*First draft*

# Universal Constants from Self-similarity and Adiabatic Invariance

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Abstract

We explore the idea that three universal constants of theoretical physics ($h$, $c$, $k_B$) reflect the asymptotic approach to *self-similarity* and *adiabatic invariance* of weakly coupled oscillators.

**Key words**: universal constants, Hamiltonian dynamics, self-similarity, adiabatic invariance, routes to chaos, universality.

## 1. Adiabatic invariants in Hamiltonian dynamics

Consider a one-dimensional oscillator whose dynamics is expressed in terms of canonically conjugate variables $(q, p)$. If the system is integrable, the energy is a constant of motion. The Hamiltonian is time-independent, the phase space orbit is a closed trajectory and both coordinate $q$ and momentum $p$ return to their starting values after each cycle. This scenario is no longer true if the Hamiltonian $H(p,q,t)$ is weakly dependent on a perturbation parameter $\lambda(t)$. In this case, the energy depends on time according to

$$E(t) = H(q, p, \lambda(t)) \quad (1)$$

Let us assume that $\lambda(t)$ varies slowly and monotonically over many oscillation cycles of period $\tau_S$. By definition, the *action-variable* is described by [1-2][1]

$$S(E,\lambda) = \frac{1}{2\pi}\int_C p(q,E,\lambda)\,dq \tag{2}$$

Analysis shows that

$$\frac{d\langle S\rangle}{dt} = 0 \tag{3}$$

The time-independence of $\langle S\rangle$ is based on the approximation that terms of the order $(d\lambda/dt)^2 << 1$ can be safely dropped. Condition (3) is a fundamental tenet of Hamiltonian dynamics and it states that, under the above assumptions, the action variable is an *adiabatic invariant*. The message it conveys is that the action can arbitrarily vary throughout the oscillation cycle, yet its average stays constant.

A textbook example of adiabatic invariance is provided by the unperturbed harmonic oscillator with Hamiltonian

$$H(p,q) = E = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 q^2 \tag{4}$$

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[1] Hereafter, we denote the action variable by "$S$" instead of the notation "$I$" used in standard discussions of action-angle variables.

for which the action integral represents the invariant given by

$$S = \frac{1}{2\pi}\iint p\,dq = \frac{E}{\omega_0} \tag{5}$$

Another classical example of adiabatic invariance is the *variable-length pendulum*, whose action is independent of the pendulum length ($l$) and reads [2-3]

$$S = \frac{E}{\nu} \tag{6}$$

Relation (6) holds under the assumption that the change in length is slow and that the pendulum performs small-amplitude oscillations with frequency $\nu$.

It is instructive to note that (6) ties in with the blackbody radiation problem that led Planck to the revolutionary discovery of quanta [3, 12]. In that problem, energy resonators confined in an cavity with volume $V$ at equilibrium temperature $T$ act as a collection of independent pendulums. The adiabatic contraction or expansion of the cavity yields an invariant similar to (6) given by

$$S_0 = \frac{\nu_n}{T} \tag{7}$$

in which $\nu_n = nc/L$ is the resonator frequency and $L$ is the linear dimension of the cavity. Relations (6) and (7) lie at the foundation of the Wien-Planck scaling law for the radiation spectrum, namely,

$$\frac{E_n}{\nu_n} = F(\frac{\nu_n}{T}) \tag{8}$$

where the explicit solution of the functional $F$ led Planck to the idea of *action quantization*. The adiabatic invariant (6) enters the Bohr-Sommerfeld quantization rule for bounded periodic systems, which assumes the form

$$S = \frac{1}{2\pi}\int_C p\,dq = n\hbar \tag{9}$$

Concerning (1)–(3), a sensible question to ask is: What happens if the perturbation $\lambda(t)$ is not strictly monotonic but undergoes a slow sinusoidal oscillation of period $\tau_\lambda >> \tau_S$ ?. Since, by (3), the action variable can sustain instantaneous fluctuations that leaves the average unchanged, the overall dynamics of $S$ and $\lambda$ may be naturally modelled as a system of *coupled oscillators*. Building this model forms the object of the next section.

## 2. The quasiperiodic behavior of driven oscillators

A convenient and often used representation of coupled oscillators is through the use of *iterated maps on the unit interval* [4]. Let the sinusoidal cycles of $\lambda$ be described by the period $\tau_\lambda$ and the fluctuations of $S$ by a characteristic period $\tau_S << \tau_\lambda$. If the two oscillators are decoupled, the successive points of the so-called Poincaré map are given by [5-6]

$$S_n = S(t = n\,\tau_\lambda) \tag{10}$$

$$S_{n+1} - S_n = 2\pi\Omega \tag{11}$$

or

$$S_{n+1} = S_n + 2\pi\Omega \tag{12}$$

in which the "bare" winding number is the ratio

$$\Omega = \frac{\tau_S}{\tau_\lambda} \tag{13}$$

When there is coupling between $S$ and $\lambda$, (12) turns into the *sine circle map* [6]

$$S_{n+1} = S_n + 2\pi\Omega - K\sin(S_n) \tag{14}$$

which can be alternatively presented as

$$\theta_{n+1} = \theta_n + \Omega - (\frac{K}{2\pi})\sin(2\pi\theta_n) \tag{15}$$

Here, $n \in \mathbf{N}$ is the iteration index, $K$ denotes the coupling strength and

$$S_n = 2\pi\theta_n \tag{16}$$

The motion is deemed *periodic* and occurs on a two-dimensional torus in phase space if (13) is a ratio of *rational* numbers. Otherwise, the motion is *quasiperiodic* and is everywhere dense on the torus in the sense that the trajectory returns arbitrarily close to its starting point an unbounded number of times. Moreover, the motion is also *ergodic* as long-term averages of motion on the torus equal area averages over the torus with uniform density [6].

The winding number of the map (14) or (15) is defined as

$$W(K) = \lim_{n\to\infty} \frac{\theta_n - \theta_0}{n} = \lim_{n\to\infty} \frac{S_n - S_0}{2\pi n} \tag{17}$$

and characterizes the average number of $2\pi$ rotations per iteration. It is important to note that (17) is different from (13) when $K \neq 0$. By contrast, $W(K=0) = \Omega$ when there is no coupling between $S$ and $\lambda$, in which case $W(0)$ denotes the angular increment at each iteration step $n$.

Analysis shows that the transition to chaos in either (14) or (15) relates to a sequence of irrational winding numbers $\{W_n\}$ that converges geometrically according to the series [6-7]

$$W_n = W_\infty - c\,\overline{\delta}^{-n} = \overline{W} - c\,\overline{\delta}^{-n} \tag{18}$$

where the convergence rate is determined by the *golden mean* $\overline{W}$ via

$$\delta = -\overline{W}^{-2} \ , \ \overline{W} = \frac{\sqrt{5}-1}{2} \tag{19}$$

A similar scaling behavior holds for the sequence of bare winding numbers $\{\Omega_n\}$ and one finds, for a given $|K|<1$,

$$\lim_{n\to\infty}\Omega_n(K)=\Omega_\infty=\overline{W} \tag{20}$$

Relations (18) – (20) reflect the approach to *self-similar* behavior of winding numbers (13) and (17) in the limit $n>>1$, that is,

$$\lim_{n\to\infty}\frac{\Omega_{n+1}-\Omega_n}{\Omega_n-\Omega_{n-1}}=-\overline{W}^{-2}\quad,\quad |K|<1 \tag{21}$$

$$\lim_{n\to\infty}\frac{W_{n+1}-W_n}{W_n-W_{n-1}}=-\overline{W}^{-2}\quad,\quad |K|\to 1 \tag{22}$$

## 3. Emergence of Planck's constant

3.1 ) Consider first the case where $1<<n<\infty$ and the coupling strength is nearly-vanishing $|K|<<1$. By (11) and (20) we obtain the *first interpretation of action quantization* in the form

$$\boxed{\delta S_{n+1,n}=S_{n+1}-S_n=2\pi\overline{W}} \tag{23}$$

Note that the condition $|K|<<1$ is consistent with the separation of the high and low energy scales in effective Quantum Field Theory [8].

3.2) Next, consider again the case $1<<n<\infty$ and refer to (17)-(19). The *second interpretation of action quantization* assumes the form

$$\boxed{\delta S_{n,0}=S_n-S_0=n\,(2\pi\overline{W})} \tag{24}$$

There is a couple of relevant observations concerning (23) and (24):

a) It is apparent from (13) that $\Omega \to 0$ when $\tau_\lambda >> \tau_S$ and $\tau_\lambda \to \infty$ . If, in addition, the coupling $K$ drops to zero in (14), the background perturbation becomes insignificant and one recovers the least-action principle of classical mechanics ( $\delta S_{n+1,n} = 0$ ).

b) According to (23) and (24), Planck's constant emerges as a fixed number related to the golden-mean. The precise numerical value of this constant is irrelevant as Planck's constant can be set to $\hbar = 1$ in natural units.

## 4. Universal constants from self-similarity

The classical harmonic oscillator equation derived from (4) is given by ( $m = 1$ )

$$\ddot{q} + \omega_0^2 q = 0 \tag{25}$$

By (5) and (9), the angular frequency $\omega_0$ has the dimension of energy in natural units, which means

$$[\omega_0] = E \ , \ \ (\hbar = 1) \tag{26}$$

A remarkable property of (25) is that it stays *self-similar* (covariant) under arbitrary scaling operations of the coordinate $q$ having the form $q \to s \cdot q$ with $s \in \mathbf{R}$. Self-similarity is only possible if $\omega_0$ is insensitive to scaling, which amounts to

$$\frac{\partial \omega_0}{\partial s} = 0, \quad \forall s \in \mathbf{R} \tag{27}$$

The requirement (27) is formally equivalent to demanding that (25) preserves its form upon multiplying $\omega_0$ by a scalar independent of $s$, or

$$\omega_0 \to \kappa \cdot \omega_0 \ , \quad \kappa \in \mathbf{R}\, , \quad \kappa \neq s \tag{28}$$

In light of the adiabatic invariance condition (5), (28) implies that $\kappa$ is *also an adiabatic invariant* whose presence has no effect on (25). This can be readily seen from (5), i.e.

$$\omega_0 \to \kappa \cdot \omega_0 \Rightarrow \frac{S}{\kappa} = \frac{E}{\kappa \cdot \omega_0} \tag{29}$$

Based on these remarks, it makes sense to attempt connecting $\kappa$ with the light speed in vacuo ($c$) and Boltzmann's constant $(k_B)$, respectively. Towards this end, let us first consider the wave equation for a monochromatic electromagnetic field in the vacuum [9]

$$\Delta f + \frac{\omega_0^2}{c^2} f = 0 \tag{30}$$

where $f$ is either one of the vector components of the field, $f = (\mathbf{E}, \mathbf{H})$. Since (30) shares the same form with (25), with the second time derivative replaced by the Laplacian, comparative inspection of (28)-(29) and (30) yields the *electromagnetic adiabatic invariant*

$$\boxed{\kappa_{EM} = 1/c} \tag{31}$$

Turning next to the Boltzmann's constant, we again refer to (5) and write it as

$$S = \frac{E}{\omega_0} = \frac{E}{2\pi}\tau_S \tag{32}$$

It is known that the quantum mechanical amplitude is the integral over all paths weighted by $e^{iS}$. The technique of Wick rotation enables a conversion from $e^{iS}$ to the Euclidean weight $e^{-S_E}$ via the concept of Euclidean time $t_E$ defined as [8]

$$t = -it_E,\ \ t_E > 0 \tag{33}$$

It is also known that Euclidean time relates to the canonical temperature parameter via

$$|t_E| = \beta = \frac{1}{k_B T} \tag{34}$$

On account of (33) and (34), (32) yields

$$S_E = \frac{E}{2\pi k_B T} \tag{35}$$

It is apparent from (6)-(8) that energy, frequency and temperature are interchangeable concepts with the same dimension in natural units. Hence, by (28) and (35) we obtain

$$S_E = \frac{E}{2\pi k_B T} = \frac{E}{\kappa \cdot \omega_0} = \frac{E}{\kappa \cdot 2\pi \nu_0} \tag{36}$$

which leads to the *Boltzmann adiabatic invariant*

$$\boxed{\kappa_B = k_B} \tag{37}$$

## 5. Open questions

We caution the reader that our analysis is strictly a tentative exploration which requires independent scrutiny and validation. Here is a sample of questions open for further clarification:

1) The approach appears to work for massless oscillations in the classical vacuum, that is, for free radiation modes. One needs to explain why collective excitations in condensed matter and plasma physics are exempt and do not lead to universal constants.

2) Not all adiabatic invariants generate universal constants. Two prime examples are the magnetic moment of a gyrating particle in a magnetic field, as well as the longitudinal invariant of a particle trapped in a magnetic mirror [10-11].

3) Can Newton's constant be linked to a similar treatment?

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