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Cosmological and Fermi Scales from Hamiltonian Chaos

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Abstract

A lesser-known property of Hamiltonian dynamics is that it can be formally mapped to the Riemannian geometry of classical gravitation. Taking advantage of this property, we explore here the possibility that the onset of Hamiltonian chaos in the ultraviolet (UV) sector of field theory generates the cosmological and Fermi scales. This finding supports the conjecture that both Standard and the ΛCDM models emerge as non-trivial attractors of the UV to infrared (IR) flow.

Key words: Hamiltonian flow, Hamiltonian chaos, cosmological constant, Fermi scale, Lyapunov stability, non-equilibrium thermodynamics.

1. Introduction

Recent decades have convincingly shown that *nonintegrability* and *chaos* are landmark features of nonlinear or far-of-equilibrium dynamical systems. Typical fingerprints of chaotic behavior include bifurcations and universality, the emergence of non-trivial attractors in phase space and the route to turbulence of fluid-like dynamics, including Hamiltonian and Renormalization Group flows. Today, the science of chaos and complexity is a mature discipline with a far-reaching array of applications ranging from fundamental and applied science to engineering, medicine, social sciences, financial affairs, weather forecasting and internet dynamics.

Given the ubiquity of nonlinear dynamics in foundational physics, it is truly surprising that, as of today and aside from a handful of exceptions [for instance,], the onset of chaos in standard cosmology and the UV sector of Quantum Field Theory remains insufficiently studied and understood.

Motivated by advances in the analysis of nonintegrable systems, we have recently conjectured that non-equilibrium dynamics and complex behavior are prone to develop near or above the Fermi scale (M_{EW}) [ ]. This regime is best characterized by the concept of non-vanishing Kolmogorov (K) entropy and the emergence of a spacetime having continuous dimensionality $(\varepsilon = 4 - D << 1)$. The surge of K-entropy in nonintegrable systems and unstable systems outside equilibrium is associated with *increasing complexity in phase space*. A prerequisite of this process is the mechanism of *decoherence,* which drives the transition from quantum to classical behavior. The expectation is that global thermalization of instabilities occurs at the endpoint of the phase space flow, a state corresponding to the onset of effective field theory [ ].

The object of this report is to examine the surprising possibility that the emergence of both cosmological constant and the Fermi scale follow from the onset of chaos in the UV sector of Hamiltonian dynamics. The paper is organized as follows: next section lays out the background and underlying assumptions on which the analysis is built; sections three to four elaborate on the emergence of cosmological and the Fermi scales from the K-entropy. Last section presents a summary and a discussion of main results. To make the paper self-contained, the Appendix section includes a brief exposition on the Riemannian theory of Hamiltonian flows.

The reader is cautioned upfront on the provisional nature of these ideas. Follow up studies are needed to independently validate, rebut, or build up this line of reasoning in a more comprehensive and robust way.

2. Theoretical background and working assumptions

An intriguing (yet far less appreciated) property of multidimensional nonlinear equations is that the spread of nearby geodesics on Riemannian manifolds, on the one hand, and the exponential separation of nearby orbits in phase space, on the other, are complementary descriptions of *dynamic instability* [ ].

To illustrate this point, consider a two-dimensional manifold and the Jacobi equation (JE) defining the separation of nearby Riemannian geodesics $\zeta(s)$ as a function of the local Gaussian curvature of the manifold $\kappa(s)$. JE is given by [ ]

$$\frac{d^2\zeta(s)}{ds^2}+\kappa(s)\zeta(s)=0 \tag{1}$$

where the arclength s plays the role of a time-like parameter. The divergence of the affine parameter $\zeta(s)$ along s reflects the geodesic *sensitivity to initial conditions*. There is a pair of disjoint subspaces defining the solution space of (1) for solutions with *exponentially expanding* separation $\zeta(s)$

$$|\zeta(s)|\geq\frac{1}{2}|\zeta(0)|\exp(\sqrt{2\kappa}s) \tag{2a}$$

and for *exponentially contracting* solutions [ ]

$$|\zeta(s)|\leq\frac{1}{2}|\zeta(0)|\exp(-\sqrt{2\kappa}s) \tag{2b}$$

Equations (2) determine the local stability of geodesics in the following sense: by convention, on spherical surfaces, $\kappa(s)>0$ and (2b) means stability, whereas, by (2a), hyperbolic surfaces with $\kappa(s)<0$ mean instability.

Secondly, recall that the transition to chaos in systems of nonlinear equations and iterated maps is quantified via positive Lyapunov exponents and nonzero K-entropy [ ]. A generic feature of these systems is that the exponential divergence of unstable - and possibly fluctuating - nearby geodesics is encoded in

$$\delta x(s) \propto \delta x(0) \exp[\lambda(s) \cdot s] \ , \ \lambda(s) > 0 \tag{3}$$

K-entropy is computed from the sum of all positive Lyapunov exponents integrated over phase space as in

$$S_K = \int_{\Sigma} \sum_i \lambda_i \, d\rho \ , \ \ \lambda_i > 0 \tag{4}$$

in which $d\rho$ stands for the differential measure of phase space. Relations (1)-(3) imply that the square root of curvature in spacetime *mirrors the role* of a local Lyapunov exponent in the corresponding phase space. This observation leads to the symbolic mapping

$$\boxed{\lambda(s) \Leftrightarrow \sqrt{\kappa(s)}} \tag{5}$$

Retracing the footsteps of [ ], we now introduce the following assumptions and approximations:

A1) The focus is on *low-dimensional nonlinear systems* exhibiting *dissipative* behavior. The rationale for choosing low-dimensional systems echoes the *center manifold theory,* where a multivariable system of differential equations reduces in the long run to a lower dimensional system of universal equations dependent on a single emerging variable [ ].

A2) Lyapunov stability is applied to geodesics having a limited extent in spacetime or phase space, and can be either stable, unstable, or fluctuating. To characterize instability and for the sake of simplicity, we assume that there are n Lyapunov exponents, of which $n-1$ are negligible in comparison with a single maximal exponent denoted as $\lambda_0 > 0$.

A3) The focus is exclusively on weak and slowly varying gravitational fields, as typically described in introductory textbooks on General Relativity (GR). This assumption enables the local approximation of GR as a Hamiltonian (energy conserving) field theory.

A4) In line with A3), we exclusively consider the vacuum regime of nearly vanishing curvature and assume that this regime is characterized by nonzero fluctuations in the curvature sign. Tracing Einstein's equation implies that the net result of these fluctuations is a *positive Ricci curvature* and a *positive cosmological constant* related through $R = 4\Lambda$.

3. Cosmological constant from K-entropy

Ramping up of K-entropy (S_K) upon lowering the observation scale μ follows from the continuous dimensional deviation of spacetime near the Fermi scale, as encoded in [ ]

$$\varepsilon(\mu) = 4 - D(\mu) \propto \frac{m^2(\mu)}{\Lambda_{UV}^2} << 1 \tag{6}$$

Here, the observation scale is taken to be dimensionless, m is a mass parameter and Λ_{UV} a large UV cutoff. The K-entropy (S_K), the Hausdorff dimension of phase space trajectories (D_H) and dimensional deviation (ε) are related through [ ]

$$D_H(\mu) = -\frac{S_K(\mu)}{\log \varepsilon(\mu)} \Rightarrow S_K(\mu) = \log[\varepsilon(\mu)^{-D_H(\mu)}] \tag{7}$$

By (4) and on account of assumption A2), the phase space density of K entropy is given by

$$\frac{dS_K(\mu)}{d\rho(\mu)} \propto \lambda_0 \tag{8}$$

where $\rho = \rho(\mu)$ runs with μ in what represents a *Liouville flow*. From (7) and (8) we obtain

$$\frac{dS_K(\mu)}{d\mu} = -\frac{D_H(\mu)}{\varepsilon(\mu)} \frac{\beta_\varepsilon(\varepsilon)}{\beta_\rho(\rho)} \propto \lambda_0 \tag{9}$$

where the beta-functions of the dimensional and Liouville flow are, respectively,

$$\beta_\varepsilon(\varepsilon) = \frac{d\varepsilon}{d\mu}, \ \ \beta_\rho(\rho) = \frac{d\rho}{d\mu} \tag{10}$$

By analogy with the Renormalization Group equations, it is convenient to cast the dimensional flow as a power series, namely,

$$\beta_\varepsilon(\mu) = \sum_{j=1}^{\infty} a_j \varepsilon^j(\mu) \tag{11}$$

Taking the lowest bound value of $\varepsilon \to \varepsilon_{\min}$ and up to a first-order approximation, combined use of (5), (9)-(11) and A2)-A4) yields

$$\boxed{-D_H(\mu)\frac{(a_1 + a_2\varepsilon_{\min})}{\beta_\rho(\rho)} \Leftrightarrow \sqrt{\kappa_{\min}} = \sqrt{4\Lambda}} \tag{12}$$

Furthermore, under the assumption that $\beta_\rho(\mu) \neq 0$, it is apparent that (12) maps the *minimal numerical value of dimensional parameter* $\varepsilon_{\min} << 1$ *to the nonzero value of the cosmological constant* Λ. The conservative limit of effective field theories matches the case $a_1 = \varepsilon_{\min} = \beta_\rho(\mu) = 0$.

It is worthwhile pausing for a moment to reflect upon the meaning of this result. As free parameter of Einstein's equations, the cosmological constant written as $\Lambda_{cc} = \Lambda^2$ denotes the energy density of classical vacuum and fixes the non-vanishing curvature of empty space. Relation (12) suggests that Λ may be interpreted as *evidence for the minimal fractality of spacetime near or above the Fermi scale*. It is conceivable that (12) may set the stage for revisiting the ΛCDM model as embodiment of non-equilibrium thermodynamics. As section four attempts to show, (12) also hints to a fresh perspective on the mass and flavor composition of high-energy physics.

4. Cantor Dust and the Fermi Scale from K-entropy

Dimensional deviation is a continuous and arbitrarily small number, which may be thought of as an *infinite string* of component deviations as in

$$\varepsilon = \sum_1^\infty \varepsilon_i = \frac{1}{\Lambda_{UV}^2} \sum_1^\infty m_i^2 << 1 \tag{13}$$

Since the cosmological constant of (12) carries the dimension of mass squared, i.e. $[\Lambda] = M^2$ and by (13), it is tempting to speculate that $\varepsilon_{\min}$ can be cast in the following form

$$\varepsilon_{\min} = \sum_1^\infty \varepsilon_{i,\min} = \frac{1}{\Lambda_{UV}^2} \sum_1^\infty m_{i,\min}^2 \propto \frac{\Lambda}{\Lambda_{UV}^2} \tag{14a}$$

or

$$\boxed{\sum_1^\infty \frac{m_{i,\min}^2}{\Lambda} = O(1)} \tag{14b}$$

The string of component deviations $\varepsilon_{i,\min}$ acts as an infinite ensemble of scalar fields clustered into a large-scale Cantor Dust structure arising from *topological condensation* [ ]. By (14), one concludes that, on scales larger than the Fermi scale, *the energy content of the cosmological constant comes from the cumulative contribution of energies stored in the Cantor Dust.* Although a highly speculative scenario, this finding points nevertheless to an attractive path towards unifying Dark Energy and Dark Matter into a single and coherent framework.

A key observation is that the analysis developed so far is not limited to cosmological constant and the geometry of General Relativity. With reference to the Appendix section, a remarkable property of classical Hamiltonian dynamics is that it can be formally mapped to Riemannian geometry. By (A2) and interpreting the kinetic terms of the Hamiltonian as mass parameters, enables one to turn them into Riemannian metric coefficients, that is,

$$m_{ij} \Leftrightarrow g_{ij} = 2\left[E - V(\varphi)\right]a_{ij} \tag{15}$$

By analogy with (13)-(14), the approach to Hamiltonian chaos above the Standard Model scale may be characterized by a relationship that replicates (14), with mass parameters replaced by (15) and with the cosmological constant scale Λ replaced by the Fermi scale. It is plausible that, proceeding along this path, leads to the "sum-of-squares" relationship of high-energy physics, written as

$$\boxed{\sum_{1}^{N} \frac{m_{i,SM}^{2}}{M_{EW}^{2}} = O(1)} \tag{16}$$

in which the sum is extended over the entire mass spectrum of the Standard Model ($i = 1, 2, ..., N$) [ ].

One may reasonably question at this point if there is a deeper motivation behind (14b) and (15). The answer is likely to lie in the concept of *generating function,* a key descriptor of chaos theory applied to iterated maps, and defined as [ ]

$$\Gamma(q) = \sum_{j} p_j^q r_j^{\tau(q)} \,, \;\; j = 1, 2, ..., n >> 1 \tag{17}$$

subject to the normalization condition

$$\sum_{j} p_j = 1 \tag{18}$$

Here, p_j is the relative frequency with which the iterated map falls in the j^{th} interval of the phase space, while q and $\tau(q)$ are continuous scaling exponents ($-\infty < q < \infty$). In the limit of many map iterations, (17) approaches unity ($\Gamma(q) \to 1$) and, in this limit, it is apparent that (14b) and (15) share the same *universality class* defined by $q = 0,\ \tau(0) = 2$.

5. Discussion

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APPENDIX

Riemannian geometry of Hamiltonian flows

A conservative system of classical fields is defined by the Hamiltonian

$$H = \frac{1}{2} a_{ij}\, \dot{\varphi}_i\, \dot{\varphi}_j + V(\varphi_1, \varphi_2, ..., \varphi_N) \qquad \text{(A1)}$$

with $H = E$ being a constant of motion. The configuration space M of the system includes N local coordinates $(\varphi_1, \varphi_2, ..., \varphi_N)$ and can be associated with a Riemannian metric using the substitution

$$g_{ij} = 2\left[E - V(\varphi)\right] a_{ij} \qquad \text{(A2)}$$

The minimum action principle reads

$$\delta I = \delta [\int_{\gamma(t)} \pi_i\, d\varphi_i] = \delta [\int_{\gamma(t)} \frac{\partial L}{\partial \dot{\varphi}_i} \dot{\varphi}_i\, dt] = 0 \qquad \text{(A3)}$$

in which the kinetic energy of the system of fields takes the form

$$T = \frac{1}{2}\dot{\varphi}_i \frac{\partial L}{\partial \dot{\varphi}_i} \qquad \text{(A4)}$$

(A3) becomes, accordingly

$$\delta \int_{\gamma(t)} 2T dt = \delta \int_{\gamma(t)} \sqrt{g_{ij}\dot{\varphi}_i \dot{\varphi}_j}\, dt = \delta \int_{\gamma(t)} ds = 0 \qquad \text{(A5)}$$

It follows from (A5) that natural motions of the Hamiltonian system are *geodesics of* M whose differential arclength is ds. The geodesic equation is given by

$$\frac{d^2\varphi^i}{ds^2} + \Gamma^i_{jk}\frac{d\varphi^j}{ds}\frac{d\varphi^k}{ds} = 0 \qquad \text{(A6)}$$

where Γ^i_{jk} represent Christoffel coefficients of the metric.

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<u>References</u>

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