IDEAS OF E. CARTAN AND S. LIE IN MODERN GEOMETRY: G-STRUCTURES AND DIFFERENTIAL EQUATIONS. LECTURE 3

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Problem:
How to make Cartan reduction in a particular case.

We will show how to find a differential invariant of a G-structure by example of a contact 2-distribution in $\mathbb{R}^3$, using the Cartan reduction method step by step.

Two-dimensional distribution Δ in M^3

Definition 1. Let M a three-dimensional manifold. A 2-dimensional distribution Δ on M is an assignment of a plane to each point of M, i.e. Δ is a sub-bundle of the tangent bundle TM. The assignment is smooth in sense that in a neighborhood U of each point $p \in M$ there are vector fields $\{X_1, X_2\}$ such that:

(1) For any point $q \in U$ the vectors $X_1(q)$, $X_2(q)$ are linear independent.
(2) For each point $q \in U$, Δ_q is the plane spanned by the two vectors $X_1(q)$, $X_2(q)$.

Sub-riemannian surface $\mathcal{S}$ on M. Let M a three-dimensional manifold. A distribution Δ is called *integrable* or *holonomic distribution* if for each point $p \in M$ there exists a surface Σ passing through p which is tangent to Δ: $T_q\Sigma = \Delta(q)$ for each $q \in \Sigma$. A 2-distribution Δ on M is holonomic if the *commutator* of vector fields X_1 and X_2,

$$[X_1, X_2]^i = X_1^s \frac{\partial X_2^i}{\partial x^s} - X_2^s \frac{\partial X_1^i}{\partial x^s} \tag{1}$$

generating Δ belong to Δ, i.e. $[X_1, X_2](p) \in \Delta(p)$. In this case the family of these surfaces form a *foliation* of M.

In another case, when $[X_1, X_2] \notin \Delta$ for all points in M, we say the distribution is *non-integrable* or *non-holonomic*.

Definition 2. A sub-riemannian surface in M is a non-holonomic distribution Δ with a scalar product $\langle \cdot, \cdot \rangle$ on $\Delta(p)$ for each $p \in M$. We denote by $\mathcal{S} = (M, (\Delta, \langle \cdot, \cdot \rangle))$ a sub-riemannian surface in M.

In this lecture we will show how to use the Cartan ideas for to find some invariants of a sub-riemannian surface $\mathcal{S}$.

Example 1 (Heisenberg distribution). Let $\mathcal{S} = (\mathbb{R}^3, \Delta, \langle\cdot,\cdot\rangle)$ be the sub-riemannian surface where

$$\Delta = span\{X_1, X_2\}, \qquad \begin{cases} X_1 = (1,0,-y) = \dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial z} \\ X_2 = (0,1,x) = \dfrac{\partial}{\partial y} + x\dfrac{\partial}{\partial z} \end{cases} \tag{2}$$

and the scalar product is the induced from $\mathbb{R}^3$. This distribution is non-holonomic because

$$\begin{aligned} [X_1, X_2] &= Y \\ Y &= X_3^s \frac{\partial}{\partial x^s} \\ \text{where} \quad Y^s &= X_1^k \frac{\partial X_2^s}{\partial x^k} - X_2^k \frac{\partial X_1^s}{\partial x^k} \\ \therefore \quad Y &= (0,0,2) = 2\frac{\partial}{\partial z} \notin \Delta \end{aligned} \tag{3}$$

Exercise 1. Let $\mathcal{S} = (\mathbb{R}^3, \Delta, \langle\cdot,\cdot\rangle)$ be the sub-riemannian surface where

- **(a):** $X_1 = (1,0,x_2)$, $X_2 = (0,1,x_1)$ in $\mathbb{R}^3$.
- **(b):** $X_1 = (1,0,-x_2)$, $X_2 = (0,1,0)$ in $\mathbb{R}^3$.

and the scalar product is the induced from $\mathbb{R}^3$. Is the distribution $\Delta = span\{X_1, X_2\}$ holonomic or not?

Cartan reduction for a sub-Riemannian surfaces $\mathcal{S}$ in $\mathbb{R}^3$

Let M be a three-dimensional manifold. Consider a sub-riemannian surface $\mathcal{S} = (M, \Delta, \langle\cdot,\cdot\rangle)$. For each $p \in M$ we always can take a local orthonormal frame field (e_1, e_2) of Δ. If we take $e_3 = [e_1, e_2]$ then (e_1, e_2, e_3) is a local frame field of M.

Let $B(M)$ be the bundle of positively oriented coframes of M.

First step: Adapting a coframe.

Definition 3. Given an oriented sub-riemannian surface $\mathcal{S} = (\Delta, \langle\cdot,\cdot\rangle)$ on a three-dimensional manifold M, we say that a co-frame $\eta = (\eta^1, \eta^2, \eta^3)$ of $B(M)$, is *adapted* to Δ if for all $p \in M$,

1. $(\eta^1|_{\Delta_p}, \eta^2|_{\Delta_p})$ is a positively oriented co-frame of Δ_p;
2. $\eta^3(W) = 0$ for any $W \in \Delta_p$;
3. $\langle W, W\rangle = [\eta^1(W)]^2 + [\eta^2(W)]^2$ for any $W \in \Delta_p$.

Definition 4 (Sub-bundle B_0)**.** The sub-bundle $B_0 \subset B$ consists of all co-frames adapted to $\mathcal{S}$. The subgroup of matrices of $GL(3)$ which transform adapted coframes into adapted coframes is

$$G_0 = \left\{ \left(\begin{array}{ccc} \cos\varphi_1 & -\sin\varphi_1 & \varphi_2 \\ \sin\varphi_1 & \cos\varphi_1 & \varphi_3 \\ 0 & 0 & \varphi_4 \end{array} \right) \middle| \varphi_4 \neq 0 \right\} \tag{4}$$

With this construction we show that any sub-riemannian surface $(\Delta, \langle\cdot,\cdot\rangle)$ defines a G_0-structure on M.

Remark 1. The quantities φ_i ($i \in \{1,2,3,4\}$) are real variables and can be considered as coordinates on G_0, so the dimension of G_0 is 4, and then the dimension of B_0 is 7.

Since G_0 is a Lie group with identity element I, one can construct the associated Lie algebra $\mathfrak{g}_0$ as the tangent space of G_0 in I. As coordinates of I are $\varphi_1 = \varphi_2 = \varphi_3 = 0$ and $\varphi_4 = 1$, taking the tangent vectors to the coordinate curves, we obtain that the Lie algebra $\mathfrak{g}_0$ associated to the Lie group G_0 is the set of matrices of type

$$\begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ -\alpha_1 & 0 & \alpha_3 \\ 0 & 0 & \alpha_4 \end{pmatrix} \tag{5}$$

Example 2. Find an adapted frame and coframe for the Heisenberg distribution of example (1),

$$X_1 = (1,0,-y), \quad X_2 = (0,1,x) \tag{6}$$

This distribution is defined in whole $\mathbb{R}^3$ and is given by the vector fields $\{X_1, X_2\}$. Using Gram-Schmidt algorithm we found one orthonormal frame for Δ, (e_1, e_2) and complete it with $e_3 = [e_1, e_2]$ to obtain the frame $e = (e_1, e_2, e_3)$ for $\mathbb{R}^3$. So $e = (e_1, e_2, e_3)$ is an adapted frame for $\mathbb{R}^3$,

$$\begin{cases} e_1 = \dfrac{1}{\sqrt{1+y^2}}\dfrac{\partial}{\partial x} - \dfrac{y}{\sqrt{1+y^2}}\dfrac{\partial}{\partial z} \\ e_2 = \dfrac{xy}{\sqrt{1+y^2}\sqrt{1+x^2+y^2}}\dfrac{\partial}{\partial x} + \dfrac{\sqrt{1+y^2}}{\sqrt{1+x^2+y^2}}\dfrac{\partial}{\partial y} + \dfrac{x}{\sqrt{1+y^2}\sqrt{1+x^2+y^2}}\dfrac{\partial}{\partial z} \\ e_3 = \dfrac{y}{(1+x^2+y^2)^{3/2}}\dfrac{\partial}{\partial x} - \dfrac{x}{(1+x^2+y^2)^{3/2}}\dfrac{\partial}{\partial y} - \dfrac{2+3x^2+3y^2}{(1+x^2+y^2)^{3/2}}\dfrac{\partial}{\partial z} \end{cases}$$

The dual co-frame is

$$\begin{cases} \eta^1 = \dfrac{(2+3y^2)\,dx}{2\sqrt{1+y^2}} - \dfrac{3xydy}{2\sqrt{1+y^2}} + \dfrac{ydz}{2\sqrt{1+y^2}} \\ \eta^2 = -\dfrac{xydx}{2\sqrt{1+y^2}\sqrt{1+x^2+y^2}} + \dfrac{(2+3x^2+2y^2)\,dy}{2\sqrt{1+y^2}\sqrt{1+x^2+y^2}} - \dfrac{xdz}{2\sqrt{1+y^2}\sqrt{1+x^2+y^2}} \\ \eta^3 = -\dfrac{y}{2}\sqrt{1+x^2+y^2}dx + \dfrac{x}{2}\sqrt{1+x^2+y^2}dy - \dfrac{1}{2}\sqrt{1+x^2+y^2}dz \end{cases}$$

that is an adapted co-frame to Δ.

Remark 2. We can write that the Heisenberg distribution is $\Delta = \ker(\eta^3)$.

Exercise 2. Find an adapted frame and coframe for the Cartan distribution of exercise (1)

$$X_1 = (1,0,-y), \quad X_2 = (0,1,0) \tag{7}$$

Definition 5 (Tautological forms)**.** The 1-forms θ^i on B_0 such that $\theta^i(\eta^a)(X) = \eta^i(d\pi(X))$ are called tautological forms.

An adapted co-frame $\eta = (\eta^1, \eta^2, \eta^3)$ defined on a neighborhood $U \subset M$ defines a trivialization

$$U \times G_0 \leftrightarrow B_0|_U, \quad (x,g) \leftrightarrow g^{-1}\eta_x. \tag{8}$$

In terms of this trivialization, the tautological forms can be written as

$$\theta_{(x,g)} = g^{-1}(\eta_x \circ d\pi) \tag{9}$$

Derivation equations.

Theorem 1. *Let $\mathcal{S} = (\Delta, \langle\cdot,\cdot\rangle)$ be a sub-riemannian surface in M. Let $B(M)$ be the principal bundle of positive oriented co-frames on M and $B_0(M)$ the principal sub-bundle of $B(M)$ with the group G_0 defined in (4) consisting of co-frames adapted to $\mathcal{S}$. The exterior derivatives of the tautological forms can be written as follows:*

$$\begin{pmatrix} d\theta^1 \\ d\theta^2 \\ d\theta^3 \end{pmatrix} = \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ -\alpha_1 & 0 & \alpha_3 \\ 0 & 0 & \alpha_4 \end{pmatrix} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} + \begin{pmatrix} T^1_{23} & T^1_{31} & T^1_{12} \\ T^2_{23} & T^2_{31} & T^2_{12} \\ T^3_{23} & T^3_{31} & T^3_{12} \end{pmatrix} \begin{pmatrix} \theta^2\wedge\theta^3 \\ \theta^3\wedge\theta^1 \\ \theta^1\wedge\theta^2 \end{pmatrix} \tag{10}$$

or in contracted form,

$$\boxed{d\theta^i = \omega^i_s \wedge \theta^s + T^i_{ab}\theta^a \wedge \theta^b} \tag{11}$$

Remark 3. (1) These equations (10) are a part of all structure equations. They represent only the equations containing the derivatives of the tautological forms θ^i in terms of themselves. Recall that the cotangent space T^*B_0 has dimension 7. One co-frame of T^*B_0 is:

$$\{\theta^1, \theta^2, \theta^3, d\alpha_1, d\alpha_2, d\alpha_3, d\alpha_4\} \tag{12}$$

(2) The 1-form $\omega = \omega^b_a$ is a pseudoconnection form.
(3) The coefficients T^i_{ab} are called the torsion coefficients.
(4) We would like to construct the invariants from the components of the connection form ω and the torsion T.

Obviously these components in general case are functions, they depends of the points (x, g) on a fibre T_xB. If all of them were constant we would finish the problem, however in the general case they are not. For this reason we have to continue.

Contact distribution. A distribution Δ on M is a contact distribution if for all $p \in M$ the plane $\Delta(p)$ is given by the zeros of a 1-form η^3. Is clear that they will also given by the zeros of $\lambda\eta^3$. Thus, $\{\lambda\eta^3\}$ all give same the same $\Delta(p)$. The Heisenberg distribution and the Cartan distribution are both contact distribution.

The property that a contact distribution Δ in $\mathbb{R}^3$ is non-integrable if

$$d\eta^3 \wedge \eta^3 \neq 0 \tag{13}$$

This property also can be formulated in terms of the tautological form: a contact distribution is a non-integrable if

$$d\theta^3 \wedge \theta^3 \neq 0 \tag{14}$$

In our case, using the equation (10), for a contact distribution in $\mathbb{R}^3$ we have,

$$d\theta^3 \wedge \theta^3 = T^3_{12}\theta^1 \wedge \theta^2 \wedge \theta^3 \tag{15}$$

Example 3. For the Heisenberg distribution treated in the example 2 we have,

$$d\theta^3 \wedge \theta^3 = -2\theta^1 \wedge \theta^2 \wedge \theta^3 \tag{16}$$

That is, $T^3_{12} = -2$

Exercise 3. For the Cartan distribution find T^3_{12}.

How the component T^3_{12} change under the action of G_0? Let us denote the action of G_0 on B^0 by R_g where $g \in G_0$. Thus the components of θ^i under action of G_0 change by the following rule,

$$R_g\theta = g^{-1}\theta \tag{17}$$

This imply that $R_g\theta^3 = g^{-1}\theta^3$, where $g \in G_0$ defined in equation (4). Therefore $R_g^* d\theta^3 = (\varphi_4)^{-1} d\theta^3$, and this imply that

$$R_g^* T^3_{12} = (\varphi_4)^{-1} T^3_{12} \tag{18}$$

For a contact distribution $T^3_{12} \neq 0$, and from (18) it follows that we can take a subbundle $B_1 \subset B_0$ with the property that $T^3_{12} = 1$. The structure group G_1 of B_1 is

$$\boxed{G_1 = \left\{ \begin{pmatrix} \cos\varphi_1 & -\sin\varphi_1 & \varphi_2 \\ \sin\varphi_1 & \cos\varphi_1 & \varphi_3 \\ 0 & 0 & 1 \end{pmatrix} \right\}} \tag{19}$$

and the Lie algebra $\mathfrak{g}_0$ associated to G_0 is the algebra of matrices,

$$\boxed{\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ -\alpha_1 & 0 & \alpha_3 \\ 0 & 0 & 0 \end{pmatrix} \right\}} \tag{20}$$

Thus we have reduced the structure group G_0 to G_1, and the dimension of B_1 is 6, because the dimension of the vertical space, isomorphic to G_1, is 3.

Second step: Reducing the structure group G_0 as much as possible. The idea is continue reducing the structure group G_0. Now we have a principal sub-bundle (B_1, G_1) of (B_0, G_0) defined by

$$B_1 = \left\{\eta = \left(\eta^1, \eta^2, \eta^3\right) \in B_0 \mid T^3_{12}(\eta) = 1\right\}$$
$$G_1 = \left\{ \begin{pmatrix} \cos\varphi_1 & -\sin\varphi_1 & \varphi_2 \\ \sin\varphi_1 & \cos\varphi_1 & \varphi_3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & 1 \end{pmatrix} \right\}$$

The structure equations are,

$$\begin{pmatrix} d\theta^1 \\ d\theta^2 \\ d\theta^3 \end{pmatrix} = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} + \begin{pmatrix} T^1_{23} & T^1_{31} & T^1_{12} \\ T^2_{23} & T^2_{31} & T^2_{12} \\ T^3_{23} & T^3_{31} & 1 \end{pmatrix} \begin{pmatrix} \theta^2\wedge\theta^3 \\ \theta^3\wedge\theta^1 \\ \theta^1\wedge\theta^2 \end{pmatrix}$$

How the components T^3_{23} and T^3_{31} are changed under the action of G_1? Since the right action R_g is defined by $R_g\theta = g^{-1}\theta$ where $g \in G_1$, then the components T^3_{23} and T^3_{31} under action of G_1 change by the following rule:

$$\boxed{R_g^* \begin{pmatrix} T^3_{23} \\ T^3_{31} \end{pmatrix} = A^{-1} \begin{pmatrix} T^3_{23} - \varphi_2 \\ T^3_{31} - \varphi_3 \end{pmatrix}}$$

So, we can define another principal sub-bundle (B_2, G_2) as follows,

$$B_2 = \left\{ \eta = \left(\eta^1, \eta^2, \eta^3\right) \in B_1 \mid T^3_{23}(\eta) = T^3_{31}(\eta) = 0 \right\}$$
$$G_2 = \left\{ \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

and the Lie algebra $\mathfrak{g}_2$ associated to G_2 is

$$\mathfrak{g}_2 = \left\{ \begin{pmatrix} 0 & \alpha_1 & 0 \\ -\alpha_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \tag{21}$$

Then the structure equations are,

$$\begin{pmatrix} d\theta^1 \\ d\theta^2 \\ d\theta^3 \end{pmatrix} = \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} + \begin{pmatrix} T^1_{23} & T^1_{31} & T^1_{12} \\ T^2_{23} & T^2_{31} & T^2_{12} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \theta^2 \wedge \theta^3 \\ \theta^3 \wedge \theta^1 \\ \theta^1 \wedge \theta^2 \end{pmatrix} \tag{22}$$

Third step: Finding invariants. We have reduced G_0 to a minimum subgroup G_2 imposing more and more conditions for the adapted co-frame η. In the ideal case we must find the unique frame of the G-structure in order to find finally invariants of G-structurs.

How the components of ω and T in the structure equations are changed? If we take another connection form ω', then the torsion map also changes and we have T' such that

$$d\theta^i = \omega'^i_m \wedge \theta^m + T'^i_{lm}\theta^l \wedge \theta^m$$

But ω and ω' are both connections on B_2, i.e. both are elements in the space of all smooth 1-forms on B_2 with values in the Lie algebra $\mathfrak{g}_2$, $\Lambda^1(B_2, \mathfrak{g}_2)$,

If σ is a fundamental vector field and μ is the difference $\mu = \omega' - \omega$ we have

$$\omega(\sigma(a)) = a, \omega'(\sigma(a)) = a \quad \text{where} \quad a \in \mathfrak{g}_2 \tag{23}$$

and

$$\mu(\sigma(a)) = 0 \tag{24}$$

Therefore μ does vanish on the vertical subbundle V. So we have,

$$\boxed{\mu = \mu^i_{js}\theta^s \Rightarrow \omega'^i_m = \omega^i_m + \mu^i_{ms}\theta^s} \tag{25}$$

Therefore,

$$\begin{aligned} d\theta^i &= \omega'^i_m \wedge \theta^m + T'^i_{lm}\theta^l \wedge \theta^m \\ &= \left(\omega^i_m + \mu^i_{ml}\theta^l\right) \wedge \theta^m + T'^i_{lm}\theta^l \wedge \theta^m \end{aligned}$$

and

$$
\begin{aligned}
d\theta^i &= \omega'^i_m \wedge \theta^m + T'^i_{lm}\theta^l \wedge \theta^m \\
&= \left(\omega^i_m + \mu^i_{ml}\theta^l\right) \wedge \theta^m + T'^i_{lm}\theta^l \wedge \theta^m \\
&= \omega^i_s \wedge \theta^s + \left(T'^i_{lm} - \mu^i_{[lm]}\right)\theta^l \wedge \theta^m = \omega^i_s \wedge \theta^s + T^i_{lm}\theta^l \wedge \theta^m \\
&\boxed{T'^i_{lm} = T^i_{lm} + \mu^i_{[lm]}}, \quad \text{where} \quad \mu^i_{[lm]} = A(\mu^i_{lm}) = \mu^i_{ml}
\end{aligned}
\tag{26}
$$

where A is the operator of alternation with restpect to the lower indices.

Finding invariants. If we take

$$\alpha' = \alpha + T^1_{12}\theta^1 + T^2_{12}\theta^2 - \frac{1}{2}\left(T^2_{31} + T^1_{23}\right)\theta^3 \tag{27}$$

and replace it in (22) we obtain,

$$T'^1_{12} = T'^2_{12} = 0, \quad \text{and} \quad T'^1_{23} = -T'^2_{31} \tag{28}$$

and differentiating $d(\theta^3)$ we obtain,

$$d(d\theta^3) = 0 \Longrightarrow T'^1_{31} = T'^2_{32} \tag{29}$$

If we denote $a_1 = T^1_{23}, a_2 = T^1_{31}$, then the structure equation will be reduce to,

$$\begin{pmatrix} d\theta^1 \\ d\theta^2 \\ d\theta^3 \end{pmatrix} = \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} + \begin{pmatrix} a_1 & a_2 & 0 \\ a_2 & -a_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \theta^2 \wedge \theta^3 \\ \theta^3 \wedge \theta^1 \\ \theta^1 \wedge \theta^2 \end{pmatrix} \tag{30}$$

One differential invariant of a contact distribution on $\mathbb{R}^3$ is,

$$\mathcal{M} = (a_1)^2 + (a_2)^2 \tag{31}$$

Example 4. Let us consider the Heisenberg distribution defined on $\mathbb{R}^3$ treated in the examples (2), and (3),

$$\eta^3 = ydx - xdy + dz \tag{32}$$

For this contact distribution, using the Cartan reduction method, we have that between another differential invariants has the invariant

$$\mathcal{M} = \frac{9}{4}\frac{(x^2 + y^2)^2}{(1 + x^2 + y^2)^4} \tag{33}$$

These calculations we obtained using Maple.

Exercise 4. Calculated the invariant $\mathcal{M}$ for the Cartan distribution treated in the second and third exercise. Is the Cartan distribution equivalent to the Heisenberg distribution?.

Summary of Lecture 3

(1) The Cartan reduction method is a tool in the modern Differential Geometry in order to determine if two geometrical structures are equivalent up to a diffeomorphism. We demonstrated this method by an example of a contact 2-distribution with a metric $\mathcal{S} = (\Delta, \langle \cdot, \cdot \rangle)$ in a three-dimensional manifold M and found a differential invariant for this geometrical structure.

(2) The method consist of three steps:
  (a) At the first step one defines an adapted coframe for the distribution $\mathcal{S}$ and constructs a sub-bundle (B_0, G_0) of the principal $GL(3)^+$-bundle of all oriented positively coframes on M. We write the structure equations which express the exterior derivatives of tautological forms in terms of themselves and a connection form.
  (b) At the second step we reduce the group G_0 as much as possible adding new conditions for the adapted coframe such that the structure group becomes smaller and smaller. Thus we get a sub-bundle (B_2, G_2).
  (c) At the third step we construct invariants from the torsion coefficients.

Answers to exercises

1 **(a):** Holonomic distribution because $[X_1, X_2] = 0$.
(b): Non-holonomic distribution because $[X_1, X_2] = (0, 0, 1)$. This distribution is called Cartan distribution.

2 Adapted frame

$$e = \begin{pmatrix} \dfrac{1}{\sqrt{1+y^2}} & 0 & \dfrac{y}{\sqrt{1+y^2}} \\ 0 & 1 & 0 \\ -\dfrac{y}{\sqrt{1+y^2}} & 0 & \dfrac{1}{\sqrt{1+y^2}} \end{pmatrix} \tag{34}$$

The adapted co-frame $\eta = (\eta^1, \eta^2, \eta^3)$ for Δ is:

$$\begin{cases} \eta^1 = \dfrac{1}{\sqrt{1+y^2}} dx - \dfrac{y}{\sqrt{1+y^2}} dz \\ \eta^2 = dy \\ \eta^3 = -\dfrac{y}{\sqrt{1+y^2}} dx + \dfrac{1}{\sqrt{1+y^2}} dz \end{cases} \tag{35}$$

3 $T^3_{12} = -1$

4

$$\mathcal{M} = \frac{1}{4} \frac{(2y^2-1)^2}{(1+y^2)^4} \tag{36}$$

The Cartan and Heisenberg distributions are not equivalents.

References

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