\mathbb{Q}}$,*
2. *$m_{X*}\alpha = m^{k+s}\alpha$,*

(3) $m_X^*\alpha = m^{2n-k-s}\alpha$,
(4) $F(\alpha) \in L_{n-k+p-s}H_{2n-2s-k}(\widehat{X})_{\mathbb{Q}}$,
(5) $F(\alpha) \in L_{n-k+p-s}H_{2n-2s-k}(\widehat{X})^s_{\mathbb{Q}}$.

Proof. We show this in the following way: (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (3).
(1) $\Leftrightarrow$ (2) is from the definition of $L_pH_k(X)^s_{\mathbb{Q}}$.
(2) $\Rightarrow$ (3). Note first that $m_X^*\alpha \in L_pH_k(X)_{\mathbb{Q}}$. Suppose $m_X^*\alpha \in L_pH_k(X)^{s'}_{\mathbb{Q}}$ and from the definition we get $m_{X*}(m_X^*\alpha) = m^{k+s'}m_X^*\alpha$. Since $m_{X*}(m_X^*\alpha) = (\deg m_X)\alpha = m^{2n}\alpha$, we obtain that $m_X^*\alpha = m^{2n-k-s'}\alpha \in L_pH_k(X)^s_{\mathbb{Q}}$. This implies that $s = s'$.
(3) $\Rightarrow$ (2). From $m^{2n}\alpha = \deg(m_X)\alpha = m_{X*}(m_X^*\alpha) = m_{X*}(m^{2n-k-s}\alpha) = m^{2n-k-s}m_{X*}\alpha$ we get $m_{X*}\alpha = m^{k+s}\alpha$.
(3) $\Rightarrow$ (4). We write $F_X(\alpha) = \sum_q \beta_q$ with $\beta_q \in L_qH_{k+2(q-p)}(\widehat{X})_{\mathbb{Q}}$. Then

$$\begin{array}{rcl} \sum_{q=p-[\frac{k}{2}]}^n \beta_q & = & F_X(\alpha) \\ & = & \frac{1}{m^{k+s}}F(m_{X*}\alpha) \\ & = & \frac{1}{m^{k+s}}m_{\widehat{X}}^*F(\alpha) \\ & = & \frac{1}{m^{k+s}}\sum_{q=0}^n m_{\widehat{X}}^*\beta_q \\ & = & \frac{1}{m^{k+s}}\sum_{q=0}^n m^{n-q+p}\beta_q \\ & = & m^{n-k-q+p-s}\beta_q. \end{array}$$

Comparing coefficients this implies that

$$F(\alpha) = \beta_{n-k+p-s} \in L_{n-k+p-s}H_{2n-k-2s}(\widehat{X})_{\mathbb{Q}}.$$

(4) $\Rightarrow$ (5). By Lemma 6.3, we have $F(\alpha) \in L_{n-k+p-s}H_{2n-2s-k}(\widehat{X})^s_{\mathbb{Q}}$.
(5) $\Rightarrow$ (3). For every $m \in \mathbb{Z}$, we have

$$\begin{array}{rcll} m_X^*\alpha & = & m_X^*(-1)^n(-1)_X^*F_{\widehat{X}}F_X\alpha & \text{(by Theorem 4.1)} \\ & = & (-1)^n(-1)_X^*F_{\widehat{X}}m_{\widehat{X}*}F_X\alpha & \text{(by Proposition 5.2)} \\ & = & (-1)^n(-1)_X^*F_{\widehat{X}}m_{\widehat{X}*}\beta_{n-k+p-s} & \text{(by Statement (5))} \\ & = & m^{-k-s}(-1)^n(-1)_X^*F_{\widehat{X}}m_{\widehat{X}*}m_{\widehat{X}}^*\beta_{n-k+p-s} & \text{(by Lemma 6.3)} \\ & = & m^{2n-k-s}(-1)^n(-1)_X^*F_{\widehat{X}}\beta_{n-k+p-s} & \\ & = & m^{2n-k-s}(-1)^n(-1)_X^*F_{\widehat{X}}F_X(\alpha) & \\ & = & m^{2n-k-s}\alpha. & \end{array}$$

□

Proof of Theorem 6.1. Suppose $\alpha \in L_pH_k(X)_{\mathbb{Q}}$ and write $F_X(\alpha) = \sum \beta_q$ with $\beta_q \in L_qH_{k+2(q-p)}(\widehat{X})_{\mathbb{Q}}$. By Lemma 6.3 we have $\beta_q \in L_qH_{k+2(q-p)}(\widehat{X})^{n-k-q+p}_{\mathbb{Q}}$. By applying Proposition 6.4 to β_q, we get

$$F_{\widehat{X}}(\beta_q) \in L_pH_k(X)^{n-k-q+p}_{\mathbb{Q}}.$$

Now by Theorem 4.1,

$$\begin{array}{rcl} \alpha & = & (-1)^n(-1)^*F_{\widehat{X}} \circ F_X(\alpha) \\ & = & (-1)^n(-1)^*\sum_{q=p-[\frac{k}{2}]}^n \beta_q \in \bigoplus_{q=p-[\frac{k}{2}]}^n L_pH_k(X)^{n-k-q+p}_{\mathbb{Q}}. \end{array}$$

This implies the assertion since $n-k+[\frac{k}{2}] = n-[\frac{k+1}{2}]$. □

The decomposition of Equation (4) is compatible with many natural maps.

Proposition 6.5. *Let $f : X \to Y$ be a group homomorphism between abelian varieties. Then induced map f_* preserves the decomposition of Lawson homology groups with rational coefficients, i.e.,*

$$f_*(L_pH_k(X)^s_{\mathbb{Q}}) \subseteq L_pH_k(Y)^s_{\mathbb{Q}},\ \forall s \in \mathbb{Z}.$$

Proof. Since $f : X \to Y$ is a group homomorphism, one has $f(m \cdot x) = m \cdot f(x)$ and so $m_Y \circ f = f \circ m_X$. Hence we have $m_{Y*} \circ f_* = f_* \circ m_{X*}$. This completes the proof of the proposition. □

Proposition 6.6. *The decomposition in Equation (4) is compatible with the cycle class map $\Phi_{p,k,\mathbb{Q}}$.*

Proof. It follows from the fact that the multiplication m_X by m on X commutes with the Fourier-Mukai transform F_X (cf. Proposition 4.2) and the cycle class map $\Phi_{p,k,\mathbb{Q}}$, the latter is the general fact that the cycle class map $\Phi_{p,k,\mathbb{Q}}$ is a natural transformation between the Lawson homolgy (cf. [FM], [L2, Ch. IV]). □

Set

$$L_pH_k(X)^s_{hom,\mathbb{Q}} := \{\alpha \in L_pH_k(X)_{hom,\mathbb{Q}} | m_{X*}\alpha = m^{k+s}\alpha, \forall\ m \in \mathbb{Z}\}.$$

From Proposition 6.6, we have

$$L_pH_k(X)^s_{hom,\mathbb{Q}} = L_pH_k(X)^s_{\mathbb{Q}}, \quad s \neq 0 \tag{5}$$

and

$$L_pH_k(X)^0_{hom,\mathbb{Q}} = L_pH_k(X)^0_{\mathbb{Q}} \cap L_pH_k(X)_{hom,\mathbb{Q}}. \tag{6}$$

On the image of the natural transform $\Phi_{p,k} \otimes \mathbb{Q} : L_pH_k(X)_{\mathbb{Q}} \to H_k(X)_{\mathbb{Q}}$, we have the following result.

Corollary 6.7. *Let X be an abelian variety of dimension n. Then we have*

$$T_pH_k(X)_{\mathbb{Q}} \cong T_{n+p-k}H_{2n-k}(\widehat{X})_{\mathbb{Q}}$$

Proof. Note that we have the following commutative diagram

$$\begin{array}{ccc} L_pH_k(X)^0_{\mathbb{Q}} & \xrightarrow{F_X} & L_{n+p-k}H_{2n-k}(\widehat{X})^0_{\mathbb{Q}} \\ \downarrow{\scriptstyle \Phi_{p,k}\otimes\mathbb{Q}} & & \downarrow{\scriptstyle \Phi_{n+p-k,2n-k}\otimes\mathbb{Q}} \\ T_pH_k(X)_{\mathbb{Q}} & \xrightarrow{\bar{F}_X} & T_{n+p-k}H_{2n-k}(\widehat{X})_{\mathbb{Q}}, \end{array}$$

where $\bar{F}_X$ is the restriction of the Fourier-Mukai transform F_X on the rational homology groups of X. Since $F_X : H_k(X, \mathbb{Q}) \to H_k(\widehat{X}, \mathbb{Q})$ is isomorphism, $\bar{F}_X : T_pH_k(X)_{\mathbb{Q}} \to T_{n+p-k}H_{2n-k}(\widehat{X})_{\mathbb{Q}}$ is injective and so

$$\dim_{\mathbb{Q}} T_pH_k(X)_{\mathbb{Q}} \leq \dim_{\mathbb{Q}} T_{n+p-k}H_{2n-k}(\widehat{X})_{\mathbb{Q}}.$$

Since $F_{\widehat{X}}$ is also an isomorphism on the rational homology groups, we obtain

$$\dim_{\mathbb{Q}} T_{n+p-k}H_{2n-k}(\widehat{X})_{\mathbb{Q}} \leq \dim_{\mathbb{Q}} T_pH_k(X)_{\mathbb{Q}}.$$

Hence

$$\dim_{\mathbb{Q}} T_pH_k(X)_{\mathbb{Q}} = \dim_{\mathbb{Q}} T_{n+p-k}H_{2n-k}(\widehat{X})_{\mathbb{Q}}$$

and so $\bar{F}_X : T_pH_k(X)_{\mathbb{Q}} \to T_{n+p-k}H_{2n-k}(\widehat{X})_{\mathbb{Q}}$ is an isomorphism. □

The motivation of our decomposition follows from that of the Chow group theory. Recall that there is also an eigenspace decomposition for $\mathrm{Ch}_p(X)$ of for every p, due to Beauville [B2]. If we set

$$\mathrm{Ch}_p(X)^s_{\mathbb{Q}} := \{\alpha \in \mathrm{Ch}_p(X)_{\mathbb{Q}} | m_{X*}\alpha = m^{2p+s}\alpha \text{ for all } n \in \mathbb{Z}\},$$

then there is a direct sum decomposition for the Chow group of X with rational coefficients

$$\mathrm{Ch}_p(X)_{\mathbb{Q}} = \bigoplus_{s=-p}^{n-p} \mathrm{Ch}_p(X)^s_{\mathbb{Q}}.$$

Beauville conjectures that $\mathrm{Ch}_p(X)^s_{\mathbb{Q}} = 0$ for $s < 0$ ([B1, B2]). Similarly, we have the following analogue of Beauville's conjecture for the Lawson homology of abelian varieties.

Conjecture 6.8. *For an abelian variety X, one has $L_pH_k(X)^s_{\mathbb{Q}} = 0$ for $s < 0$.*

From Theorem 4.1 and Proposition 6.4, we have

Corollary 6.9. *Let X be an abelian variety of dimension n. The Fourier-Mukai transformation F_X induces an isomorphism*

$$L_pH_k(X)^s_{\mathbb{Q}} \cong L_{n-k+p-s}H_{2n-2s-k}(\widehat{X})^s_{\mathbb{Q}}$$

for all integer s.

Proof. To see the injectivity of F_X, let $\alpha \in L_pH_k(X)^s_{\mathbb{Q}}$ such that

$$F_X(\alpha) = 0 \in L_{n-k+p-s}H_{2n-2s-k}(\widehat{X})^s_{\mathbb{Q}}. \tag{7}$$

Now we apply $F_{\widehat{X}}$ to both sides of Equation (7), we get

$$F_{\widehat{X}} \circ F_X(\alpha) = F_{\widehat{X}}(0). \tag{8}$$

The right side of Equation (8) is obviously equal to zero. By Theorem 4.1, the left side of equation (8) is $(-1)^n(-1)^*_X(\alpha)$. Since $(-1)^*_X$ is an isomorphism, we get $\alpha = 0$.

To see the surjectivity of F_X, for $\beta \in L_{n-k+p-s}H_{2n-2s-k}(\widehat{X})^s_{\mathbb{Q}}$, we have $F_{\widehat{X}}(\beta) \in L_pH_k(X)^s_{\mathbb{Q}}$ by Proposition 6.4. Set $\alpha := (-1)^n(-1)^*_{\widehat{X}}$. Then $F_X(\alpha) = \beta$ by applying Theorem 4.1 to $\widehat{X}$. □

From the explanation in the beginning of this section, if we define

$$H_k(X)^s_{\mathbb{Q}} := \{\alpha \in H_k(X)_{\mathbb{Q}} | m_{X*}\alpha = m^{k+s}\alpha, \forall\, m \in \mathbb{Z}\},$$

then by applying F_X on singular homology with rational coefficients, one gets

$$H_k(X)^s_{\mathbb{Q}} = \begin{cases} H_k(X)_{\mathbb{Q}}, & s = 0; \\ 0, & s \neq 0. \end{cases} \tag{9}$$

Now we will check the situation of the conjecture for low dimensional abelian varieties. For those X of $\dim X \leq 2$, Friedlander's result [F1, Th.4.6] and the Dold-Thom theorem imply that Conjecture 6.8 holds.

Example 6.10. *Conjecture 6.8 holds for abelian variety X of dimension 3 except possibly for $p = 1, k \geq 4$ and $s = 3 - k$.*

Proof. From the above discussion, we only need to consider abelian varieties X of dimension three. That is, we need to show $L_pH_k(X)^s_{\mathbb{Q}} = 0$ except for $p = 1, k \geq 4$ and $s = 3 - k$.

There are different cases according to p and k. It is trivial when $p < 0$ or $p > 3$.

(1) $p = 0$. In this case, one has the Dold-Thom isomorphism $L_0H_k(X) \cong H_k(X)$ and therefore by Equation (9) we have

$$L_0H_k(X)^s_{\mathbb{Q}} \cong H_k(X)^s_{\mathbb{Q}} = \begin{cases} H_k(X)_{\mathbb{Q}}, & s = 0; \\ 0, & s \neq 0. \end{cases}$$

So we have $L_0H_k(X)^s_{\mathbb{Q}} = 0$ for $s < 0$ for all $k \geq 0$.

(2) $p = 2$. In this case, we obtain from Friedlander's theorem (cf. [F1, Th.4.6]) that

$$L_2H_k(X)^s_{\mathbb{Q}} \cong H_k(X)^s_{\mathbb{Q}} = \begin{cases} H_k(X)_{\mathbb{Q}}, & s = 0; \\ 0, & s \neq 0. \end{cases}$$

for $k \geq 5$. For $k = 4$, $L_2H_k(X)^s_{\mathbb{Q}} \subseteq H_k(X)^s_{\mathbb{Q}} = 0$ for $s < 0$ by Equation (9). So we have $L_2H_k(X)^s_{\mathbb{Q}} = 0$ for $s < 0$ for all $k \geq 4$.

(3) $p = 3$. In this case, by definition, the only nontrivial $L_3H_k(X)$ occurs in the case that $k = 6$. When $k = 6$, we have $L_3H_6(X)_{\mathbb{Q}} \cong H_6(X)_{\mathbb{Q}}$. So $L_3H_6(X)^s_{\mathbb{Q}} \cong H_6(X)^s_{\mathbb{Q}} = 0$ for $s < 0$ by Equation (9).

(4) $p = 1$.

(a) $k = 2$. By applying Corollary 6.9 to X, we obtain that

$$L_1H_2(X)^s_{\mathbb{Q}} \cong L_{2-s}H_{4-2s}(\widehat{X})^s_{\mathbb{Q}}.$$

So if $s < 0$, then $2 - s > 2$ and hence $L_{2-s}H_{4-2s}(\widehat{X})^s_{\mathbb{Q}} = 0$ by applying the above cases to $\widehat{X}$.

(b) $k = 3$. Again by applying Corollary 6.9 to X, we get

$$L_1H_3(X)^s_{\mathbb{Q}} \cong L_{1-s}H_{3-2s}(\widehat{X})^s_{\mathbb{Q}}.$$

So if $s < 0$, then $1 - s \geq 2$ and hence $L_{1-s}H_{3-2s}(\widehat{X})^s_{\mathbb{Q}} = 0$ by the applying the above cases to $\widehat{X}$.

(c) $k \geq 4$. By applying Corollary 6.9 to X, we get

$$\begin{aligned} L_1H_k(X)^s_{\mathbb{Q}} &\cong L_{4-k-s}H_{6-k-2s}(\widehat{X})^s_{\mathbb{Q}} \\ &= 0, \text{if } s \neq 3 - k. \end{aligned}$$

□

From the discuss above, we see that the mysterious part of $L_1H_k(X)_{\mathbb{Q}}$ for an abelian threefold X is $L_1H_k(X)^{3-k}_{\mathbb{Q}} = \{\alpha \in L_1H_k(X)_{\mathbb{Q}} | m_{X*}\alpha = m^3\alpha\}$. For $k = 2$, it is the Griffiths group with rational coefficients of 1-cycles on X; For $k = 3$, conjecturally it is $T_1H_3(X, \mathbb{Q})$; For $k \geq 4$, conjecturally it is zero.

Now we describe the relation between of Conjecture 6.8 and a weak version conjecture by Suslin (see the appendix below for the statement). In the usual (or strong) version of the Suslin conjecture, varieties are only required to be quasi-projective. Moreover, the strong version Suslin conjecture also includes the statement that the cycle class map $\Phi_{p,k}$ from Lawson homology to the singular homology is injective for $k = n + p - 1$.

Proposition 6.11. *Conjecture 6.8 is equivalent to the (weak version) Suslin conjecture for Lawson homology with rational coefficients on abelian varieties.*

Proof. Let X be an abelian variety of dimension n. First we assume Conjecture 6.8, i.e.

$$L_pH_k(X)^s_{\mathbb{Q}} = 0$$

for all $k \geq 2p$ and $s < 0$. Then for $k \geq n+p$, we obtain that

(10)

$$\begin{array}{rcll} L_pH_k(X)_{\mathbb{Q}} & = & \bigoplus_{s=p-k}^{n-[\frac{k+1}{2}]} L_pH_k(X)^s_{\mathbb{Q}} & \text{(by Theorem 6.1)} \\ & = & \bigoplus_{s=0}^{n-[\frac{k+1}{2}]} L_pH_k(X)^s_{\mathbb{Q}} & \text{(by Conjecture 6.8)} \\ & \cong & \bigoplus_{s=0}^{n-[\frac{k+1}{2}]} L_{n-k+p-s}H_{2n-2s-k}(\widehat{X})^s_{\mathbb{Q}} & \text{(by Corollary 6.9)} \\ & \cong & \bigoplus_{s=0}^{n-[\frac{k+1}{2}]} H_{2n-2s-k}(\widehat{X})^s_{\mathbb{Q}} & (\text{since } n-k+p-s \leq 0) \\ & = & H_{2n-k}(\widehat{X})^0_{\mathbb{Q}} & \text{(by Equation (9))} \\ & = & H_{2n-k}(\widehat{X})_{\mathbb{Q}} & \text{(by Equation (9))} \\ & \cong & H_k(X)_{\mathbb{Q}} & \text{(by Fourier Inversion)} \end{array}$$

and this is exactly the Suslin conjecture for X with rational coefficients for $k \geq n+p$.

Conversely, we assume that the Suslin conjecture holds for X. Then by Equation (10), we have $L_pH_k(X)^s_{\mathbb{Q}} = 0$ for $p-k \leq s < 0$. This together with Corollary 6.2 implies that $L_pH_k(X)^s_{\mathbb{Q}} = 0$ for all $s < 0$. □

Remark 6.12. *The weak version Suslin conjecture relates to a Hard Lefschetz type conjecture for Lawson homology (cf.* [FM], [X]*). Moreover, by applying an action of $SL(2,\mathbb{Z})$ on the space of algebraic cycles modulo the algebraic equivalence, as observed by Beauville, Xu gives further direct sum decomposition of $L_pH_k(X)^s_{\mathbb{Q}}$ in terms of primitive elements.*

The following question was asked by Friedlander and Lawson in the terminology of their morphic cohomology (for definition, see section 8 below). By Friedlander-Lawson's duality theorem (cf. [FL2]) between the Morphic cohomology and Lawson homology, an equivalent version in terms of Lawson homology is given in the following.

Question 6.13 (Compare with Question 9.7 in [FL1])**.** *For a smooth projective variety X of dimension n, is $\Phi_{p,k} : L_pH_k(X) \to H_k(X)$ surjective for $k \geq n+p$?*

Obviously, the Suslin conjecture says Friedlander-Lawson's answer to the above question is yes. The answer is also yes to the question for abelian varieties, as given in the morphic cohomology version (cf. [FL2, Cor. 9.5]). The affirmative answer to Question 6.13 can be obtained by using the Friedlander-Lawson's duality theorem. Now we give an alternative proof of the surjectivity of $\Phi_{p,k}$, without using the Friedlander-Lawson's duality theorem.

Proposition 6.14 (Friedlander-Lawson)**.** *For an abelian variety X of dimension n, the cycle class map $\Phi_{p,k} : L_pH_k(X) \to H_k(X)$ surjective for $k \geq n+p$.*

Proof. From the proof of Proposition 6.11, we see that for any abelian variety X of dimension n, $\Phi_{p,k} : L_pH_k(X)_{\mathbb{Q}} \to H_k(X)_{\mathbb{Q}}$ is surjective for all $k \geq n+p$. The

details are given as follows: Since $k \geq n+p$, we have
(11)

$$\begin{array}{rcll} L_pH_k(X)_{\mathbb{Q}} & = & \bigoplus_{s=p-k}^{n-[\frac{k+1}{2}]} L_pH_k(X)^s_{\mathbb{Q}} & \text{(by Theorem 6.1)} \\ & \supseteq & \bigoplus_{s=0}^{n-[\frac{k+1}{2}]} L_pH_k(X)^s_{\mathbb{Q}} & \\ & \cong & \bigoplus_{s=0}^{n-[\frac{k+1}{2}]} L_{n-k+p-s}H_{2n-2s-k}(\widehat{X})^s_{\mathbb{Q}} & \text{(by Corollary 6.9)} \\ & \cong & \bigoplus_{s=0}^{n-[\frac{k+1}{2}]} H_{2n-2s-k}(\widehat{X})^s_{\mathbb{Q}} & \text{(since } n-k+p-s\leq 0) \\ & = & H_{2n-k}(\widehat{X})^0_{\mathbb{Q}} & \text{(by Equation (9))} \\ & = & H_{2n-k}(\widehat{X})_{\mathbb{Q}} & \text{(by Equation (9))} \\ & \cong & H_k(X)_{\mathbb{Q}} & \text{(by Fourier Inversion)} \end{array}$$

This implies that $\Phi_{p,k} : L_pH_k(X)_{\mathbb{Q}} \to H_k(X)_{\mathbb{Q}}$ is surjective for $k \geq p+n$. Since the Suslin conjecture with finite coefficients holds, as from the work of Milnor-Bloch-Kato conjecture (now a theorem by by Voevodsky, Rost and others), we obtain that $\Phi_{p,k} : L_pH_k(X) \to H_k(X)$ is also surjective for $k \geq p+n$. □

Remark 6.15. *From the above proposition and a recent result by Beilinson* [Bl], *one obtains an alternative proof of the Grothendieck standard conjecture of Lefschetz type for X (see the appendix below for the statement). The first proof of this Grothendieck standard conjecture of Lefschetz type for abelian varieties was obtained by Lieberman* [Li].

Proposition 6.16. *Let X be an abelian variety of dimension n. Suppose the (strong version) Suslin conjecture holds for X. Then for 1-cycles, we have*

$$L_1H_k(X)^0_{\mathbb{Q}} \cong T_1H_k(X)_{\mathbb{Q}}$$

for all $k \geq 2$ and there is a finite filtration on $L_1H_k(X)_{\mathbb{Q}}$ given by

$$\begin{array}{rcl} F^0L_1H_k(X)_{\mathbb{Q}} & = & L_1H_k(X)_{\mathbb{Q}}, \\ F^1L_1H_k(X)_{\mathbb{Q}} & = & L_1H_k(X)_{hom,\mathbb{Q}}, \\ F^2L_1H_k(X)_{\mathbb{Q}} & = & \ker AJ_X, \\ F^jL_1H_k(X)_{\mathbb{Q}} & = & \bigoplus_{j\geq s} L_1H_k(X)^s_{\mathbb{Q}}, \\ F^jL_1H_k(X)_{\mathbb{Q}} & = & 0, j>>1, \end{array}$$

where AJ_X is the Abel-Jacobi for Lawson homology, as defined by the author in [H2].

Proof. From Theorem 6.1, we have a finite filtration given by

$$F^jL_pH_k(X)_{\mathbb{Q}} = \bigoplus_{j\geq s} L_pH_k(X)^s_{\mathbb{Q}}, \quad j = p-k, p-k+1, \cdots, n-[(k+1)/2].$$

Moveover, $F^{p-k}L_pH_k(X)_{\mathbb{Q}} = L_pH_k(X)_{\mathbb{Q}}$ and $F^jL_pH_k(X)_{\mathbb{Q}} = 0$ for $j > n - [(k+1)/2]$. By assumption and Proposition 6.11, we have

$$F^jL_pH_k(X)_{\mathbb{Q}} = L_pH_k(X)_{\mathbb{Q}}$$

for all $j \leq 0$.

By the (strong) Suslin conjecture for Lawson homology with rational coefficients, $\Phi_{p,k} : L_pH_k(X)_{\mathbb{Q}} \to H_k(X)_{\mathbb{Q}}$ is injective for $k = n+p-1$. By applying Theorem

6.1 to $L_pH_k(X)_{\mathbb{Q}}$ for $k = n+p-1$, we get

(12)
$$\begin{array}{rcll} L_pH_k(X)_{\mathbb{Q}} & = & \bigoplus_{s=p-k}^{n-[\frac{k+1}{2}]} L_pH_k(X)^s_{\mathbb{Q}} & \text{(by Theorem 6.1)} \\ & = & \bigoplus_{s=0}^{n-[\frac{k+1}{2}]} L_pH_k(X)^s_{\mathbb{Q}} & \text{(by Proposition 6.11)} \\ & \cong & \bigoplus_{s=0}^{n-[\frac{n+p}{2}]} L_{1-s}H_{n-p+1-2s}(\widehat{X})^s_{\mathbb{Q}} & \text{(by Corollary 6.9)} \\ & \cong & L_1H_{n-p+1}(\widehat{X})^0_{\mathbb{Q}} & \\ & & \oplus \bigoplus_{s=1}^{n-[\frac{n+p}{2}]} H_{n-p+1-2s}(\widehat{X})^s_{\mathbb{Q}} & \text{(by Dold-Thom Theorem)} \\ & = & L_1H_{n-p+1}(\widehat{X})^0_{\mathbb{Q}} & \text{(by Equation (9))} \\ & \cong & L_pH_{n+p-1}(X)^0_{\mathbb{Q}} & \text{(by Fourier Inversion)} \end{array}$$

, Therefore, $L_pH_{n+p-1}(X)^0_{\mathbb{Q}} \cong L_pH_{n+p-1}(X)_{\mathbb{Q}}$. Then the Sulin conjecture implies that

$$L_pH_{n+p-1}(X)^0_{\mathbb{Q}} \cong T_pH_{n+p-1}(X,\mathbb{Q}) \tag{13}$$

and hence $F^1L_pH_k(X)_{\mathbb{Q}} = L_pH_k(X)_{hom,\mathbb{Q}}$. In particular, one has $F^1L_1H_k(X)_{\mathbb{Q}} = L_1H_k(X)_{hom,\mathbb{Q}}$ for $p=1$.

Under the assumption of the (strong) Suslin conjecture, for $k \geq 2$, we have the following isomorphisms

$$\begin{array}{rcll} L_1H_k(X)^0_{\mathbb{Q}} & \cong & L_{n+1-k}H_{2n-k}(\widehat{X})^0_{\mathbb{Q}} & \text{(by Corollary 6.9)} \\ & \cong & T_{n+1-k}H_{2n-k}(\widehat{X},\mathbb{Q}) & \text{(by Equation (13))} \\ & \cong & T_1H_k(X,\mathbb{Q}). & \text{(by Corollary (6.7))} \end{array}$$

Note that the Abel-Jacobi map for the Lawson homology of a smooth projective variety X is a natural map

$$AJ_X : L_pH_k(X)_{hom} \longrightarrow \left\{ \bigoplus_{r>k-2p+1, r+s=k-2p+1} H^{p+r,p+s}(X) \right\}^* \Big/ H_{k+1}(X,\mathbb{Z}).$$

So if X is an abelian variety and $\alpha \in L_pH_k(X)^s_{\mathbb{Q}} \cap L_pH_k(X)_{hom,\mathbb{Q}}$, then

$$AJ_X(m_{X*}\alpha) = m_{X*}(AJ_X(\alpha)),$$

i.e. $m^{k+s}AJ_X(\alpha) = m^{k+1}AJ_X(\alpha)$ since the action of m_{X*} on $H^{k+1}(X,\mathbb{Q})$ is the multiplication by m^{k+1}. Therefore $AJ_X(\alpha) = 0$ unless $\alpha \in L_pH_k(X)^1_{\mathbb{Q}}$. □

Remark 6.17. *From the above proposition, it is reasonable to conjecture for an abelian variety X that $L_pH_k(X)^0_{\mathbb{Q}} \cong T_pH_k(X,\mathbb{Q})$ for all integers $k \geq 2p$. Then the above proposition would hold for all $p \geq 0$.*

The direct sum decomposition for the Lawson homology of an abelian variety X in Theorem 6.1 is compatible with the Pontryagin product.

Proposition 6.18. *Let X be an abelian variety of dimension n. Then the following diagram*

$$\begin{array}{ccc} L_pH_k(X)^s_{\mathbb{Q}} \otimes L_qH_l(X)^{s'}_{\mathbb{Q}} & \xrightarrow{\ *\ } & L_{p+q}H_{k+l}(X)^{s+s'}_{\mathbb{Q}} \\ \Big\downarrow{\scriptstyle i^s_{p,k}\otimes i^{s'}_{q,l}} & & \Big\downarrow{\scriptstyle i^{s+s'}_{p+q,k+l}} \\ L_pH_k(X)_{\mathbb{Q}} \otimes L_qH_l(X)_{\mathbb{Q}} & \xrightarrow{\ *\ } & L_{p+q}H_{k+l}(X)_{\mathbb{Q}} \end{array}$$

is commutative, where $i^s_{p,k} : L_pH_k(X)^s_{\mathbb{Q}} \to L_pH_k(X)_{\mathbb{Q}}$ is the inclusion of the summand in the direct sum decomposition (cf. Equation (4)).

Proof. We need to show that for $\alpha \in L_pH_k(X)^s_{\mathbb{Q}}$ and $\beta \in L_qH_l(X)^{s'}_{\mathbb{Q}}$, one has $\alpha * \beta \in L_{p+q}H_{k+l}(X)^{s+s'}_{\mathbb{Q}}$. Since $\alpha \in L_pH_k(X)^s_{\mathbb{Q}}$, we have by definition $m_{X*}(\alpha) = m^{k+s}\alpha$. Similarly, $m_{X*}(\beta) = m^{l+s'}\beta$. It is enough to show that $m_{X*}(\alpha * \beta) = m^{k+l+s+s'}\alpha * \beta$. Note that by definition the two morphisms $\mu \circ (m_X, m_X)$ and $m_X \circ \mu$ coincide, i.e., $\mu \circ (m_X, m_X) = m_X \circ \mu : X \times X \to X$. So we have the commutative diagram by the induced maps on Lawson homology groups

$$\begin{array}{ccccc} L_pH_k(X)_{\mathbb{Q}} \otimes L_qH_l(X)_{\mathbb{Q}} & \xrightarrow{\times} & L_{p+q}H_{k+l}(X \times X)_{\mathbb{Q}} & \xrightarrow{\mu_*} & L_{p+q}H_{k+l}(X)_{\mathbb{Q}} \\ \downarrow{\scriptstyle m_{X*}\otimes m_{X*}} & & & & \downarrow{\scriptstyle m_{X*}} \\ L_pH_k(X)_{\mathbb{Q}} \otimes L_qH_l(X)_{\mathbb{Q}} & \xrightarrow{\times} & L_{p+q}H_{k+l}(X \times X)_{\mathbb{Q}} & \xrightarrow{\mu_*} & L_{p+q}H_{k+l}(X)_{\mathbb{Q}}. \end{array}$$

So we have

$$\mu * \circ \times \circ (m_{X*} \otimes m_{X*}) = m_{X*} \circ \mu_* \circ \times : L_pH_k(X)_{\mathbb{Q}} \otimes L_pH_k(X)_{\mathbb{Q}} \to L_pH_k(X)_{\mathbb{Q}},$$

in other words,

$$m_{X*}(\alpha * \beta) = m_{X*}\alpha * m_*\beta.$$

Now since $\alpha \in L_pH_k(X)^s_{\mathbb{Q}}$ and $\beta \in L_qH_l(X)^{s'}_{\mathbb{Q}}$, we have

$$\begin{aligned} m_{X*}(\alpha * \beta) &= m_{X*}\alpha * m_*\beta \\ &= (m^{k+s}\alpha) * (m^{l+s'}\beta) \\ &= m^{k+s+l+s'}\alpha * \beta. \qquad \text{(by the bilinearity of } *) \end{aligned}$$

So by definition we get $\alpha * \beta \in L_{p+q}H_{k+l}(X)^{s+s'}_{\mathbb{Q}}$. □

However, we cannot hope the Pontryagin product would bring more on the Abel-Jacobi map for Lawson homology, since AJ_X is only non-trivial on $L_pH_k(X)^1_{\mathbb{Q}}$, but the Pontryagin product of $\alpha \in L_pH_k(X)^1_{\mathbb{Q}}$ and $\beta \in L_qH_l(X)^1_{\mathbb{Q}}$ is in $L_{p+q}H_{k+l}(X)^2_{\mathbb{Q}}$. So the Abel-Jacobi map AJ_X is identically zero on $L_{p+q}H_{k+l}(X)^2_{\mathbb{Q}}$.

7. Filtrations on Lawson homology

In this section we will define a filtration on Lawson homology, as an analogue of the conjectural Bloch-Beilinson filtration for Chow groups. For X a smooth projective variety, then conjecturally, for every integer k, there exists a decreasing filtration $\tilde{F}^jL_pH_k(X)_{\mathbb{Q}}$ on the Lawson homology group with rational coefficients, satisfying the following properties:

(1) $\tilde{F}^jL_pH_k(X)_{\mathbb{Q}} = 0$ for $j >> 1$.
(2) The filtration is stable under the action of correspondences:
If $\Gamma \in A_{\dim X+d}(X \times Y)$, then the maps

$$\Gamma_* : L_pH_k(X)_{\mathbb{Q}} \to L_{p+d}H_{k+2d}(Y)_{\mathbb{Q}}$$

satisfy

$$\Gamma_*(\tilde{F}^jL_pH_k(X)_{\mathbb{Q}}) \subseteq \tilde{F}^jL_{p+d}H_{k+2d}(Y)_{\mathbb{Q}}.$$

(3) The induced map

$$Gr^j_{\tilde{F}}\Gamma_* : Gr^j_{\tilde{F}}L_pH_k(X)_{\mathbb{Q}} \to Gr^j_{\tilde{F}}L_{p+d}H_{k+2d}(Y)_{\mathbb{Q}}$$

vanishes if the class Γ is zero on $H_{k+j}(X, \mathbb{Q})$, i.e.,

$$0 = [\Gamma]_* : H_{k+j}(X, \mathbb{Q}) \to H_{k+j+2d}(Y, \mathbb{Q}).$$

A candidate of such a filtration could be given in the following way by induction: Assume that we have constructed $\tilde{F}^{j-1}L_qH_l(Y)_{\mathbb{Q}}$ for every $l \geq 2q$ and every smooth projective variety Y. Then we set

$$\tilde{F}^jL_pH_k(X)_{\mathbb{Q}} := \mathrm{span}\{Im\ \Gamma_*(\tilde{F}^{j-1}L_{p+r}H_{k+2r}(Y)_{\mathbb{Q}}) | \Gamma \in A^{r+\dim X}(Y \times X)_{\mathbb{Q}}\},$$

where Γ satisfies the condition that

$$\Gamma_* : H_{k+2r}(Y, \mathbb{Q}) \to H_k(X, \mathbb{Q})$$

is zero.

For X an abelian variety, if we set as before

$$F^jL_pH_k(X)_{\mathbb{Q}} = \bigoplus_{j\geq s} L_pH_k(X)^s_{\mathbb{Q}}, \quad j = p-k, p-k+1, \cdots, n-[(k+1)/2].$$

As an analogue in Chow theory ([Mr]), it is reasonable to conjecture that the two filtrations for abelian varieties coincide.

8. Morphic Cohomology

There is a cohomological version of Lawson homology, i.e., the Friedlander-Lawson morphic cohomology, is defined to be the homotopy group of algebraic cocycles. The topological group $\mathcal{Z}^q(X)$ of all algebraic cocycles of codimension-q on X is defined as a homotopy quotient completion (cf. [FL1, Definition 2.8])

$$\mathcal{Z}^q(X) := [\mathfrak{Mor}(X, \mathcal{C}_0(\mathbb{P}^q))/\mathfrak{Mor}(X, \mathcal{C}_0(\mathbb{P}^{q-1}))]^+ = \mathfrak{Mor}(X, \mathcal{Z}_0(\mathbb{C}^q)).$$

The $(2q-k)$-th homotopy group of the space of algebraic cocycles instead of algebraic cycles, is defined to be *the Friedlander-Lawson morphic cohomology group* and is denoted by $L^qH^k(X)$.

Let X be an abelian variety. There is also an eigenspace decomposition of $L^qH^k(X)_{\mathbb{Q}}$ for each pair of q, k such that $k \leq 2q$.

$$L^qH^k(X)^s_{\mathbb{Q}} := \{\alpha \in L^qH^k(X)_{\mathbb{Q}} | m_X^*\alpha = m^{k-s}\alpha\}$$

Proposition 8.1 (Refined Friedlander-Lawson duality for Abelian varieties)**.** *There is an induced isomorphism*

$$L^qH^k(X)^s_{\mathbb{Q}} \cong L_{n-q}H_{2n-k}(X)^s_{\mathbb{Q}}$$

from the Friedlander-Lawson duality for all integers s and $k \leq 2q$, where $n = \dim X$.

Proof. Let $\mathcal{D} : L^qH^k(X) \to L_{n-q}H_{2n-k}(X)$ be the the Friedlander-Lawson duality homomorphism (cf. [FL2]). We denote by the same notation for the coefficient extension map $\mathcal{D} : L^qH^k(X)_{\mathbb{Q}} \to L_{n-q}H_{2n-k}(X)_{\mathbb{Q}}$. Since $\mathcal{D}$ is an isomorphism, it is enough to show that the image of $L^qH^k(X)^s_{\mathbb{Q}}$ under $\mathcal{D}$ is $L_{n-q}H_{2n-k}(X)^s_{\mathbb{Q}}$. First note that there is a commutative diagram

$$(14) \qquad \begin{array}{ccc} L^qH^k(X) & \xrightarrow{\mathcal{D}} & L_{n-q}H_{2n-k}(X) \\ \downarrow{\scriptstyle m_X^!} & & \downarrow{\scriptstyle m_{X*}} \\ L^qH^k(X) & \xrightarrow{\mathcal{D}} & L_{n-q}H_{2n-k}(X), \end{array}$$

where $m_X^!$ is the Gysin map induced by the map $m_X : X \to X$ (cf. [FL2, Prop.5.5]). By the Fulton's excess formula for the morphic cohomology (also Lawson homology), we have $m_X^! m_X^* = \deg m_X$ (cf. [HL]). The latter is equal to m^{2n}.

Now or $\alpha \in L^qH^k(X)^s_{\mathbb{Q}}$, by definition we have $m_X^*(\alpha) = m^{k-s}\alpha$. Hence $\alpha = \frac{1}{m^{k-s}} \cdot m_X^*\alpha$ and

$$m_X^!\alpha = \frac{1}{m^{k-s}} \cdot m^! m_X^*\alpha = \frac{1}{m^{k-s}} \cdot m^{2n}\alpha = m^{2n-k+s}\alpha. \tag{15}$$

From Equation (14), we have $\mathcal{D} \circ m_X^! = m_{X*} \circ \mathcal{D}$. So by this as well as Equation (15), one gets

$$m_{X*}(\mathcal{D}\alpha) = \mathcal{D}(m_X^!\alpha) = \mathcal{D}(m^{2n-k+s}\alpha) = m^{2n-k+s}\mathcal{D}\alpha. \tag{16}$$

That is to say, $\mathcal{D}\alpha \in L_{n-q}H_{2n-k}(X)^s_{\mathbb{Q}}$. This gives us an isomorphism

$$L^qH^k(X)^s_{\mathbb{Q}} \cong L_{n-q}H_{2n-k}(X)^s_{\mathbb{Q}}$$

for all integers s and $k \leq 2q$. □

By this proposition, all the results in terms of Lawson homology in section 6 have the corresponding morphic cohomological version obtained by replacing $L^qH^k(X)^s_{\mathbb{Q}}$ by $L_{n-q}H_{2n-k}(X)^s_{\mathbb{Q}}$. For example, we have a decomposition for the morphic cohomology of an abelian variety.

Proposition 8.2. *Let X be an abelian variety of dimension n. Then we have the following decomposition*

$$L^qH^k(X)_{\mathbb{Q}} \cong \bigoplus_{s=k-q-n}^{[\frac{k}{2}]} L^qH^k(X)^s_{\mathbb{Q}}.$$

Proof. It follows from Theorem 4.1 and Proposition 8.1. □

9. Semi-topological K-theory

Recall that the (singular) **semi-topological K-theory** (denoted by $K_*^{sst}(-)$) was introduced and developed by Friedlander and Walker in a sequence of papers (cf. [FW1], [FW2], [FW3], [FW4] and reference therein).

Let $\mathcal{K}^{sst}(X)$ be a homotopy-theoretic group completion of a space of maps of X to an infinite Grassmannian, topologized as in [FW3]. The **semi-topological K-group** $K_j^{sst}(X)$ of X is defined to be the j-th homotopy group of $\mathcal{K}^{sst}(X)$. The rational K^{sst}-groups is denoted by

$$K_j^{sst}(X)_{\mathbb{Q}} := K_j^{sst}(X) \otimes \mathbb{Q}.$$

One of the fundamental result in semi-topological K-theory is that there is a natural isomorphism between rational K^{sst}-groups and certain direct sum of rational morphic groups:

Theorem 9.1 (Friedlander-Walker [FW3])**.** *There is a natural isomorphism*

$$K_j^{sst}(X)_{\mathbb{Q}} \cong \bigoplus_{q\geq 0} L^qH^{2q-j}(X)_{\mathbb{Q}}, \quad j \geq 0$$

for any smooth (quasi-)projective variety X.

Now let X be an abelian variety. From Theorem 9.1 and Proposition 8.1, we have the following result.

Corollary 9.2. *Let X be an abelian variety of dimension X. Then*

$$K_j^{sst}(X)_{\mathbb{Q}} \cong \bigoplus_{q\geq 0} \bigoplus_{s=q-n-j}^{[\frac{2q-j}{2}]} L^q H^{2q-j}(X)^s_{\mathbb{Q}},$$

In other words, $K_j^{sst}(X)_{\mathbb{Q}}$ is decomposed to be the direct sum of eigenspaces of the map $m_X^* : K_j^{sst}(X)_{\mathbb{Q}} \to K_j^{sst}(X)_{\mathbb{Q}}$.

Proof. We have the following isomorphisms

$$\begin{aligned} K_j^{sst}(X)_{\mathbb{Q}} &\cong \bigoplus_{q\geq 0} L^q H^{2q-j}(X)_{\mathbb{Q}} && \text{(by Theorem 9.1)} \\ &\cong \bigoplus_{q\geq 0} \bigoplus_{s=q-n-j}^{[\frac{2q-j}{2}]} L^q H^{2q-j}(X)^s_{\mathbb{Q}}. && \text{(by Proposition 8.1)} \end{aligned}$$

□

Note that in Corollary 9.2, there is only finite many nonzero direct summands on the right side of the equation since, for fixed j, $L^q H^{2q-j}(X)$ vanishes when q large. In particular, for an abelian variety of dimension three, we give explicitly the following equations.

Example 9.3. *Let X be an abelian variety of* $\dim X = 3$. *Then we have*

$$K_0^{sst}(X)_{\mathbb{Q}} \cong H^0(X,\mathbb{Q}) \oplus NS(X)_{\mathbb{Q}} \oplus L_1H_2(X)^0_{\mathbb{Q}} \oplus \mathrm{Griff}_1(X)_{\mathbb{Q}} \oplus H^6(X,\mathbb{Q}). \tag{17}$$

$$K_1^{sst}(X)_{\mathbb{Q}} \cong H^1(X,\mathbb{Q}) \oplus L_1H_3(X)^0_{\mathbb{Q}} \oplus L_1H_3(X)^1_{\mathbb{Q}} \oplus H^5(X,\mathbb{Q}). \tag{18}$$

and there is surjective map

$$K_j^{sst}(X)_{\mathbb{Q}} \twoheadrightarrow K_j^{top}(X)_{\mathbb{Q}},\ \forall j \geq 2. \tag{19}$$

The weak version Suslin conjecture for Lawson homology with rational coefficients implies that

$$K_j^{sst}(X)_{\mathbb{Q}} \cong K_j^{top}(X)_{\mathbb{Q}},\ \forall j \geq 2 \tag{20}$$

where $K_j^{top}(X)_{\mathbb{Q}}$ is the j-th topological K-group with rational coefficients.

Proof. From Corollary 9.2, one has

$$K_0^{sst}(X)_{\mathbb{Q}} \cong H^0(X,\mathbb{Q}) \oplus NS(X)_{\mathbb{Q}} \oplus L^2H^4(X)^0_{\mathbb{Q}} \oplus L^2H^4(X)^1_{\mathbb{Q}} \oplus H^6(X,\mathbb{Q}).$$

Note that $L^2H^4(X)^0_{\mathbb{Q}} \cong L_1H_2(X)^0_{\mathbb{Q}}$ is a finite dimensional $\mathbb{Q}$-vector space. Now Equation (17) follows from a fact that $L^2H^4(X)^1_{\mathbb{Q}} \cong L^2H^4(X)_{hom,\mathbb{Q}} = \mathrm{Griff}_1(X)_{\mathbb{Q}}$ by Beauville (cf. [B2, Prop.6]). Equation (18) follows from Corollary 9.2 and Proposition 8.1. Equation (19) follows from Corollary 9.2 and the surjectivity of $\Phi_{p,k} : L_pH_k(X)_{\mathbb{Q}} \to H_k(X)_{\mathbb{Q}}$ for all $k \geq n+p$. Equation (20) follows from Corollary 9.2 the Atiyah-Hirzebruch isomorphism between the topological K-group with rational coefficients and the singular cohomology with rational coefficients, and the fact the morphic cohomology is isomorphic to the singular cohomology for X in these cases under the assumption of the weak Suslin conjecture. □

If we denote by $K_j^{sst}(X)_{\mathbb{Q}}^s = \{\alpha \in K_j^{sst}(X)_{\mathbb{Q}} | m_X^*(\alpha) = m^s\alpha\}$, then

$$\begin{cases} K_0^{sst}(X)_{\mathbb{Q}}^0 & \cong & H^0(X,\mathbb{Q}) \cong \mathbb{Q}; \\ K_0^{sst}(X)_{\mathbb{Q}}^2 & \cong & NS(X)_{\mathbb{Q}}; \\ K_0^{sst}(X)_{\mathbb{Q}}^3 & \cong & \mathrm{Griff}^2(X)_{\mathbb{Q}} = \mathrm{Griff}_1(X)_{\mathbb{Q}}; \\ K_0^{sst}(X)_{\mathbb{Q}}^4 & \cong & L^2H^4(X)^0 \cong L_1H_2(X)^0; \\ K_0^{sst}(X)_{\mathbb{Q}}^6 & \cong & H^6(X,\mathbb{Q}) \cong \mathbb{Q}; \\ K_0^{sst}(X)_{\mathbb{Q}}^s & = & 0, \text{all other } s. \end{cases}$$

$$\begin{cases} K_1^{sst}(X)_{\mathbb{Q}}^1 & \cong & H^1(X,\mathbb{Q}); \\ K_1^{sst}(X)_{\mathbb{Q}}^3 & \cong & L^2H^3(X)_{\mathbb{Q}}^0 \cong L_1H_3(X)_{\mathbb{Q}}^0; \\ K_1^{sst}(X)_{\mathbb{Q}}^4 & \cong & L^2H^3(X)_{\mathbb{Q}}^0 \cong L_1H_3(X)_{\mathbb{Q}}^0; \\ K_1^{sst}(X)_{\mathbb{Q}}^5 & \cong & H^5(X,\mathbb{Q}); \\ K_1^{sst}(X)_{\mathbb{Q}}^s & = & 0, \text{all other } s. \end{cases}$$

and

$$K_j^{sst}(X)_{\mathbb{Q}}^s \cong H^s(X,\mathbb{Q}), \forall s \in \mathbb{Z}, j \geq 2.$$

10. Appendix

In this appendix we will discuss and summary for convenience the relations between a few conjectures in Lawson homology theory. Some of them are known or implied in the literatures before.

Let X be a smooth complex projective variety. It was shown in [FM, §7] that the subspaces $T_pH_k(X,\mathbb{Q})$ form a decreasing filtration (called the *topological filtration*):

$$\cdots \subseteq T_pH_k(X,\mathbb{Q}) \subseteq T_{p-1}H_k(X,\mathbb{Q}) \subseteq \cdots \subseteq T_0H_k(X,\mathbb{Q}) = H_k(X,\mathbb{Q})$$

and $T_pH_k(X,\mathbb{Q})$ vanishes if $2p > k$.

Denote by $G_pH_k(X,\mathbb{Q}) \subseteq H_k(X,\mathbb{Q})$ the $\mathbb{Q}$-vector subspace of $H_k(X,\mathbb{Q})$ generated by the images of mappings $H_k(Y,\mathbb{Q}) \to H_k(X,\mathbb{Q})$, induced from all morphisms $Y \to X$ of varieties of dimension $\leq k-p$.

The subspaces $G_pH_k(X,\mathbb{Q})$ also form a decreasing filtration (called the *geometric filtration*):

$$\cdots \subseteq G_pH_k(X,\mathbb{Q}) \subseteq G_{p-1}H_k(X,\mathbb{Q}) \subseteq \cdots \subseteq G_0H_k(X,\mathbb{Q}) \subseteq H_k(X,\mathbb{Q})$$

Denote by $\tilde{F}_pH_k(X,\mathbb{Q}) \subseteq H_k(X,\mathbb{Q})$ the maximal sub-(Mixed) Hodge structure of span $k-2p$. (See [G2] and [FM].) The sub-$\mathbb{Q}$ vector spaces $\tilde{F}_pH_k(X,\mathbb{Q})$ form a decreasing filtration of sub-Hodge structures:

$$\cdots \subseteq \tilde{F}_pH_k(X,\mathbb{Q}) \subseteq \tilde{F}_{p-1}H_k(X,\mathbb{Q}) \subseteq \cdots \subseteq \tilde{F}_0H_k(X,\mathbb{Q}) \subseteq H_k(X,\mathbb{Q})$$

and $\tilde{F}_pH_k(X,\mathbb{Q})$ vanishes if $2p > k$. This filtration is called the *Hodge filtration.*

It was shown by Friedlander and Mazur that

$$T_pH_k(X,\mathbb{Q}) \subseteq G_pH_k(X,\mathbb{Q}) \subseteq \tilde{F}_pH_k(X,\mathbb{Q}) \tag{21}$$

holds for any smooth projective variety X and $k \geq 2p \geq 0$.

Friedlander and Mazur proposed the following conjecture which relates Lawson homology theory to the central problems in the algebraic cycle theory.

Conjecture 10.1 (Friedlander-Mazur conjecture, [FM])**.** *For any smooth projective variety X, one has*

$$T_pH_k(X,\mathbb{Q}) = G_pH_k(X,\mathbb{Q}).$$

The Friedlander-Mazur conjecture remains open for general threefolds. However, it has been verified for some cases. For example, it was shown to hold for general abelian varieties (cf. [F2]) or abelian varieties for which the the generalized Hodge conjecture holds (cf. [A]); It was also shown to hold for threefold X with $h^{2,0}(X) = 0$, in particular, the complete intersection of dimension three (cf. [H1]). It also holds for *any* abelian threefold. To see the last statement, we note that for X a threefold X, it is enough to show Conjecture 10.1 for the $p = 1$. Note that by Proposition 6.14, $T_1H_k(X,\mathbb{Q}) = G_1H_k(X,\mathbb{Q})$ for $k \geq 4$. By Proposition 1.15 in [H1], one has $T_1H_3(X,\mathbb{Q}) = G_1H_3(X,\mathbb{Q})$. Moreover, Friedlander and Mazur proved that $T_1H_2(X,\mathbb{Q}) = G_1H_2(X,\mathbb{Q})$ (cf. [FM, §7]).

Conjecture 10.2 (The generalized Hodge conjecture, [G2] and [FM])**.** *For any smooth projective variety X, one has*

$$G_pH_k(X,\mathbb{Q}) = \tilde{F}_pH_k(X,\mathbb{Q}).$$

There is a corresponding conjecture in terms of morphic cohomology (cf. [FW4]). The equivalence between the homological version and the cohomological version is given by using Friedlander-Lawson duality isomorphism (cf. [FL2]). In [FW4], the cohomological version of Conjecture 10.1 and 10.2 combine to one conjecture $T_pH_k(X,\mathbb{Q}) = \tilde{F}_pH_k(X,\mathbb{Q})$ which they called the (strong) Friedlander-Mazur conjecture.

Now let $X \subset \mathbb{P}^N$ be a smooth variety of dimension n and let H be a hyperplane section such that $Y := X \cap H$ is smooth. The Lefschetz operator $L : H^i(X,\mathbb{Q}) \to H^{i+2}(X,\mathbb{Q})$ is defined by $L(\alpha) = \alpha \cup [Y]$, where $[Y]$ denotes the homology class of Y in $H^2(X,\mathbb{Q})$. The Hard Lefschetz Theorem says

$$L^{n-i} : H^i(X,\mathbb{Q}) \to H^{2n-i}(X,\mathbb{Q})$$

is an isomorphism. Note that this L_X is given by the algebraic cycle $\Delta(Y)$, in other words, $L(-) = p_{2*}\big(p_1^*(-) \cap \Delta(Y)\big)$.

Conjecture 10.3 (Grothendieck standard conjecture of Lefschetz type, [G1])**.** *The inverse $\Lambda^{n-i} : H^{2n-i}(X,\mathbb{Q}) \to H^i(X,\mathbb{Q})$ to L^{n-i} is given by an algebraic cycle for each $0 \leq i \leq n$.*

In this case Λ^{n-i} is also called *algebraic* for each $0 \leq i \leq n$.

Conjecture 10.4 (Hard Lefschetz conjecture for Lawson homology, [FM])**.** *Let X be a smooth projective variety of dimension n and let h be a hyperplane section. Then the intersection*

$$h^k\bullet : L_pH_{n+k}(X)_{\mathbb{Q}} \to L_{p-k}H_{n-k}(X)_{\mathbb{Q}},\ \alpha \mapsto [h]^k \bullet \alpha,$$

is injective for $k \geq 1$, where $[h]$ is viewed as the class of h in $L_{n-1}H_{2n-2}(X)_{\mathbb{Q}}$.

The following conjecture is essential to the structure of the Lawson homology for a smooth quasi-projective variety.

Conjecture 10.5 (The Suslin conjecture for Lawson Homology with coefficient A, [FHW])**.** *For any abelian group A and any smooth quasi-projective variety X of dimension n, the map $\Phi_{p,k} : L_pH_k(X,A) \to H_k(X,A)$ is an isomorphism for $k \geq n+p$ and is an injection for $k = n+p-1$.*

This is an analogue of the Beilinson-Lichtenbaum conjecture in motivic cohomology theory. When A is a finite abelian group, the Milnor-Bloch-Kato conjecture(or theorem) implies that the Suslin conjecture for Lawson Homology with coefficient A.

The weak version Suslin conjecture for Lawson homology only requires X be projective and $\Phi_{p,k}$ be isomorphic for $k \geq n+p$. Clearly, the weak version Suslin conjecture is much weaker than the (strong) Suslin conjecture. However, it is mentioned separately since the weak version Suslin conjecture for Lawson homology with integer coefficients implies the Grothendieck standard conjecture of Lefschtez type (see below).

These conjectures (Conjecture 10.1-10.5) hold for all smooth projective varieties of dimension less or equal than two. However, as far as I know, all of them are still open even for a general smooth projective variety of dimension three. Known cases and statements on these conjectures can be found in the literature (cf. [Bl], [F2], [FHW], [FM], [FW4]), [H1], [LF], [Pe], [X], [Vo], etc. and the references therein).

Lemma 10.6. *The maps $\Phi_{p,k} : L_pH_k(X,\mathbb{Q}) \to H_k(X,\mathbb{Q})$ are surjective for all smooth projective variety X and $k \geq p + \dim X$ is equivalent to the Friedlander-Mazur conjecture holds for all smooth projective variety.*

Proof. On one side, it is clear that if the Friedlander-Mazur conjecture for a smooth projective variety X, then $T_pH_k(X,\mathbb{Q}) = G_pH_k(X,\mathbb{Q})$ for $k \geq p + \dim X$. Since $k \geq p + \dim X$, $G_pH_k(X,\mathbb{Q}) = H_k(X,\mathbb{Q})$ and so $T_pH_k(X,\mathbb{Q}) = H_k(X,\mathbb{Q})$, i.e., $\Phi_{p,k} : L_pH_k(X,\mathbb{Q}) \to H_k(X,\mathbb{Q})$ is surjective.

One the other side, we need to show that for any smooth projective variety Y, $T_pH_k(Y,\mathbb{Q}) = G_pH_k(Y,\mathbb{Q})$ for all $k \geq 2p$. By assumption, we only need to show $T_pH_k(Y,\mathbb{Q}) = G_pH_k(Y,\mathbb{Q})$ for all $2p \leq k \leq n+p-1$. This was done in the proof of Theorem 1.20 in [H1], where we base on a stronger assumption (i.e. the Suslin conjecture). However, the injection for $\Phi_{p,k}$ is not really used in that proof. □

Proposition 10.7. *The Friedlander-Mazur conjecture is equivalent to the Grothendieck standard conjecture of Lefschtez type. More precise, the Friedlander-Mazur conjecture holds for all smooth projective varieties if and only if Λ is algebraic for every smooth projective varieties.*

Proof. On one side, we note that the Grothendieck standard conjecture of Lefschtez type implies the Friedlander-Mazur conjecture (cf. [F2]). On the other side, the Friedlander-Mazur conjecture implies that $\Phi_{p,k} : L_pH_k(X,\mathbb{Q}) \to H_k(X,\mathbb{Q})$ is surjective for all $k \geq p + \dim X$. The surjectivity of $\Phi_{p,k}$ for all $k \geq p + \dim X$ is equivalent to the Grothendieck standard conjecture of Lefschtez type (cf. [Bl]). □

Proposition 10.8. *The generalized Hodge conjecture holds for all smooth projective varieties implies the Conjecture 10.1. More precisely,*

$$\text{“}G_pH_k(X,\mathbb{Q}) = \tilde{F}_pH_k(X,\mathbb{Q}),\ \textit{all } X\text{”} \Leftrightarrow \text{“}T_pH_k(Y,\mathbb{Q}) = \tilde{F}_pH_k(Y,\mathbb{Q}),\ \textit{all } Y\text{”}.$$

Proof. The part “ $\Leftarrow$ ” follows directly from the assumption and Equation (21). For the part “ $\Rightarrow$ ”, we assume the generalized Hodge conjecture holds for all smooth projective varieties, in particular, it holds for $X \times X$ in the case that $k = 2p$, which is the classical Hodge conjecture. It is known the Hodge conjecture for $X \times X$ implies that the Grothendieck standard conjecture for X, which holds for all smooth projective varieties implies that Conjecture 10.1 holds for all smooth projective varieties. □

Remark 10.9. *From the above discuss we see that the generalized Hodge conjecture and the Suslin conjecture for Lawson homology with integer coefficients dominate many key problems in the theory of Lawson homology. So solutions to those problems for general smooth projective varieties is the most difficult problems in this field. An alternative way to deal with problems in Lawson homology theory would be the study on those varieties carrying special structures.*

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School of Mathematics, Sichuan University, Chengdu 610064, P. R. China

Current Address: Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: huwenchuan@gmail.com