GROUPS OF NONLINEAR CONGRUENCES AND GEODESIC LINES ON THE ORDERED SET OF POSITIVE DEFINITE MATRICES

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Abstract. A family of nonlinear transformation groups of Hermitian matrices preserving the basic linear congruences properties is constructed. Each group generates a structure of a partially ordered vector multispace on the set of Hermitian matrices. Determinant inequalities which allow as to find geodesic lines for an analog of indefinite metric on positive definite matrix set are obtained.

- $\mathbb{H}_n$ stands for the set of all Hermitian $n \times n$ matrices with complex entries.
- $\mathbb{P}_n$ stands for the subset of all positive semidefinite $n \times n$ matrices.
- $\mathbb{P}_n^+$ stands for the subset of all positive definite $n \times n$ matrices.
- $GL(n, \mathbb{C})$ stands for the group of all invertible $n \times n$ matrices with respect to the ordinary multiplication of matrices.
- $SL(n, \mathbb{C})$ stands for the group of $n \times n$ matrices with a unit determinant.
- $U(n)$ stands for the group of unitary $n \times n$ matrices.
- $SU(n)$ stands for the group of unitary $n \times n$ matrices with a unit determinant.

1 Introduction

Invariance and causality are fundamental principles of physics. Coordinate transformations play an important role, if they preserve some notion of positivity. This explains a special interest of mathematicians and physicists in a partially ordered set of positive definite matrices. The congruence transformation $X' = Q \cdot X \cdot Q^*$, where $Q \in GL(n, \mathbb{C})$ and Q^* is Hermitian conjugate matrix, is the linear transformation of $\mathbb{H}_n$. The congruences preserve positive definiteness and the standard partial order relation $X \leq Y$ (in the sense of quadratic forms). Riemannian metrics on the set of positive definite matrices and automorphisms of corresponding structures are intensively studied (see [1]-[4], [13, chapter 6]).

In this article we introduce the simplest nonlinear transformations of $\mathbb{H}_n$. These transformations preserve the ordinary congruence basic properties. The nonlinear congruent transformations define the structure of so-called ordered vector multispace on $\mathbb{H}_n$. They are order automorphisms for this structure. In addition, these transformations define an invariant scalar function on a pair of positive definite matrices (an analogue of the point-dependent indefinite metric). The triangle inverse inequality is performed on the causally connected point triple. If a pair of points is causally connected, then there is a structure of a vector space on $\mathbb{H}_n$ such that the equation of geodesic (of maximum "length") is a linear equation. The geodesic line point determines the addition operation choice in its equation. It's addition operation is the operation of vector multispace.

2 Groups of non-linear congruences

2.1. Recall that *inertia* of $X \in \mathbb{H}_n$ is the triple of nonnegative integers

$$In(X) = \{\pi(X), \varsigma(X), \nu(X)\},$$

where $\pi(X)$, $\varsigma(X)$, $\nu(X)$ are the numbers of positive, zero, and negative eigenvalues of X (counted with multiplicity). The transformation

$$Y = l_Q(X) = Q \cdot X \cdot Q^*, \quad Q \in GL(n, \mathbb{C}) \tag{2.1}$$

of $\mathbb{H}_n$ onto itself is called *a congruence*.

We list the characteristic properties of congruences:

(i). *The set of all congruences* (2.1) *form a arcwise connected linear transformation group of* $\mathbb{H}_n$ *with respect to the superposition of mappings*;

(ii). $In(l_Q(X)) = In(X)$ *for all* $X \in \mathbb{H}_n$. *In particular* $l_Q(X) \geq 0 \Leftrightarrow X \geq 0$;

(iii). *If* $Q \in SL(n, \mathbb{C})$, *then* $\det l_Q(X) = \det X$ *for all* $X \in \mathbb{H}_n$.

2.2. Suppose F is an arbitrary one-to-one mapping of $\mathbb{H}_n$ onto itself, F^{-1} is the reverse mapping for F, and $f \circ g$ is the superposition of mappings. Let l_Q be congruence, i.e. $l_Q(X) = Q \cdot X \cdot Q^*$, and $X \in \mathbb{H}_n$. The set of all mappings

$$\varphi_Q(X) = F \circ l_Q \circ F^{-1}(X), \quad Q \in GL(n, \mathbb{C}) \tag{2.2}$$

is a group of transformations of $\mathbb{H}_n$ with respect to the superposition of mappings. This group is denoted by $G_F(n)$.

We will to call the group $G_F(n)$ is *regular* if $In \varphi_Q(X) \equiv In X$ for all $Q \in GL(n, \mathbb{C})$ and $\det \varphi_Q(X) \equiv \det X$ for all $Q \in SL(n, \mathbb{C})$.

2.3 Lemma. *The group* $G_F(n)$ *is regular if and only if*

$$In\, X_1 = In\, X_2, \ \det X_1 = \det X_2 \ \ implies \ \ In\, F(X_1) = In\, F(X_2), \ \det F(X_1) = \det F(X_2). \tag{2.3}$$

Proof. Let $G_F(n)$ be regular. For any X_1, X_2, $In\, X_1 = In\, X_2$, $\det X_1 = \det X_2$ there exists a congruence l_Q such that $X_2 = l_Q(X_1)$, $Q \in SL(n, \mathbb{C})$. The mapping $X_2' = \varphi_Q(X_1')$ closes the commutative diagram

$$
\begin{array}{ccc}
X_1 \in \mathbb{H}_n & \xrightarrow{\ l_Q\ } & X_2 \in \mathbb{H}_n \\
F^{-1} \uparrow & & F \downarrow \\
X_1' \in \mathbb{H}_n & \xrightarrow{\ \varphi_Q\ } & X_2' \in \mathbb{H}_n
\end{array}.
$$

Since group $G_F(n)$ is regular and $Q \in SL(n, \mathbb{C})$, then $In\, F(X_1) = In\, F(X_2)$, $\det F(X_1) = \det F(X_2)$. Sufficiency is obvious.∎.

Let f be one-to-one mapping of $\mathbb{R}$ onto itself. For any matrix $X \in \mathbb{H}_n$ with spectral value decomposition $X = U \cdot \Lambda \cdot U^*$, where $\Lambda = \mathrm{diag}\{\lambda_1, \cdots, \lambda_n\}$ and U is unitary, we define

$$f(X) = U \cdot \mathrm{diag}\{f(\lambda_1), \cdots, f(\lambda_n)\} \cdot U^*. \tag{2.4}$$

It is clear that $Y = f(X)$ is one-to-one mapping of $\mathbb{H}_n$ onto itself.

2.4 Theorem. *Let* f *be one-to-one mapping of* $\mathbb{R}$ *onto itself, continuous over all of* $\mathbb{R}$, *and* $f(-1) = -f(1)$. *The group* $G_f(n)$ *is regular if and only if*

$$f(x) \equiv C_0 \cdot |x|^{\delta} \cdot \mathrm{sgn}\, x, \quad \delta > 0, \ C_0 \neq 0. \tag{2.5}$$

Proof. Let $G_f(n)$ be regular. Since f is one-to-one on $\mathbb{R}$ and $f(-1)=-f(1)$, then $f(1)=C_0\neq 0$. The function $h(x)=f(x)/f(1)$ satisfies the condition (2.3) and $h(1)=1$. Choosing $X_1=\mathrm{diag}\{-x,\ 1,\ 1,\cdots,1\}$, $X_2=\mathrm{diag}\{x,\ -1,\ 1,\cdots,1\}$, we get $\det h(X_1)=\det h(X_2)$, i.e. $h(-x)=-h(x)$.

Let $X_1=\mathrm{diag}\{\lambda\mu,\ 1,\ 1,\cdots,1\}$, $X_2=\mathrm{diag}\{\lambda,\ \mu,\ 1,\cdots,1\}$, where $\lambda,\mu>0$. By Lemma 2.3, we have

$$h(\lambda\mu)=h(\lambda)h(\mu),\ \ \lambda,\mu>0. \tag{2.6}$$

In the class of continuous functions on the positive semi-axis the solutions of Cauchy functional equation (2.6) are functions $h(x)=x^{\alpha}$, $\alpha\in\mathbb{R}$ and only they. By condition $h(x)$ is continuous at point 0 and one-to-one on $\mathbb{R}$. Then $\alpha>0$. Using condition $h(-x)=-h(x)$ we obtain (2.5). Sufficiency is obvious.∎.

Now we shall give the following definition.

2.5 Definition. Let $\tau_{\delta}(x)=|x|^{\delta}\cdot\mathrm{sgn}\,x$, $\delta>0$ and $\tau_{\delta}^{-1}(y)=|y|^{1/\delta}\cdot\mathrm{sgn}\,y$. The set of all transformations

$$X'=\varphi_Q(X)=\tau_{\delta}[Q\cdot\tau_{\delta}^{-1}(X)\cdot Q^*],\ \ Q\in GL(n,\mathbb{C}),\ \ X,X'\in\mathbb{H}_n, \tag{2.7}$$

is called the *group of δ-congruences* and denote by $G_{\delta}(n)$. If $Q\in SL(n,\mathbb{C})$, then the mapping (2.7) is called *unimodular δ-congruence*. The subgroup of unimodular δ-congruences denote by $SG_{\delta}(n)$.

2.6 Remark. If $\delta=1$, then group $G_{\delta}(n)$ is a group of ordinary congruences.

2.7. *The following results are easily checked directly from the definition:*

(i). $Q,R\in GL(n,\mathbb{C})\ \Rightarrow\ \varphi_Q\circ\varphi_R=\varphi_{Q\cdot R},\ \varphi_Q^{-1}=\varphi_{Q^{-1}}$;

(ii). $Q\in GL(n,\mathbb{C})\ \Rightarrow\ In\,\varphi_Q(X)=In\,X\ \ for\ all\ X\in\mathbb{H}_n$. *In particular* $\varphi_Q(X)\geq 0\ \Leftrightarrow\ X\geq 0$;

(iii). $Q\in SL(n,\mathbb{C})\ \Rightarrow\ \det\varphi_Q(X)=\det X\ \ for\ all\ X\in\mathbb{H}_n$;

(iv). *If* $\alpha\in\mathbb{R}$, *then* $\varphi_Q(\alpha\cdot X)=\alpha\cdot\varphi_Q(X)$.

Proof (iv): If $\alpha\in\mathbb{R}$, then $\varphi_Q(\alpha\cdot X)=\tau_{\delta}[Q\cdot\tau_{\delta}^{-1}(\alpha\cdot X)\cdot Q^*]=\tau_{\delta}\circ\tau_{\delta}^{-1}(\alpha)\cdot\varphi_Q(X)=\alpha\cdot\varphi_Q(X)$.∎.

For any matrix $Q\in GL(n,\mathbb{C})$ with singular value decomposition $Q=U\cdot\Lambda\cdot V$ (see [11], p. 150), where U,V is unitary and $\Lambda=\mathrm{diag}\{\alpha_1,\cdots,\alpha_n\}>0$, we have (see **2.7** (i))

$$\varphi_Q=\varphi_{U\cdot\Lambda\cdot V}=\varphi_U\circ\varphi_{\Lambda}\circ\varphi_V. \tag{2.8}$$

Next we will need the following properties of mappings $\varphi_Q\in G_{\delta}(n)$.

2.8 Proposition. *The mappings φ_Q of group $G_{\delta}(n)$ satisfy the conditions:*

(i). *If* $W\in U(n)$, *then* $\varphi_W(X)\equiv WXW^*$;

(ii). *If* $\delta\neq 1$ *and* $\Lambda=\mathrm{diag}\{\alpha_1,\cdots,\alpha_n\}>0$, *then* $\varphi_{\Lambda}(X+Y)=\varphi_{\Lambda}(X)+\varphi_{\Lambda}(Y)$ *for all* $X,Y\in\mathbb{H}_n$ *if and only if* $\alpha_1=\alpha_2=\cdots=\alpha_n$;

(iii). *If* $0\neq\varsigma\in\mathbb{C}$, *then* $\varphi_{\varsigma\cdot Q}(X)=|\varsigma|^{2\delta}\cdot\varphi_Q(X)\quad for\ all\ \ X\in\mathbb{H}_n$;

(iv). *If* $\delta\neq 1$, *then* $\varphi_Q(X+Y)=\varphi_Q(X)+\varphi_Q(Y)$ *for all* $X,Y\in\mathbb{H}_n$ *if and only if* $\varsigma\cdot Q\in U(n)$ *for some complex number* $\varsigma\neq 0$;

(v). *Let* $In\,Y=In\,X$. *There exists* $\varphi_Q\in G_{\delta}(n)$ *such that* $Y=\varphi_Q(X)$. *If moreover* $\det Y=\det X$, *then* $\varphi_Q\in SG_{\delta}(n)$.

Proof (i) - (v):

(i). By definition (2.4), we have $\varphi_W(X) = \tau_\delta[W\,\tau_\delta^{-1}(X)\,W^*] = W[\tau_\delta \circ \tau_\delta^{-1}(X)]W^* = W X W^*$;

(ii). The general case is reduced to a case of 2×2 matrices. Let

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} > 0.$$

Then

$$\varphi_\Lambda(X) + \varphi_\Lambda(Y) = \begin{pmatrix} \alpha_1^{2\delta} & 0 \\ 0 & \alpha_2^{2\delta} \end{pmatrix} + \begin{pmatrix} 0 & \alpha_1^\delta \alpha_2^\delta \\ \alpha_1^\delta \alpha_2^\delta & 0 \end{pmatrix} = \begin{pmatrix} \alpha_1^{2\delta} & \alpha_1^\delta \alpha_2^\delta \\ \alpha_1^\delta \alpha_2^\delta & \alpha_2^{2\delta} \end{pmatrix}; \tag{2.9}$$

$$\varphi_\Lambda(X + Y) = \varphi_\Lambda\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 2^{1-\delta}(\alpha_1^2 + \alpha_2^2)^{\delta-1} \cdot \begin{pmatrix} \alpha_1^2 & \alpha_1\alpha_2 \\ \alpha_1\alpha_2 & \alpha_2^2 \end{pmatrix}. \tag{2.10}$$

Comparing (2.9) and (2.10), we obtain $2^{1-\delta}(\alpha_1^2 + \alpha_2^2)^{\delta-1} = \alpha_i^{\delta-1}\alpha_j^{\delta-1}$, $i,j = 1,2$. If $\delta \neq 1$, then $\alpha_1 = \alpha_2$. Sufficiency is obvious;

(iii). We have

$$\varphi_{\varsigma \cdot Q}(X) \overset{def}{=} \tau_\delta[\varsigma Q \cdot \tau_\delta^{-1}(X) \cdot \bar\varsigma Q^*] = \tau_\delta(|\varsigma|^2) \cdot \tau_\delta[Q \cdot \tau_\delta^{-1}(X) \cdot Q^*] = |\varsigma|^{2\delta} \cdot \varphi_Q(X);$$

(iv) follows from (i), (ii), (iii) and (2.8);

(v). Let $X, Y \in \mathbb{H}_n$ and $In X = In Y$. There are exists $U, V \in U(n)$ such that $X = U \cdot \Lambda_X \cdot U^*$, $Y = V \cdot \Lambda_Y \cdot V^*$, where

$$\Lambda_X = \mathrm{diag}\{\lambda_1, \cdots, \lambda_n\}, \quad \Lambda_Y = \mathrm{diag}\{\mu_1, \cdots, \mu_n\} \quad \text{and} \quad \mathrm{sgn}\,\lambda_k = \mathrm{sgn}\,\mu_k, \; k = 1, \cdots, n.$$

By definition, put

$$Q = V \cdot \Delta \cdot U^*, \quad \Delta = diag\{\alpha_1, \cdots, \alpha_n\}, \text{ where } \alpha_k = |\mu_k / \lambda_k|^{1/2\delta} \text{ if } \lambda_k \neq 0, \text{ otherwise } \alpha_k = 1.$$

Then

$$\varphi_Q(X) = \tau_\delta[V \cdot \Delta \cdot U^* \cdot \tau_\delta^{-1}(X) \cdot U \cdot \Delta \cdot V^*] = V \cdot \tau_\delta[\Delta \cdot \tau_\delta^{-1}(\Lambda_X) \cdot \Delta] \cdot V^* = V \cdot \tau_\delta[\tau_\delta^{-1}(\Lambda_Y)] \cdot V^* = Y.$$

If $\det Y = \det X$, then, as is easy to see, the determinant of Δ can be chosen equal to 1. ∎.

3 Vector Multispace

First we consider the question of a suitable definition of a vector multispace.

3.1 Proposition. *Let L be a vector space with the operations $x + y$, $\alpha \cdot x$. Each one-to-one mapping φ of the set L onto itself defines on L new operations $x \overset{\varphi}{\oplus} y$, $\alpha \overset{\varphi}{\times} x$ such that:*

(i). *The set L with operations $x \overset{\varphi}{\oplus} y$, $\alpha \overset{\varphi}{\times} x$ is a new vector space L_φ;*

(ii). *The mapping $\varphi : L \to L_\varphi$ is a linear operator.*

Proof. By definition, put

$$x \overset{\varphi}{\oplus} y = \varphi[\varphi^{-1}(x) + \varphi^{-1}(y)], \quad \alpha \overset{\varphi}{\times} x = \varphi[\alpha \cdot \varphi^{-1}(x)]. \tag{3.1}$$

The set L is new vector space with operations (3.1). Moreover:

$$\varphi(x + y) = \varphi\left[\varphi^{-1}\big(\varphi(x)\big) + \varphi^{-1}\big(\varphi(y)\big)\right] = \varphi(x) \overset{\varphi}{\oplus} \varphi(y); \quad \varphi(\alpha \cdot x) = \varphi\left[\alpha \cdot \varphi^{-1}\big(\varphi(x)\big)\right] = \alpha \overset{\varphi}{\times} \varphi(x). \blacksquare.$$

Let G be a group of nonlinear transformations of the vector space L.

The *vector multispace of pair* $\{L,G\}$ is the space L with the set operations (3.1), where φ runs over all group G.

Remark. Each mapping $\varphi \in G$ is automorphism of vector multispace of pair $\{L,G\}$:

$$\varphi(x \overset{\psi}{\oplus} y) = \varphi \circ \psi \left[\psi^{-1} \circ \varphi^{-1}(\varphi(x)) + \psi^{-1} \circ \varphi^{-1}(\varphi(y)) \right] = \varphi(x) \overset{\varphi \circ \psi}{\oplus} \varphi(y) ; \tag{3.2}$$

$$\varphi(\alpha \overset{\psi}{\times} x) = \varphi \circ \psi \left[\alpha \cdot \psi^{-1} \circ \varphi^{-1}(\varphi(x)) \right] = \alpha \overset{\varphi \circ \psi}{\times} \varphi(x) . \blacksquare . \tag{3.3}$$

The set $\mathbb{H}_n$ is n^2- dimensional vector space over $\mathbb{R}$ with respect to the ordinary addition of matrices and multiplication matrix by a number. Let $G = G_\delta(n)$ be a group of δ-congruences.

3.2 Definition. The δ- *multispace* $\mathbb{H}_\delta(n)$ is the vector multispace of pair $\{\mathbb{H}_n, G_\delta(n)\}$. $\square$.

Now proceed on to research the operations of multispace $\mathbb{H}_\delta(n)$.

Symbols $\alpha \overset{Q}{\times} X$, $X \overset{Q}{\oplus} Y$ by denote the operations (3.1) corresponding to the mapping $\varphi_Q \in G_\delta(n)$. From the homogeneity of transformations φ_Q (see **2.7** (iv)) we obtain

3.3 Proposition. *There exists a unique multiplication operation by a number in $\mathbb{H}_\delta(n)$.*

Proof. By definition $\alpha \overset{Q}{\times} X = \varphi_Q[\alpha \cdot \varphi_Q^{-1}(X)] = \alpha \cdot \varphi_Q[\varphi_Q^{-1}(X)] = \alpha \cdot X . \blacksquare$.

3.4 Proposition. *The addition's operations of multispace $\mathbb{H}_\delta(n)$ satisfy the following conditions:*

(i). *If $Q,R \in GL(n,\mathbb{C})$, then $\varphi_R(X \overset{Q}{\oplus} Y) = \varphi_R(X) \overset{R \cdot Q}{\oplus} \varphi_R(Y)$ for all $X,Y \in \mathbb{H}_\delta(n)$;*

(ii). *If $W \in U(n)$, then $X \overset{W}{\oplus} Y = X + Y$ for all $X,Y \in \mathbb{H}_\delta(n)$;*

(iii). *If $Y = \beta \cdot X$, then $X \overset{Q}{\oplus} Y = X + Y$ for all $Q \in GL(n,\mathbb{C})$;*

(iv). *If $Q \in GL(n,\mathbb{C})$ and $0 \neq \varsigma \in \mathbb{C}$, then $X \overset{\varsigma \cdot Q}{\oplus} Y = X \overset{Q}{\oplus} Y$ for all $X,Y \in \mathbb{H}_\delta(n)$;*

(v). *If $X,Y \geq 0$, then $X \overset{Q}{\oplus} Y \geq 0$ for all $Q \in GL(n,\mathbb{C})$.*

Proof (i) – (v):

(i) follows from (3.2);

(ii). By Proposition 2.8 (i) $X \overset{W}{\oplus} Y = \varphi_W[\varphi_W^{-1}(X) + \varphi_W^{-1}(Y)] = W[W^* X W + W^* Y W] W^* = X + Y$;

(iii). We have $X \overset{Q}{\oplus} Y = X \overset{Q}{\oplus} (\beta \cdot X) = (1 + \beta) \cdot X = X + \beta \cdot X = X + Y$;

(iv). Using Propositions 2.7 (iv) and 2.8 (iii), we get

$$X \overset{\varsigma \cdot Q}{\oplus} Y = \varphi_{\varsigma \cdot Q}[\varphi_{\varsigma \cdot Q}^{-1}(X) + \varphi_{\varsigma \cdot Q}^{-1}(Y)] = |\varsigma|^{2\delta} \cdot \varphi_Q \left[|\varsigma|^{-2\delta} \cdot \varphi_Q^{-1}(X) + |\varsigma|^{-2\delta} \cdot \varphi_Q^{-1}(Y) \right] = X \overset{Q}{\oplus} Y ;$$

(v). Since the mappings φ_Q and φ_Q^{-1} preserve inertia, then $X \overset{Q}{\oplus} Y = \varphi_Q[\varphi_Q^{-1}(X) + \varphi_Q^{-1}(Y)] \geq 0 . \blacksquare$.

From Proposition 3.4 (iv) we obtain

3.5 Corollary. *Let $SG_\delta(n)$ is a group of unimodular δ- congruences (Definition 2.5). The multispaces of pairs $\{\mathbb{H}_n, SG_\delta(n)\}$ and $\{\mathbb{H}_n, G_\delta(n)\}$ are the same.* $\blacksquare$.

Let $Q = |Q|_L \cdot W$ is polar value decomposition for $Q \in GL(n,\mathbb{C})$, where $|Q|_L = (Q \cdot Q^*)^{1/2} > 0$ uniquely determined and $W \in U(n)$.

3.6 Proposition. *Let* $\delta > 0$. *If* $Q \in GL(n, \mathbb{C})$, *then* $X \overset{Q}{\oplus} Y = X \overset{|Q|_L}{\oplus} Y$ *for all* $X, Y \in \mathbb{H}_\delta(n)$.

Proof. If $Q = |Q|_L \cdot W$, then

$$X \overset{Q}{\oplus} Y = \varphi_Q \left[\varphi_Q^{-1}(X) + \varphi_Q^{-1}(Y) \right] = \varphi_{|Q|_L} \circ \varphi_W \left[\varphi_W^{-1}(\varphi_{|Q|_L}^{-1}(X)) + \varphi_W^{-1}(\varphi_{|Q|_L}^{-1}(Y)) \right] =$$

$$= \varphi_{|Q|_L} W \left[W^* \varphi_{|Q|_L}^{-1}(X) W + W^* \varphi_{|Q|_L}^{-1}(Y) W \right] W^* = X \overset{|Q|_L}{\oplus} Y . \blacksquare .$$

3.7 Corollary. *Let* $\delta \neq 1$ *and* $R, Q \in SL(n, \mathbb{C})$. *The addition's operations* $X \overset{R}{\oplus} Y$, $X \overset{Q}{\oplus} Y$ *are the same if and only if* $|R|_L = |Q|_L$.

Proof. Supposing $X \overset{R}{\oplus} Y = X \overset{Q}{\oplus} Y$ for all $X, Y \in \mathbb{H}_n$, we obtain

$$\varphi_Q^{-1} \circ \varphi_R [\varphi_R^{-1}(X) + \varphi_R^{-1}(Y)] = \varphi_Q^{-1}(X) + \varphi_Q^{-1}(Y) = \varphi_Q^{-1} \circ \varphi_R [\varphi_R^{-1}(X)] + \varphi_Q^{-1} \circ \varphi_R [\varphi_R^{-1}(Y)],$$

i.e. $\varphi_Q^{-1} \circ \varphi_R = \varphi_{Q^{-1} \cdot R}$ is linearly mapping. By Proposition 2.8 (iv) $Q^{-1}R = W \in U(n)$, i.e. $R = Q \cdot W$. Then $|R|_L = (Q \cdot W \cdot W^* \cdot Q^*)^{1/2} = (Q \cdot Q^*)^{1/2} = |Q|_L . \blacksquare .$

Proposition 3.4(iv), Corollary 3.7 and Remark 2.6 mean that we have proved

3.8 Theorem. *If* $\delta > 0$, $\delta \neq 1$, *then exist is a one-to-one correspondence between the set of addition's operations of multispace* $\mathbb{H}_\delta(n)$ *and the set* $S\mathbb{P}_n^+$ *of positive definite matrices with a unit determinant. If* $\delta = 1$, *then multispace* $\mathbb{H}_1(n)$ *coincides with the vector space* $\mathbb{H}_n . \blacksquare .$

3.9 Remark. A group $SG_\delta(n)$ of unimodular congruences is the smallest subgroup in $G_\delta(n)$ that defines the same set of linear operation on $\mathbb{H}_\delta(n)$ that all group $G_\delta(n)$.

4 Partially Ordered Vector Multispace

4.1. A subset C of a vector space L is *convex cone* if $\alpha x + \beta y$ belongs to C for any positive scalars α, β, and any x, y in L. A cone is called *closed* if it contains all its limit points. A cone is called *pointed* if $C \cap (-C) = \{\theta\}$.

Each closed pointed convex cone C induces (see [14]) a *partial ordering* $\leq$ on L so that $x \leq y$ if and only if $y - x \in C$. This partial ordering compatible with the vector space structure, i.e. for all x, y, z in L and $0 < \lambda$ in $\mathbb{R}$ the following two axioms are satisfied

$$x \leq y \text{ implies } x + z \leq y + z ; \qquad x \leq y \text{ implies } \lambda x \leq \lambda y .$$

A vector space with such an order is called an *ordered vector space.*

There exists a one-to-one correspondence between the partial orders on a vector space L that are compatible with the vector space structure and the closed pointed convex cones of L.

4.2. Let's list the requirements for the concept of partial order on vector multispace $\mathbb{H}_\delta(n)$:

(i). A relation of partial order must be compatible with all operations of multispace $\mathbb{H}_\delta(n)$;

(ii). A relation of partial order must be preserving when transformations of group $G_\delta(n)$.

4.3 Proposition. *The set* $\mathbb{P}_n$ *of all positive semidefinite Hermitian* $n \times n$ *matrices is the unique* (*up to multiplication by* (-1)) *nontrivial closed pointed convex* (*with respect to each addition*) *cone in* $\mathbb{H}_\delta(n)$ *that is invariant under all transformations of group* $G_\delta(n)$.

Proof. The set of all matrices of the same inertia is the smallest (with respect to the set-theoretic inclusion) subset in $\mathbb{H}_\delta(n)$ that invariant under the all transformations of the group $G_\delta(n)$. Among a nontrivial invariant subsets in $\mathbb{H}_\delta(n)$ (the matrices with the same nonzero inertia) only the set $\mathbb{P}_n^+$ of positive definite and the set $-\mathbb{P}_n^+$ of negatively definite matrices are convex (with respect to the each addition) cones. Hence closure $\mathbb{P}_n$ of the set $\mathbb{P}_n^+$ is the desired closed pointed convex cone in $\mathbb{H}_\delta(n)$. $\blacksquare$.

Given the requirements of **4.2** (i) we introduce the following definition.

4.4 Definition. The vector multispace $\mathbb{H}_\delta(n)$ is called the *ordered vector multispace*, if defined partial order $\leq_\delta$ on $\mathbb{H}_\delta(n)$ such that

$$X \leq_\delta Y \qquad \text{if and only if} \qquad Y \overset{Q}{\ominus} X \in \mathbb{P}_n \qquad \text{for all} \quad Q \in GL(n,\mathbb{C}). \qquad (4.1)$$

4.5 Remark. Considering Corollary 3.8, we can replace the group $GL(n,\mathbb{C})$ by the unimodular group $SL(n,\mathbb{C})$ in condition (4.1).

There are several conditions that characterize partial ordering $\leq_\delta$ on the set of positive definite matrices $\mathbb{P}_n^+$ at different values $\delta > 0$.

4.6 Theorem. *Let* $X, Y \in \mathbb{P}_n^+$. *If* $0 < \delta \leq 1$, *then the following conditions are equivalent:*

(i). $X \leq_\delta Y$;

(ii). $Y \overset{\tau_{2\delta}^{-1}(X)}{\ominus} X \geq 0$;

(iii). $Y^{1/\delta} \geq X^{1/\delta}$.

Proof. (i) $\Rightarrow$ (ii). Let $X \leq_\delta Y$, i.e. $\forall Q \in GL(n,\mathbb{C}): Y \overset{Q}{\ominus} X \geq 0$. Then $Y \overset{\tau_{2\delta}^{-1}(X)}{\ominus} X \geq 0$.

(ii) $\Rightarrow$ (iii). If $Y \overset{\tau_{2\delta}^{-1}(X)}{\ominus} X \geq 0$, then

$$\varphi_{X^{1/2\delta}}[\varphi_{X^{1/2\delta}}^{-1}(Y) - \varphi_{X^{1/2\delta}}^{-1}(X)] \geq 0 \;\Rightarrow\; \varphi_{X^{1/2\delta}}[(X^{-1/2\delta}Y^{1/\delta}X^{-1/2\delta})^\delta - I_n] \geq 0 \;\Rightarrow$$

$$\Rightarrow\; (X^{-1/2\delta}Y^{1/\delta}X^{-1/2\delta})^\delta - I_n \geq 0 \;\Rightarrow\; X^{-1/2\delta}Y^{1/\delta}X^{-1/2\delta} \geq I_n \;\Rightarrow\; Y^{1/\delta} \geq X^{1/\delta}.$$

(iii) $\Rightarrow$ (i). If $Y^{1/\delta} \geq X^{1/\delta}$, then $Q^{-1}Y^{1/\delta}Q^{-1*} \geq Q^{-1}X^{1/\delta}Q^{-1*}$ for all $Q \in GL(n,\mathbb{C})$. Since $0 < \delta \leq 1$, then

$$f(X) = X^\delta \text{ is monotone on } \mathbb{P}_n^+. \text{ Hence}$$

$$(Q^{-1}Y^{1/\delta}Q^{-1*})^\delta \geq (Q^{-1}X^{1/\delta}Q^{-1*})^\delta \;\Rightarrow\; \varphi_Q[\varphi_Q^{-1}(Y) - \varphi_Q^{-1}(X)] \geq 0 \;\Rightarrow\; X \leq_\delta Y. \blacksquare.$$

We need the following Lemma to characterize the partial order at $\delta > 1$.

4.7 Lemma. *Let* $\delta > 0$ *and* $X, Y \in \mathbb{P}_n^+$. *There exists a transformation* $\varphi_{Q_0} \in SG_\delta(n)$ *such that*

$$\varphi_{Q_0}(X) = \lambda_0 \cdot I_n, \quad \varphi_{Q_0}(Y) = \Lambda_\mu = \text{diag}\{\mu_1, \cdots, \mu_n\},$$

where $\lambda_0 = \sqrt[n]{\det X}$ *and* $\det Y = \det \Lambda_\mu$.

Proof. Since $In X = In(\lambda_0 I_n)$ and $\det X = \det(\lambda_0 I_n)$, then (Proposition 2.8 (v)) there is exist $R \in SL(n,\mathbb{C})$ such that

$$\varphi_R(X) = \lambda_0 \cdot I_n, \quad \varphi_R(Y) = Y' > 0. \qquad (4.3)$$

Let $Q_0 = W \cdot R$, where $W \cdot Y' \cdot W^* = \Lambda_\mu = \text{diag}\{\mu_1, \cdots, \mu_n\}$, and $W \in SU(n)$. Then

$$\varphi_{Q_0}(X) = \varphi_{W \cdot R}(X) = W \cdot (\lambda_0 \cdot I_n) \cdot W^* = \lambda_0 \cdot I_n, \quad \varphi_{Q_0}(Y) = \varphi_{W \cdot R}(Y) = W \cdot Y' \cdot W^* = \Lambda_\mu . \blacksquare .$$

4.8 Theorem. *Let* $\delta > 1$. *If* $X, Y \in \mathbb{P}_n^+$, *then*

$$X \leq_\delta Y \qquad \text{if and only if} \quad Y = \beta \cdot X, \quad \beta \geq 1. \tag{4.4}$$

Proof. Let $Q = Q_0^{-1} \cdot U_n \cdot \Lambda_n$, where Q_0 satisfies conditions of Lemma 4.7 and

$$U_n = \begin{pmatrix} U_2 & 0 \\ 0 & I_{n-2} \end{pmatrix}, \ U_2 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}; \quad \Lambda_n = \begin{pmatrix} \Lambda_2 & 0 \\ 0 & I_{n-2} \end{pmatrix}, \ \Lambda_2 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} > 0 .$$

Since $Q = Q_0^{-1} \cdot U_n \cdot \Lambda_n \in SL(n, \mathbb{C})$, then

$$\det(Y \overset{Q}{\ominus} X) = \det \varphi_{Q_0^{-1} \cdot U_n \cdot \Lambda_n} \left[\varphi_{U_n \cdot \Lambda_n}^{-1}\left(\varphi_{Q_0}(Y) \right) - \varphi_{U_n \cdot \Lambda_n}^{-1}\left(\varphi_{Q_0}(X) \right) \right] =$$

$$= \det \left[\varphi_{\Lambda_n}^{-1}(U_n^* \cdot \Lambda_\mu^{(n)} \cdot U_n) - \lambda_0 \cdot \varphi_{\Lambda_n}^{-1}(I_n) \right] = (\mu_3 - \lambda_0) \cdots (\mu_n - \lambda_0) \cdot \det \left[\varphi_{\Lambda_2}^{-1}(U_2^* \cdot \Lambda_\mu^{(2)} \cdot U_2) - \lambda_0 \cdot \varphi_{\Lambda_2}^{-1}(I_2) \right],$$

where $\Lambda_\mu^{(n)} = \varphi_{Q_0}(Y) = \mathrm{diag}\{\mu_1, \cdots, \mu_n\}$, $\Lambda_\mu^{(2)} = \mathrm{diag}\{\mu_1, \mu_2\}$, $\lambda_0 \cdot I_n = \varphi_{Q_0}(X)$.

By condition $Y \overset{Q}{\ominus} X \geq 0$ for all $Q \in SL(n, \mathbb{C})$. Then $\mu_k \geq \lambda_0$, $k = 1, \cdots, n$. We to prove that from conditions $X \leq_\delta Y$ and $\delta > 1$, we obtaining $\mu_2 = \mu_1$. We have

$$g(\alpha) = \det \left[\varphi_{\Lambda_2}^{-1}(U_2^* \cdot \Lambda_\mu^{(2)} \cdot U_2) - \lambda_0 \cdot \varphi_{\Lambda_2}^{-1}(I_2) \right] =$$

$$= \det \left[2^{-\delta} \begin{pmatrix} \alpha^{-2}(\mu_1^{1/\delta} + \mu_2^{1/\delta}) & (\mu_1^{1/\delta} - \mu_2^{1/\delta}) \\ (\mu_1^{1/\delta} - \mu_2^{1/\delta}) & \alpha^2(\mu_1^{1/\delta} + \mu_2^{1/\delta}) \end{pmatrix}^\delta - \lambda_0 \cdot \begin{pmatrix} \alpha^{-2\delta} & 0 \\ 0 & \alpha^{2\delta} \end{pmatrix} \right].$$

The analysis shows that

$$g(\alpha) \sim -\frac{\lambda_0}{2^\delta} \alpha^{-4(\delta-1)}(\mu_1^{1/\delta} + \mu_2^{1/\delta})^{\delta-2}(\mu_1^{1/\delta} - \mu_2^{1/\delta})^2 \qquad \text{at} \quad \alpha \to 0+ .$$

If $\mu_1^{1/\delta} - \mu_2^{1/\delta} \neq 0$ and $\delta > 1$, then $\lim_{\alpha \to 0+} g(\alpha) = -\infty$, which contradicts the condition $Y \overset{Q}{\ominus} X \geq 0$. Then $\mu_2 = \mu_1$. Similarly we get $\mu_k = \mu_1$ for $k = 3, \cdots, n$. Hence $\Lambda_\mu^{(n)} = \mu_1 \cdot I_n$ and

$$Y = \varphi_{Q_0}^{-1}(\Lambda_\mu^{(n)}) = \varphi_{Q_0}^{-1}(\mu_1 \cdot I_n) = \beta \cdot \varphi_{Q_0}^{-1}(\lambda_0 \cdot I_n) = \beta \cdot X, \quad \beta = \mu_1 / \lambda_0 \geq 1 .$$

Sufficiency is obvious. $\blacksquare$.

The partial order relation $X \leq_\delta Y$ satisfies the condition **4.2**(ii).

4.9 Proposition. *Let* $X, Y > 0$, $\varphi_Q \in G_\delta(n)$. *Then* $X \leq_\delta Y$ *in* $\mathbb{H}_\delta(n)$ *if and only if* $\varphi_Q(X) \leq_\delta \varphi_Q(Y)$.

Proof. If $0 < \delta \leq 1$, then

$$X \leq_\delta Y \iff X^{1/\delta} \leq Y^{1/\delta} \iff Q \cdot X^{1/\delta} \cdot Q^* \leq Q \cdot Y^{1/\delta} \cdot Q^* \iff [\varphi_Q(X)]^{1/\delta} \leq [\varphi_Q(Y)]^{1/\delta} \iff \varphi_Q(X) \leq_\delta \varphi_Q(Y) .$$

If $\delta > 1$, then $X \leq_\delta Y \iff Y = \beta \cdot X, \beta \geq 1 \iff \varphi_Q(Y) = \beta \cdot \varphi_Q(X) \iff \varphi_Q(Y) \leq_\delta \varphi_Q(X) . \blacksquare$.

In conclusion, we present some inequalities for the Q - sum of positive definite matrices. We will need the following F. Hansen's result.

Theorem (see [5]). *Let* Y *and* Q *be bounded linear operators on Hilbert space* H. *We suppose that* $Y \geq 0$ *and* $\|Q\| \leq 1$. *If* f *is an operator monotone function defined on* $[0, \infty)$, *then*

$$Q \cdot f(Y) \cdot Q^* \leq f(Q \cdot Y \cdot Q^*) . \blacksquare \tag{4.5}$$

4.10 Proposition. *Let* $X \geq 0$ *and* $QQ^* \leq I_n$. *If* $0 < \delta_1 \leq \delta_2 \leq 1$, *then*

$$(Q \cdot X^{1/\delta_2} \cdot Q^*)^{\delta_2} \leq (Q \cdot X^{1/\delta_1} \cdot Q^*)^{\delta_1}. \tag{4.6}$$

Proof. If $X \geq 0$, then $Y = X^{1/\delta_1} \geq 0$. The function $f(x) = x^{\delta_1/\delta_2}$ is an operator monotone function on $[0, \infty)$. By inequality (4.5) we obtain

$$Q \cdot (X^{1/\delta_1})^{\delta_1/\delta_2} \cdot Q^* \leq (Q \cdot X^{1/\delta_1} \cdot Q^*)^{\delta_1/\delta_2} \;\Rightarrow\; Q \cdot X^{1/\delta_2} \cdot Q^* \leq (Q \cdot X^{1/\delta_1} \cdot Q^*)^{\delta_1/\delta_2}.$$

Since $f(x) = x^{\delta_2}$ is an operator monotone function on $[0, \infty)$, then (4.6) is true. $\blacksquare$.

4.11 Corollary. *Suppose* $X \geq 0$ *and* $\alpha_1 \geq \cdots \geq \alpha_n > 0$ *are singular numbers of the matrix* $Q \in GL(n, \mathbb{C})$. *If* $0 < \delta_1 \leq \delta_2 \leq 1$, *then*

$$(Q \cdot X^{1/\delta_2} \cdot Q^*)^{\delta_2} \leq \alpha_1^{2(\delta_2 - \delta_1)} \cdot (Q \cdot X^{1/\delta_1} \cdot Q^*)^{\delta_1}; \quad (Q^{-1} \cdot X^{1/\delta_2} \cdot Q^{-1*})^{\delta_2} \leq \alpha_n^{-2(\delta_2 - \delta_1)} \cdot (Q^{-1} \cdot X^{1/\delta_1} \cdot Q^{-1*})^{\delta_1}.$$

Proof. If $Q \in GL(n, \mathbb{C})$, then $Q_1 = \alpha_1^{-1} Q$ satisfy the condition $Q_1 Q_1^* \leq I_n$. By Proposition 4.10 we have

$$(Q_1 \cdot X^{1/\delta_2} \cdot Q_1^*)^{\delta_2} \leq (Q_1 \cdot X^{1/\delta_1} \cdot Q_1^*)^{\delta_1} \;\Rightarrow\; \alpha_1^{-2\delta_2} \cdot (Q \cdot X^{1/\delta_2} \cdot Q^*)^{\delta_2} \leq \alpha_1^{-2\delta_1} \cdot (Q \cdot X^{1/\delta_1} \cdot Q^*)^{\delta_1}.$$

Similarly for the matrix $Q_2 = \alpha_n \cdot Q^{-1}$, $Q_2 Q_2^* \leq I_n$. $\blacksquare$.

Let $0 < \delta \leq 1$ and $X, Y \geq 0$. We introduce the notations

$$(Q \cdot X^{1/\delta} \cdot Q^*)^{\delta} \overset{def}{=} \varphi_{\delta,Q}(X); \quad \varphi_{\delta,Q}\Big[\varphi_{\delta,Q}^{-1}(X) + \varphi_{\delta,Q}^{-1}(Y)\Big] \overset{def}{=} X \overset{Q}{\oplus}_{\delta} Y.$$

4.12 Theorem. *Suppose* $X, Y \geq 0$ *and the matrix* $Q \in GL(n, \mathbb{C})$ *has singular numbers* $\alpha_1 \geq \cdots \geq \alpha_n > 0$. *If* $0 < \delta_1 \leq \delta_2 \leq 1$, *then*

$$X \overset{Q}{\oplus}_{\delta_2} Y \leq \left(\alpha_1 / \alpha_n\right)^{2(\delta_2 - \delta_1)} \cdot (X \overset{Q}{\oplus}_{\delta_1} Y). \tag{4.7}$$

Proof. Using corollary 4.11 we obtain

$$X \overset{Q}{\oplus}_{\delta_2} Y = \varphi_{\delta_2,Q}\Big[\varphi_{\delta_2,Q}^{-1}(X) + \varphi_{\delta_2,Q}^{-1}(Y)\Big] \leq \alpha_1^{2(\delta_2-\delta_1)} \cdot \varphi_{\delta_1,Q}\Big[\varphi_{\delta_2,Q^{-1}}(X) + \varphi_{\delta_2,Q^{-1}}(Y)\Big] \leq$$

$$\leq \alpha_1^{2(\delta_2-\delta_1)} \cdot \varphi_{\delta_1,Q}\Big[\alpha_n^{-2(\delta_2-\delta_1)} \cdot \varphi_{\delta_1,Q^{-1}}(X) + \alpha_n^{-2(\delta_2-\delta_1)} \cdot \varphi_{\delta_1,Q^{-1}}(Y)\Big] =$$

$$= (\alpha_1 / \alpha_n)^{2(\delta_2-\delta_1)} \cdot \varphi_{\delta_1,Q}\Big[\varphi_{\delta_1,Q}^{-1}(X) + \varphi_{\delta_1,Q}^{-1}(Y)\Big] = (\alpha_1 / \alpha_n)^{2(\delta_2-\delta_1)} \cdot (X \overset{Q}{\oplus}_{\delta_1} Y). \blacksquare.$$

5 Chains in Ordered Vector Multispace $\mathbb{H}_{\delta}(n)$

A linearly ordered subset in a partially ordered set is called a *chain*. Let X_0 be fixed positive definite matrix of the partially ordered vector multispace $\mathbb{H}_{\delta}(n)$. Consider the set of all chains in $\mathbb{P}_n^+ \subseteq \mathbb{H}_{\delta}(n)$ passing through the point X_0.

5.1 Proposition. *If* $\delta > 1$, *then through each point* $X_0 \in \mathbb{P}_n^+$ *in* $\mathbb{H}_{\delta}(n)$ *passes a unique chain.*

Proof. By Theorem 4.8 we have $X_0 \leq_{\delta} X$ if and only if $X = \beta \cdot X_0$, $\beta \geq 1$. Therefore, there is a unique chain in $\mathbb{P}_n^+$ passing through the point X_0 is determined by the equation

$$X = s \cdot X_0, \; s > 0. \blacksquare. \tag{5.1}$$

5.2. Let $0 < \delta \leq 1$ and X_0, X_1 are fixed positive definite matrices satisfying the condition $X_0 \leq_{\delta} X_1$. By Definition 4.4 and Remark 4.5 $X \leq_{\delta} Y$ if and only if $Y \overset{Q}{\ominus} X \geq 0$ for all

$Q \in SL(n,\mathbb{C})$. Denote by $S_k(A)$ the elementary symmetric polynomial of degree $k = 1, \cdots, n$ from the eigenvalues of the matrix A. By the definition $Y \overset{Q}{\ominus} X \geq 0$ if and only if the eigenvalues of $Y \overset{Q}{\ominus} X$ are nonnegative numbers. It is easy to show that

$$X_0 \leq_\delta X_1 \iff S_k(X_1 \overset{Q}{\ominus} X_0) \geq 0, \ k = 1, \cdots, n \quad \text{for all} \quad Q \in SL(n,\mathbb{C}). \tag{5.2}$$

The map $X' = \varphi_R(X)$ of the group $SG_\delta(n)$ take each $S_k(X_1 \overset{Q}{\ominus} X_0)$ to $S_k[\varphi_R(X_1 \overset{Q}{\ominus} X_0)]$.

5.3 Proposition. *Let* $0 < \delta \leq 1$ *and* $X_0, X_1 \in \mathbb{H}_\delta(n)$. *The number* $S_n(X_1 \overset{Q}{\ominus} X_0)$ *is an invariant of the group* $SG_\delta(n)$. *For* $k = 1, \cdots, n-1$ *the number* $S_k(X_1 \overset{Q}{\ominus} X_0)$ *is an invariant of the group* $SG_\delta(n)$ *if and only if* $S_k(X_1 \overset{Q}{\ominus} X_0) = 0$.

Proof. The invariance of $S_n(X_1 \overset{Q}{\ominus} X_0) = \det(X_1 \overset{Q}{\ominus} X_0)$ is obvious (see **2.7**(iii)). Consider one of the numbers $S_k(X_1 \overset{Q}{\ominus} X_0)$, where $k = 1, \cdots, n-1$. If $S_k(X_1 \overset{Q}{\ominus} X_0) = 0$, then $rank(X_1 \overset{Q}{\ominus} X_0) < k$. The transformation φ_R of the group $SG_\delta(n)$ preserve the inertia. Then $rank \, \varphi_R(X_1 \overset{Q}{\ominus} X_0) < k$, i.e. $S_k[\varphi_R(X_1 \overset{Q}{\ominus} X_0)] = 0$. If $S_k(X_1 \overset{Q}{\ominus} X_0) \neq 0$, then transformations of the group $SG_\delta(n)$ change its value. Indeed, because the eigenvalues of $X_1 \overset{Q}{\ominus} X_0$ do not change during transformation $\varphi_W(X) = W \cdot X \cdot W^*$, $W \in SU(n)$, we can assume that

$$X_1 \overset{Q}{\ominus} X_0 = \Lambda_\mu = diag\{\mu_1, \cdots, \mu_n\}, \tag{5.3}$$

where $\mu_1 \cdots \mu_k \neq 0$. Let $R = diag\{1, \cdots, 1, \alpha_k, \alpha_{k+1}, 1, \cdots, 1,\}$, $\alpha_k \cdot \alpha_{k+1} = 1$. It is easy to see that the value of $S_k[\varphi_R(X_1 \overset{Q}{\ominus} X_0)]$ can be made arbitrarily large by selecting the appropriate number α_k. ∎.

5.4. Let $0 < \delta \leq 1$ and X_0, Y_0 be positive definite matrices in $\mathbb{H}_\delta(n)$. If $X_0 \leq_\delta Y_0$, then the points X_0, Y_0 can be connected by a set of chains. By Proposition 5.3 the number $s(Q) = S_n(X_1 \overset{Q}{\ominus} X_0) = \det(X_1 \overset{Q}{\ominus} X_0)$ is an invariant of group $SG_\delta(n)$ for each matrix $Q \in GL(n,\mathbb{C})$. Among all chains connecting points X_0, Y_0 we are interested in an extreme chain for which the function $s(Q) = \det(X_1 \overset{Q}{\ominus} X_0)$ takes an extreme value. We will need some determinant inequalities.

6 Determinant inequalities for positive definite matrices

In this section, we prove for vector multispace $\mathbb{H}_\delta(n)$ the Minkowski's determinant inequality and the following theorem.

6.1 Theorem. *Let* $0 < \delta \leq 1$ *and* X, Y *be positive defined* $n \times n$*- matrices. If* $X \leq_\delta Y$, *then*

$$\forall Q \in GL(n,\mathbb{C}): \ \det(Y \overset{Q}{\ominus} X) \geq \det(Y \overset{\tau_{2\delta}^{-1}(X)}{\ominus} X). \tag{6.1}$$

The inequality holds for any pair of positive definite matrices at $n = 2$.

6.2. Recall some definitions. For any $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, let $x_\downarrow = (x_{[1]}, x_{[2]}, \cdots, x_{[n]})$ denote the *decreasing rearrangement* of x, i. e., $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$. The elementwize vector ordering $x_j \leq y_j$, $j = 1, \cdots, n$, is denoted by $x \leq y$. By definition, put $\log x = (\log x_1, \cdots, \log x_n)$.

For $x, y \in \mathbb{R}^n$ y is said to be *majorizes* x (see [10], [12]) if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \quad 1 \leq k \leq n-1 \quad \text{and} \quad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}, \tag{6.2}$$

this relation is denoted by $x \prec y$.

Let $X \in \mathbb{P}_n^+$. The eigenvalues of X are denoted by $\lambda(X) = (\lambda_1, \lambda_2, \cdots, \lambda_n)_{\downarrow}$, i.e. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$.

Let $\mathbb{D}_+^n(1) = \{x \in \mathbb{R}^n : x_1 \geq 1, \cdots, x_n \geq 1\}$. For a set $\mathbb{A} \subseteq \mathbb{R}^n$

$$x \prec y \quad \text{on} \quad \mathbb{A}$$

means $x, y \in \mathbb{A}$ and $x \prec y$.

Proposition 6.3 is an analogue of Schur's result (see [10], p. 78) for log- majorization.

6.3 Proposition. *If* $\log(x_1, \cdots, x_n) \prec \log(y_1, \cdots, y_n)$ *on* $\mathbb{D}_+^n(1)$, *then*

$$(x_1 - 1) \cdots (x_n - 1) \geq (y_1 - 1) \cdots (y_n - 1). \tag{6.3}$$

Proof. Let $\tilde{z}_k = \prod_{j=1}^{k} z_j$, $k = 1, \cdots, n$. Then $\log x \prec \log y$ on $\mathbb{D}_+^n(1)$ if and only if

$$x_1 \geq \cdots \geq x_n \geq 1, \quad y_1 \geq \cdots \geq y_n \geq 1, \quad \tilde{x}_k \leq \tilde{y}_k, \quad k = 1, \cdots, n-1 \quad \text{and} \quad \tilde{x}_n = \tilde{y}_n.$$

Let $\hat{y}$ be a constant vector in $\mathbb{D}_+^n(1)$. On the set $\mathbb{D} = \{x \in \mathbb{D}_+^n(1) : \log x \prec \log \hat{y}\}$ consider the function

$$\phi(x_1, \cdots, x_n) = (x_1 - 1) \cdots (x_n - 1) = (\tilde{x}_1 - 1)(\tilde{x}_2 / \tilde{x}_1 - 1) \cdots (\tilde{x}_{n-1} / \tilde{x}_{n-2} - 1)(\tilde{x}_n / \tilde{x}_{n-1} - 1).$$

Since $\tilde{y}_n$ is constant and $\tilde{x}_n = \tilde{y}_n$, then $\partial \varphi / \partial \tilde{x}_n \equiv 0$. Therefore $\log x \prec \log y$ on $\mathbb{D}$ implies $\phi(x) \geq \phi(y)$ if and only if

$$\phi(x_1, \cdots, x_n) = (\tilde{x}_1 - 1)(\tilde{x}_2 / \tilde{x}_1 - 1) \cdots (\tilde{x}_{n-1} / \tilde{x}_{n-2} - 1)(\tilde{x}_n / \tilde{x}_{n-1} - 1)$$

is decreasing in $\tilde{x}_j$, $j = 1, \cdots, n-1$, over the region where $x \in \mathbb{D}$. Since

$$\frac{\partial \varphi}{\partial \tilde{x}_k} = -\frac{x_k - x_{k+1}}{x_1 \cdots x_k} (x_1 - 1) \cdots (x_{k-1} - 1)(x_{k+2} - 1) \cdots (x_n - 1) \leq 0 \quad \text{on} \quad \mathbb{D}_+^n(1),$$

then (6.3) is true. ∎.

The following two well-known theorems generalize the inequality Lieb and Tearing ([6], [7]).

6.4 Theorem (see [8], Theorem 6.ii). *Let* X, Y *are positive defined Hermitian* $n \times n$ *matrices. If* $0 < \alpha < \beta$, *then*

$$\log \lambda^{1/\alpha}(X^\alpha Y^\alpha) \prec \log \lambda^{1/\beta}(X^\beta Y^\beta). \quad \blacksquare. \tag{6.4}$$

6.5 Theorem (see [9], Theorem 2.1). *Let* A, B *are positive definite* $n \times n$ *matrices. If* $0 < \delta \leq 1$, *then*

$$\operatorname{tr}(A^\delta \cdot B^\delta \cdot A^\delta) \leq \operatorname{tr}(A \cdot B \cdot A)^\delta, \tag{6.5}$$

where $\operatorname{tr} X$ *stands for matrix trace.* ∎.

To prove Theorem 6.1, we need several lemmas.

6.6 Lemma. *If* A, B *is* $n \times n$ *positive defined Hermitian matrices and* $0 < \alpha \leq 1$, *then*

$$\log \lambda(A^\alpha B^\alpha A^\alpha) \prec \log \lambda(ABA)^\alpha. \tag{6.6}$$

Proof. If $A > 0$, then $\lambda(AB) = \lambda(A^{1/2} B A^{1/2})$. Taking into account Theorem 6.4, we obtain

$$\log \lambda^{1/\alpha}(A^{\alpha/2} B^\alpha A^{\alpha/2}) \prec \log \lambda^{1/\beta}(A^{\beta/2} B^\beta A^{\beta/2}). \tag{6.7}$$

Substituting A for $A^{1/2}$ in (6.7) and let $0 < \alpha < \beta = 1$. Then

$$\log \lambda^{1/\alpha}(A^\alpha B^\alpha A^\alpha) \prec \log \lambda(ABA) \iff \log \lambda(A^\alpha B^\alpha A^\alpha) \prec \log \lambda(ABA)^\alpha. \blacksquare.$$

6.7 Lemma. *The inequality* (6.1) *is equivalent to inequality*

$$\forall A > 0: \det[A^\delta(A^{-1}\hat{Y}^{1/\delta}A^{-1})^\delta A^\delta - I_n] \geq \det[\hat{Y} - I_n], \tag{6.8}$$

where $\hat{Y} = (X^{-1/2\delta}Y^{1/\delta}X^{-1/2\delta})^\delta$.

Proof. Let $Q_0 = X^{1/2\delta}$. We can write a matrix $Q \in GL(n,\mathbb{C})$ in the form $Q = Q_0 \cdot A \cdot U$, where $A > 0$, $U \in U(n)$. Let $\hat{Y} = \varphi_{Q_0}^{-1}(Y) = (X^{-1/2\delta}Y^{1/\delta}X^{-1/2\delta})^\delta$. Then

$$\det(Y \overset{Q}{\ominus} X) = \det \varphi_{Q_0 AU}[\varphi_{Q_0 AU}^{-1}(Y) - \varphi_{Q_0 AU}^{-1}(X)] = \left|\det Q_0\right|^{2\delta} \det[A^\delta(A^{-1}\hat{Y}^{1/\delta}A^{-1})^\delta A^\delta - I_n].$$

Similarly, for the matrix $Y \overset{\tau_{2\delta}^{-1}(X)}{\ominus} X$ we obtain

$$\det(Y \overset{\tau_{2\delta}^{-1}(X)}{\ominus} X) = \det(Y \overset{Q_0}{\ominus} X) = \left|\det Q_0\right|^{2\delta} \det[\varphi_{Q_0}^{-1}(Y) - \varphi_{Q_0}^{-1}(X)] = \left|\det Q_0\right|^{2\delta} \det[\hat{Y} - I_n].$$

Thus, inequality (6.1) is equivalent to inequality (6.8). $\blacksquare$.

6.8 Lemma. *Let* $0 < \delta \leq 1$, $A > 0$. *If* $Y \geq I_n$, *then* $A^\delta(A^{-1}Y^{1/\delta}A^{-1})^\delta A^\delta \geq I_n$.

Proof. If $Y \geq I_n$, then $Y^{1/\delta} \geq I_n$. We obtain $A^{-1}Y^{1/\delta}A^{-1} \geq A^{-2}$. Since $0 < \delta \leq 1$, then $f(X) = X^\delta$ is monotone on $\mathbb{P}_n$. Then $(A^{-1}\tilde{Y}^{1/\delta}A^{-1})^\delta \geq A^{-2\delta}$, i.e. $A^\delta(A^{-1}\hat{Y}^{1/\delta}A^{-1})^\delta A^\delta \geq I_n$. $\blacksquare$.

6.9 Proof of Theorem 6.1. Let $X, Y \in \mathbb{P}_n^+$ and $X \leq_\delta Y$. Taking into account Theorem 4.6, we obtain $X^{1/\delta} \leq Y^{1/\delta}$, i.e. $\hat{Y} = (X^{-1/2\delta}Y^{1/\delta}X^{-1/2\delta})^\delta \geq I_n$. Using Lemma 6.8, we get $A^\delta(A^{-1}\hat{Y}^{1/\delta}A^{-1})^\delta A^\delta \geq I_n$. The eigenvalues of $C = A^\delta(A^{-1}\hat{Y}^{1/\delta}A^{-1})^\delta A^\delta$ and $\hat{Y}$ are denoted by $\lambda(C) = (\lambda_1, \lambda_2, \cdots, \lambda_n)_\downarrow \in \mathbb{D}_+^n(1)$ and $\mu(\hat{Y}) = (\mu_1, \mu_2, \cdots, \mu_n)_\downarrow \in \mathbb{D}_+^n(1)$. Let $B = A^{-1}\hat{Y}^{1/\delta}A^{-1}$. By Lemma 6.6, $\log \lambda(A^\delta B^\delta A^\delta) \prec \log \lambda(ABA)^\delta$ so that

$$\log \lambda[A^\delta(A^{-1}\hat{Y}^{1/\delta}A^{-1})^\delta A^\delta] \prec \log \lambda[A(A^{-1}\hat{Y}^{1/\delta}A^{-1})A]^\delta = \log \mu(\hat{Y}).$$

Using Proposition 6.3, we get

$$\det[A^\delta(A^{-1}\hat{Y}^{1/\delta}A^{-1})^\delta A^\delta - I_n] = (\lambda_1 - 1)\cdots(\lambda_n - 1) \geq (\mu_1 - 1)\cdots(\mu_n - 1) = \det[\hat{Y} - I_n],$$

which, according to Lemma 6.7, is equivalent to inequality (6.1).

The condition $X \leq_\delta Y$ can be omitted for 2×2 matrices. Indeed, let $B = A^{-1}\hat{Y}^{1/\delta}A^{-1}$, $\hat{Y} = (X^{-1/2\delta}Y^{1/\delta}X^{-1/2\delta})^\delta$ are 2×2 matrices. The eigenvalues of $C = A^\delta(A^{-1}\hat{Y}^{1/\delta}A^{-1})^\delta A^\delta$ and $\hat{Y}$ are denoted by $\lambda(C) = (\lambda_1, \lambda_2)$ and $\mu(\hat{Y}) = (\mu_1, \mu_2)$. Then

$$\lambda_1 \lambda_2 = \det[A^\delta(A^{-1}\hat{Y}^{1/\delta}A^{-1})^\delta A^\delta] = \det \hat{Y} = \mu_1 \mu_2. \tag{6.9}$$

Using Theorem 6.5, we get

$$\lambda_1 + \lambda_2 = \text{tr}(A^\delta \cdot (A^{-1}\hat{Y}^{1/\delta}A^{-1})^\delta \cdot A^\delta) \leq \text{tr}(A \cdot A^{-1}\hat{Y}^{1/\delta}A^{-1} \cdot A)^\delta = \text{tr}\,\hat{Y} = \mu_1 + \mu_2. \tag{6.10}$$

Combining (6.9) and (6.10), we obtain

$$\det[A^\delta(A^{-1}\hat{Y}^{1/\delta}A^{-1})^\delta A^\delta - I_2] = \lambda_1 \lambda_2 - (\lambda_1 + \lambda_2) + 1 \geq \mu_1 \mu_2 - (\mu_1 + \mu_2) + 1 = \det[\hat{Y} - I_2]. \blacksquare.$$

In conclusion, we prove for vector multispace $\mathbb{H}_\delta(n)$ the Minkowski's determinant inequality.

6.10 Theorem (Minkowski's determinant inequality). *Let* $\delta > 0$ *and* $X, Y \in \mathbb{H}_\delta(n)$ *are positive definite* $n \times n$*-matrices. Then*

$$[\det (Y \overset{Q}{\oplus} X)]^{1/n} \geq [\det Y]^{1/n} + [\det X]^{1/n}, \quad \forall Q \in GL(n, \mathbb{C}) \tag{6.11}$$

with equality if and only if $Y = cX$ *for some* $c > 0$.

Proof. By Corollary 3.7, we can assume that $Q \in SL(n, \mathbb{C})$. Then $\det \varphi_Q(X) = \det X$ and we obtains

$$[\det (Y \overset{Q}{\oplus} X)]^{1/n} = \left[\det \varphi_Q [\varphi_Q^{-1}(Y) + \varphi_Q^{-1}(X)] \right]^{1/n} = \left[\det [\varphi_Q^{-1}(Y) + \varphi_Q^{-1}(X)] \right]^{1/n} \geq$$

$$\geq [\det \varphi_Q^{-1}(Y)]^{1/n} + [\det \varphi_Q^{-1}(X)]^{1/n} = (\det Y)^{1/n} + (\det X)^{1/n},$$

where for ordinary addition the well-known (see [11], p. 510) Minkowski inequality is used. For the case of equality in (6.11) we have $\left[\det [\varphi_Q^{-1}(Y) + \varphi_Q^{-1}(X)] \right]^{1/n} = [\det \varphi_Q^{-1}(Y)]^{1/n} + [\det \varphi_Q^{-1}(X)]^{1/n}$. Then $\varphi_Q^{-1}(Y) = c\, \varphi_Q^{-1}(X)$, i.e. $Y = cX$ for some $c > 0$. $\blacksquare$.

7 Analog of an indefinite metric on the set of positive definite matrices

By Theorem 3.8 and Proposition 3.4(iv), each addition operation in a multispace $\mathbb{H}_\delta(n)$ is induced by some positive definite matrix $A > 0$. On the set $\mathbb{P}_n^+ \subset \mathbb{H}_\delta(n)$ of positive definite matrices we introduce a ternary operation $\{X, A, Y\} \mapsto Z$ by the relation $Z = X \overset{\tau_{2\delta}^{-1}(A)}{\oplus} Y$.

7.1 Proposition. *Operation* $Z = X \overset{\tau_{2\delta}^{-1}(A)}{\oplus} Y$ *retains its form under all the transformations* $X' = \varphi_Q(X),\ \varphi_Q \in G_\delta(n)$ *in* $\mathbb{P}_n^+$*, i.e.*

$$\varphi_Q(X \overset{\tau_{2\delta}^{-1}(A)}{\oplus} Y) = X' \overset{\tau_{2\delta}^{-1}(A')}{\oplus} Y'. \tag{7.1}$$

Proof. By Propositions 3.4 (i), 3.6 we have

$$\varphi_Q(X \overset{\tau_{2\delta}^{-1}(A)}{\oplus} Y) = \varphi_Q(X) \overset{Q \cdot \tau_{2\delta}^{-1}(A)}{\oplus} \varphi_Q(Y) = X' \overset{\left| Q \cdot \tau_{2\delta}^{-1}(A) \right|_L}{\oplus} Y',$$

where $\left| Q \cdot \tau_{2\delta}^{-1}(A) \right|_L = (Q \cdot A^{1/\delta} \cdot Q^*)^{1/2} = [(Q \cdot A^{1/\delta} \cdot Q^*)^\delta]^{1/2\delta} = \tau_{2\delta}^{-1}[\varphi_Q(A)] = \tau_{2\delta}^{-1}(A')$. $\blacksquare$.

7.2 Proposition. *Let* $X, Y \in \mathbb{H}_\delta(n)$ *be positive definite matrices and* $\delta > 0$. *Then*

$$Y \overset{\tau_{2\delta}^{-1}(X)}{\ominus} X = Y \overset{\tau_{2\delta}^{-1}(Y)}{\ominus} X . \tag{7.2}$$

Proof. It is sufficient to show that for some $\varphi_Q \in SG_\delta(n)$ the equality $\varphi_Q(Y \overset{\tau_{2\delta}^{-1}(X)}{\ominus} X) = \varphi_Q(Y \overset{\tau_{2\delta}^{-1}(Y)}{\ominus} X)$ is satisfied. By Lemma 4.7 there exists a transformation $\varphi_Q \in SG_\delta(n)$ such that $\varphi_Q(X) = \lambda_0 I_n$, $\varphi_Q(Y) = \Lambda_\mu = \mathrm{diag}\{\mu_1, \cdots, \mu_n\}$. Using Proposition 7.1 we have

$$\varphi_Q(Y \overset{\tau_{2\delta}^{-1}(X)}{\ominus} X) = \Lambda_\mu \overset{\tau_{2\delta}^{-1}(\lambda_0 I_n)}{\ominus} \lambda_0 I_n; \quad \varphi_Q(Y \overset{\tau_{2\delta}^{-1}(Y)}{\ominus} X) = \Lambda_\mu \overset{\tau_{2\delta}^{-1}(\Lambda_\mu)}{\ominus} \lambda_0 I_n .$$

Since for each matrix $\Lambda_0 = \mathrm{diag}\{\alpha_1, \cdots, \alpha_n\} \in GL(n, \mathbb{C})$ the equality $\Lambda_\mu \overset{\Lambda_0}{\ominus} \lambda_0 \cdot I_n = \Lambda_\mu - \lambda_0 \cdot I_n$ is satisfied, then $\varphi_Q(Y \overset{\tau_{2\delta}^{-1}(X)}{\ominus} X) = \Lambda_\mu \overset{\tau_{2\delta}^{-1}(\lambda_0 I_n)}{\ominus} \lambda_0 I_n = \Lambda_\mu - \lambda_0 I_n = Y \overset{\tau_{2\delta}^{-1}(\Lambda_\mu)}{\ominus} X = \varphi_Q(Y \overset{\tau_{2\delta}^{-1}(Y)}{\ominus} X)$. $\blacksquare$.

7.3 Definition. Let $X \leq_\delta Y$ be positive definite matrices of the ordered vector multispace $\mathbb{H}_\delta(n)$, $0 < \delta \leq 1$. The number $\Delta\sigma(X, Y) = [\det (Y \overset{\tau_{2\delta}^{-1}(Y)}{\ominus} X)]^{1/n}$ will be called *the interval from point X to point Y*.

7.4 Theorem (Triangle Reverse Inequality). *Let X,Y,Z be positive defined matrices of the ordered vector multispace $\mathbb{H}_\delta(n)$, $0 < \delta \leq 1$. If $X \leq_\delta Y \leq_\delta Z$, then*

$$\Delta\sigma(X,Z) \geq \Delta\sigma(X,Y) + \Delta\sigma(Y,Z) \tag{7.3}$$

with equality if and only if $(Z \overset{\tau_{2\delta}^{-1}(X)}{\ominus} X) = c \cdot (Y \overset{\tau_{2\delta}^{-1}(X)}{\ominus} X)$ for some $c > 0$.

Proof. Using Minkowski's determinant inequality and Theorem 6.1 we obtain

$$\Delta\sigma(X,Z) = \left[\det (Z \overset{\tau_{2\delta}^{-1}(X)}{\ominus} X) \right]^{1/n} = \left[\det [(Z \overset{\tau_{2\delta}^{-1}(X)}{\ominus} Y) \overset{\tau_{2\delta}^{-1}(X)}{\oplus} (Y \overset{\tau_{2\delta}^{-1}(X)}{\ominus} X)]] \right]^{1/n} \geq \left[(6,11)\right] \geq [\det (Y \overset{\tau_{2\delta}^{-1}(X)}{\ominus} X)]^{1/n} +$$

$$+ [\det (Z \overset{\tau_{2\delta}^{-1}(X)}{\ominus} Y)]^{1/n} \geq \left[(6,1)\right] \geq [\det (Y \overset{\tau_{2\delta}^{-1}(X)}{\ominus} X)]^{1/n} + [\det (Z \overset{\tau_{2\delta}^{-1}(Y)}{\ominus} Y)]^{1/n} = \Delta\sigma(X,Y) + \Delta\sigma(Y,Z).$$

The equality $\Delta\sigma(X,Z) = \Delta\sigma(X,Y) + \Delta\sigma(Y,Z)$ is true if only if

$$\left[\det [(Y \overset{\tau_{2\delta}^{-1}(X)}{\ominus} X)] \overset{\tau_{2\delta}^{-1}(X)}{\oplus} (Z \overset{\tau_{2\delta}^{-1}(X)}{\ominus} Y)] \right]^{1/n} = [\det (Y \overset{\tau_{2\delta}^{-1}(X)}{\ominus} X)]^{1/n} + [\det (Z \overset{\tau_{2\delta}^{-1}(X)}{\ominus} Y)]^{1/n}.$$

By Theorem 6.10 we have $(Z \overset{\tau_{2\delta}^{-1}(X)}{\ominus} Y) = a \cdot (Y \overset{\tau_{2\delta}^{-1}(X)}{\ominus} X)$ for some $a > 0$. From here we get $(Z \overset{\tau_{2\delta}^{-1}(X)}{\ominus} X) = c \cdot (Y \overset{\tau_{2\delta}^{-1}(X)}{\ominus} X)$ for some $c > 0$. $\blacksquare$.

7.5 Corollary. *Let X_0, X_1 be positive defined matrices of the ordered vector multispace $\mathbb{H}_\delta(n)$, $0 < \delta \leq 1$. If $X_0 \leq_\delta X_1$, then there exists a unique chain of "maximum length" connecting points X_0, X_1. This maximum chain has a parametrization*

$$X(s) = (1-s) \cdot X_0 \overset{\tau_{2\delta}^{-1}(X_0)}{\oplus} s \cdot X_1, \ \ 0 \leq s \leq 1, \tag{7.4}$$

which is natural in the sense that $\Delta\sigma(X_0, X(s)) = s \cdot \Delta\sigma(X_0, X_1)$ for each $0 \leq s \leq 1$.

Proof. The maximum length is a consequence of Triangle Reverse Inequality. The interval from point X_0 to point $X(s)$ is

$$\Delta\sigma(X_0, X(s)) = [\det (X(s) \overset{\tau_{2\delta}^{-1}(X_0)}{\ominus} X_0)]^{1/n} = [\det (s(X_1 \overset{\tau_{2\delta}^{-1}(X_0)}{\ominus} X_0))]^{1/n} = s \cdot \Delta\sigma(X_0, X_1). \blacksquare.$$

7.6 Example (Causal Lorentz manifold of dimension 4). Let X_0, Y be positive defined 2×2 matrices and $0 < \delta \leq 1$. Each Hermitian 2×2 matrix A uniquely represented in the form

$$A = \begin{pmatrix} ct + z & x + iy \\ x - iy & ct - z \end{pmatrix} \tag{7.5}$$

with fixed constant $c > 0$. Hermitian matrix

$$Y \overset{\tau_{2\delta}^{-1}(X_0)}{\ominus} X_0 = \begin{pmatrix} c \cdot \tilde{\Delta}t + \tilde{\Delta}z & \tilde{\Delta}x + i\tilde{\Delta}y \\ \tilde{\Delta}x - i\tilde{\Delta}y & c \cdot \tilde{\Delta}t - \tilde{\Delta}z \end{pmatrix}$$

is an element of the tangent space at the point X_0. Here the representation (7.5) for $Y \overset{\tau_{2\delta}^{-1}(X_0)}{\ominus} X_0$ defines each of numbers $\tilde{\Delta}t, \tilde{\Delta}x, \tilde{\Delta}y, \tilde{\Delta}z$. Then

$$\det (Y \overset{\tau_{2\delta}^{-1}(X_0)}{\ominus} X_0) = c^2 \cdot \tilde{\Delta}t^2 - \tilde{\Delta}x^2 - \tilde{\Delta}y^2 - \tilde{\Delta}z^2 \tag{7.6}$$

is the invariant of the group $SG_\delta(2)$. This invariant defines a indefinite metric on the manifold of positive definite 2×2 matrices at point X_0. The partial order relation $X \leq_\delta Y$ is a model of causal relationship between "events" (7.5). Positive definite matrices model "observed" events. Parameter $\tau = s \cdot \Delta\sigma(X_0, X_1)$ (Definition 7.3) can be considered as the free particle's own time.

The group $SG_\delta(2)$ of nonlinear transformations of multispace $\mathbb{H}_\delta(2)$ is a representation of the arcwise connected component of the unit of Lorenz group. Group $SG_\delta(2)$, unlike the Lorentz group, is defined not by one, but by two parameters $c > 0$ and $0 < \delta \le 1$. Note that at $c \to \infty$ and each $0 < \delta \le 1$ the nonlinear transformations of the group $SG_\delta(2)$ are transform into linear Galilean transformations.

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