The Diophantine Equation

$\arctan\left(\frac{\mathbf{1}}{\mathbf{x}}\right)+\arctan\left(\frac{\ell}{\mathbf{y}}\right)=\arctan\left(\frac{\mathbf{1}}{\mathbf{k}}\right)$

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1 Introduction

The subject matter of this work is the two-variable diophantine equation $\arctan\left(\frac{1}{x}\right)+\arctan\left(\frac{\ell}{y}\right)=\arctan\left(\frac{1}{k}\right)$ for given positive integers k and ℓ, such that $\gcd(\ell, k^2+1)=1$ (i.e., ℓ and k^2+1 are relatively prime). The main objective is to determine all positive integer pairs (x, y) which satisfy

$$\left\{\begin{array}{l} \arctan\left(\dfrac{1}{x}\right) + \arctan\left(\dfrac{\ell}{y}\right) = \arctan\left(\dfrac{1}{k}\right) \\ x, y \in \mathbb{Z}^+,\ \gcd(\ell, k^2+1) = 1 \text{ and} \\ \text{with gcd } (\ell, y) = 1 \ \text{(i.e., } \ell \text{ and } y \text{ are} \\ \text{relatively prime)} \end{array}\right\} \quad (1)$$

This is done in Theorem 1, Section 4. As we will see, there are exacgtly N distinct solutions to (1) where N is the number of positive divisors of the integer k^2+1. The N pairs (x, y), which are solutions to (1), are expressed parametrically in terms of the positive divisors of k^2+1. Also, note that when $\ell = 1$, equation (1) is symmetric with respect to the two variables x and y. If (a, b) is a solution, then so is (b, a). The motivating force behind this work is a recent article published in the journal *Mathematics and Computer Education* (see [1]). The article, authored by Hasan Unal, is entitled "Proof without words: an arctangent equality". It consists of four illustrations, a purely geometric proof of the equality,

$$\arctan\left(\frac{1}{3}\right) + \arctan\left(\frac{1}{7}\right) = \arctan\left(\frac{1}{2}\right).$$

From the point of view of (1), the last equality says that the pair $(3, 7)$ is a solution of (1), in the case $\ell = 1$ and $k = 2$.

According to Theorem 1, $(3, 7)$ and $(7, 3)$ are the only solutions to (1) for $\ell = 1$ and $k = 2$.

This, then, is the other objective of this article. To generate more arctangent type of equalities. This is done in Section 5, where a listing of such equalities is offered; an immediate consequence of Theorem 1.

In Section 2, we list two trigonometric preliminaries: the well known identity for the tangent of the sum of two angles and a couple of basic facts regarding arctangent function.

In Section 3, we state two well known results from number theory: Euclid's lemma; and the formula that gives the number of positive divisors of a positive integer. We use these in the proof of Theorem 1.

2 Trigonometric preliminaries

(a) If θ_1 and θ_2 are two angles measured in radians, such that neither θ_1 nor θ_2, nor their sum $\theta_1 + \theta_2$ is of the form $k\pi + \dfrac{\pi}{2}$, k and integer.

Then,

$$\tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1\theta_2}$$

(b) Let f be the arctangent function, $f(x) = \arctan x$. Then,

(i) $\arctan 1 = \frac{\pi}{4}$

(ii) $\left\{\begin{array}{ll} 0 < \theta < \frac{\pi}{r} & \\ \text{and} & \\ & \theta = \arctan c \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} 0 < \theta < \frac{\pi}{4} \\ \\ 0 < c = \tan\theta < 1 \end{array}\right\}$

(iii) $\left\{\begin{array}{ll} 0 < \theta < \frac{\pi}{2} & \\ \text{and} & \\ & \theta = \arctan c \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} 0 < \theta < \frac{\pi}{2} \\ \\ 0 < c = \tan\theta \end{array}\right\}$

3 Number theory preliminaries

The following result is commonly known as Euclid's lemma, and is of great significance in number theory.

Result 1 (Euclid's lemma): *Let a, b, c be positive integers such that a is a divisor of the product bc; and with a also being relatively prime to b. Then, a is a divisor of c.*

The next result provides a formula that gives the exact number of positive divisors of a positive integer.

Result 2 (number of divisors formula) *Let $n \geq 2$ be a positive integer, and let $p_1, \ldots, p_t$ in increasing order, be the distinct prime bases that appear in the prime factorization of n, so that $n = p_1^{e_1}, \ldots p_t^{e_t}$, with the exponents $e_1, \ldots, e_t$ being positive integers. Also, let N be the number of positive divisors of n. Then,*

(i) $N = \prod_{i=1}^{t} (e_i + 1) \ldots (e_1 + 1) \ldots (e_t + 1)$.

(ii) In particular, when $e_1 = \ldots = e_t = 1$ (i.e., when n is squarefree)

$$N = 2^t$$

Both of these two results can be easily found in number theory books and texts. For example, see reference [2].

4 Theorem 1 and its proof

Theorem 1. *Let k and ℓ be fixed or given positive integers such that $gcd(\ell, k^2+1) = 1$. Consider the diophantine equation (1).*

(a) There are exactly N distinct positive integer pairs (x, y) which are solutions to equation (1) where N is the number of positive integer divisors of the integer k^2+1. Specifically, if (x, y) is a positive integer solution of (1), then
$x = k + \ell \cdot \left(\dfrac{k^2+1}{d}\right)$ and $y = k\ell + d$ *where d is a positive integer divisor of k^2+1.*

(b) If $k^2+1 = p$, a prime number, then equation (1) has exactly two distinct positive integer solutions. These are

$$(x, y) = (k + \ell(k^2+1),\ k\ell + 1), \quad (k + \ell,\ k\ell + k^2 + 1).$$

(c) If $k^2+1 = p_1 p_2$, a product of two distinct primes p_1 and p_2, equation (1) has exactly four distinct positive integer solutions. These are,

$$(x, y) = (k + \ell(k^2+1), k\ell + 1), \quad (k + \ell,\ k\ell + k^2 + 1),$$

$$(k + \ell p_2.\ k\ell + p_1), \text{ and } \qquad (k + \ell p_1,\ k\ell + p_2)$$

Proof. First note that parts (b) and (c) are immediate consequences of part (a) and Result 2. We omit the details. We prove part (a)

(a) Let d be a positive integer divisor of k^2+1. We will show that the positive integer pair, $(x_d, y_d) = \left(k + \ell \cdot \left(\dfrac{k^2+1}{d}\right),\ k\ell + d\right)$ is a solution to (1). First note that $y_d = k\ell + d$, is relatively prime to ℓ. Indeed, if y_d

and ℓ had a prime factor q in common then q would divide $y_d - k\ell = d$; and thus (since d is a divisor of k^2+1) $y_d - k\ell = d$, then q would divide k^2+1 contrary to the hypothesis that $\gcd(\ell, k^2+1) = 1$. Thus, $\gcd(\ell, y_d) = 1$.

It is clear that since k, ℓ and d are positive integers, we have $x_d > 1, y_d > 1$ and $k \geq 1$. So,

$$\left(0 < \frac{1}{x_d} < 1,\ 0 < \frac{\ell}{y_d} < 1,\ 0 < \frac{1}{k} \leq 1\right). \tag{2}$$

Let

$$\theta_1 = \arctan\left(\frac{1}{x_d}\right),\ \theta_2 = \arctan\left(\frac{1}{y_d}\right),\ \theta = \arctan\left(\frac{1}{k}\right). \tag{3}$$

Then, by (2), (3) and part (b) of the trigonometric preliminaries, we have

$$\left\{\begin{array}{l} 0 < \theta_1 < \frac{\pi}{4},\ 0 < \theta_2 < \frac{\pi}{4},\ 0 < \theta \leq \frac{\pi}{4} \\ \\ \text{and } 0 < \theta_1 + \theta_2 < \frac{\pi}{2},\ \tan\theta_1 = \frac{1}{x_d},\ \tan\theta_2 = \frac{\ell}{y_d},\ \tan\theta = \frac{1}{k} \end{array}\right\} \tag{4}$$

From (4) and part (a) of trigonometric preliminaries, it follows that

$$\begin{aligned} \tan(\theta_1+\theta_2) &= \frac{\frac{1}{x_d} + \frac{\ell}{y_d}}{1 - \frac{1}{x_d}\cdot\frac{\ell}{y_d}}; \\ \tan(\theta_1+\theta_2) &= \frac{y_d + \ell x_d}{x_d y_d - \ell}; \\ \tan(\theta_1+\theta_2) &= \frac{d\cdot(y_d + \ell x_d)}{d x_d y_d - d\ell}. \end{aligned} \tag{5}$$

By (5) and the expressions for x_d and y_d (see beginning of the proof) we get

$$
\begin{aligned}
\tan(\theta_1+\theta_2) &= \frac{d^2+k\ell d+k\ell d+\ell^2\cdot(k^2+1)}{[dk+\ell(k^2+1)](k\ell+d)-d\ell}; \\
\tan(\theta_1+\theta_2) &= \frac{d^2+2k\ell d+\ell^2\cdot(k^2+1)}{d\ell k^2+k\ell^2(k^2+1)+kd^2+\ell dk^2+d\ell-d\ell}; \\
\tan(\theta_1+\theta_2) &= \frac{d^2+2k\ell d+\ell^2\cdot(k^2+1)}{k\cdot[2dk\ell+d^2+\ell^2(k^2+1)]} = \frac{1}{k} = \tan\theta; \\
\tan(\theta_1+\theta_2) &= \tan\theta
\end{aligned}
\qquad (6)
$$

By (6) and part (b) of the trigonometric preliminaries, it follows that $\theta_1+\theta_2=\theta$, which combined with (3), clearly establishes that the pair (x_d, y_d) is a solution to (1).

Now, the converse. Suppose that (x, y) is a positive integer solution to (1).

Then

$$
\left(0<\frac{1}{x}\le 1,\quad 0<\frac{\ell}{y}\le \ell,\quad 0<\frac{1}{k}\le 1\right) \qquad (7)
$$

Using (7), the trigonometric preliminaries, parts (a) and (b) and by taking tangent of both sides of (1), we obtain,

$$
\frac{\frac{1}{x}+\frac{\ell}{y}}{1-\frac{1}{x}\frac{\ell}{y}} = \frac{1}{k}
$$

or equivalently

(Note that since $0<\dfrac{1}{k}\le 1$. The equal sides of (1) can be utmost equal to $\dfrac{\pi}{4}$)

$$
\begin{aligned}
xy-\ell &= k(y+x\ell); \\
y\cdot(x-k) &= \ell\cdot(1+kx)
\end{aligned}
\qquad (8)
$$

Equation (8) shows that y is a divisor of the product $\ell(1+kx)$. But, by (1), we know that $\gcd(\ell, y) = 1$. Thus, by Result 1 (Euclid's lemma), it follows that y must divide $1 + kx$; and so,

$$\left\{ \begin{array}{l} 1 + kx = y \cdot v \\ \\ v \text{ a positive integer} \end{array} \right\} \tag{9}$$

By (9) and (8) we have that,

$$x = \ell \cdot v + k \tag{10}$$

From (9) and (10) we further get

$$1 + k(\ell v + k) = yv;$$

or equivalently

$$k^2 + 1 = (y - \ell k) \cdot v \tag{11}$$

Since v is a positive integer, equation (11) shows that $(y - \ell k)$ is a positive integer divisor of $k^2 + 1$. Let $y - \ell k = d$, d a positive divisor of $k^2 + 1$. Then $y = \ell k + d$ and by (11) and (10) we also get

$$x = k + \ell \cdot \left(\frac{k^2 + 1}{d} \right),$$

which proves that the solution (x, y) has the required form.

Finally, we see by inspection that the N (number of positive divisors of $k^2 + 1$) positive integer solutions to (1) are distinct since, obviously, all the Ny-coordinates are distinct. The proof is complete.

□

5 A listing of nine equalities

Let k and ℓ be positive integers such that $\gcd(\ell, k^2+1) = 1$. Applying Theorem 1 with $d = 1$ and $d = k^2+1$ produces two inequalities.

1. $\arctan\left(\frac{1}{k+\ell(k^2+1)}\right) + \arctan\left(\frac{\ell}{k\ell+1}\right) = \arctan\left(\frac{1}{k}\right)$
2. $\arctan\left(\frac{1}{k+\ell}\right) + \arctan\left(\frac{\ell}{k\ell+k^2+1}\right) = \arctan\left(\frac{1}{k}\right)$

Next, applying Theorem 1 with $k = \ell = 1$, produces the equality:

3. $\arctan\left(\frac{1}{3}\right) + \arctan\left(\frac{1}{2}\right) = \frac{\pi}{4}$

For $\ell = 1$ and $k = 2$:

4. $\arctan\left(\frac{1}{3}\right) + \arctan\left(\frac{1}{7}\right) = \arctan\left(\frac{1}{2}\right)$.

For $\ell = 1$ and $k = 3$

5. $\arctan\left(\frac{1}{11}\right) + \arctan\left(\frac{1}{4}\right) = \arctan\left(\frac{1}{3}\right)$
6. $\arctan\left(\frac{1}{8}\right) + \arctan\frac{1}{5} = \arctan\left(\frac{1}{3}\right)$

For $\ell = 2$ and $k = 4$:

7. $\arctan\left(\frac{1}{38}\right) + \arctan\left(\frac{2}{9}\right) = \arctan\left(\frac{1}{4}\right)$
8. $\arctan\left(\frac{1}{6}\right) + \arctan\left(\frac{2}{25}\right) = \arctan\left(\frac{1}{4}\right)$

For $\ell = 1$ and $k = 6$:

9. $\arctan\left(\frac{1}{43}\right) + \arctan\left(\frac{1}{7}\right) = \arctan\left(\frac{1}{6}\right)$

References

[1] Unal, Hasan, *Proof without words: an arctangent equality*, Mathematics and Computer Education, Fall 2011, Vol. 45, No. 3, p 197.

[2] Rosen, Kenneth H., *Elementary Number Theory and Its Applications*, 5th edition, Pearson, Addison Wesley, 2005.

For Result 1 (Lemma 3.4 in the above book), see page 109.

For Result 2 (Theorem 7.9 in the above book), see page 252.