Algebrizing friction: a brief look at the Metriplectic Formalism

Massimo Materassi (1)
Emanuele Tassi (2)

(1) Istituto dei Sistemi Complessi ISC-CNR, Sesto Fiorentino, Firenze, Italy
E-mail: massimo.materassi@fi.isc.cnr.it
(2) Centre de Physique Théorique, CNRS -Aix-Marseille Universités, Campus de Luminy, Marseille, France
E-mail: tassi@cpt.univ-mrs.fr

Abstract: The formulation of Action Principles in Physics, and the introduction of the Hamiltonian framework, reduced dynamics to bracket algebræ of observables. Such a framework has great potentialities, to understand the role of symmetries, or to give rise to the quantization rule of modern microscopic Physics.
Conservative systems are easily algebrized via the Hamiltonian dynamics: a conserved observable H generates the variation of any quantity f via the Poisson bracket $\{f,H\}$.
Recently, dissipative dynamical systems have been algebrized in the scheme presented here, referred to as *metriplectic framework*: the dynamics of an isolated system with dissipation is regarded as the sum of a Hamiltonian component, generated by H via a Poisson bracket algebra; plus dissipation terms, produced by a certain quantity S via a new symmetric bracket. This S is in involution with any other observable and is interpreted as the *entropy* of those degrees of freedom statistically encoded in friction.
In the present paper, the metriplectic framework is shown for two original “textbook” examples. Then, dissipative Magneto-Hydrodynamics (MHD), a theory of major use in many space physics and nuclear fusion applications, is reformulated in metriplectic terms.
Keywords: Dissipative systems, Hamiltonian systems, Magneto-Hydrodynamics.

1. Introduction

Hamiltonian systems play a key role in Physics, since the dynamics of elementary particles appear to be Hamiltonian. Hamiltonian systems are endowed with a bracket algebra (that of quantum commutators, or classically of Poisson brackets): such a scheme is of exceptional clarity in terms of symmetries [1], offering the opportunity of retrieving most of the information about the system without even trying to solve the equations of motion.
Despite their central role, Hamiltonian systems are far from covering the main part of real systems: indeed, Hamiltonian systems are intrinsically conservative and reversible, while, as soon as one zooms out from the level of elementary particles, the real world appears to be made of dissipative, irreversible processes [2]. In most real systems there are couplings bringing energy from processes at a certain time- or space-scale, treated deterministically, to processes evolving at much “smaller” and “faster” scales, to be treated statistically, as “noise”. This is exactly what *friction* does, and this transfer appears to be *irreversible*.

A promising attempt of algebrizing the classical Physics of dissipation appears to be the *Metriplectic Formalism* (MF) exposed here [3, 4]. The MF applies to *closed systems with dissipation*, for which the energy conservation and entropy growth hold: the MF satisfies these two conditions [5]. The first important ingredient of the MF is the *metriplectic bracket* (MB):

$$\langle\langle f,g\rangle\rangle = \{f,g\} + (f,g),$$

where the first term $\{f,g\}$ is a Poisson bracket, while the term (f,g) is a *symmetric bracket*, bilinear and semi-definite. The total energy is represented by a Hamiltonian H which has *zero symmetric bracket with any quantity* (i.e. $(f,H) = 0$ for all f). The total entropy is mimicked by an observable S that has *zero Poisson bracket with any quantity* (i.e. $\{f,S\} = 0$ for all f). Then, a free energy F is defined as

$$F = H + \alpha S,$$

α being a coefficient that will disappear from the equations of motion, due to the suitable definition of (f,g); it coincides with minus the equilibrium temperature of the system (see below in the examples). The dynamics of any f reads:

$$\dot{f} = \langle\langle f,F\rangle\rangle = \{f,H\} + \alpha(f,S).$$

This dynamics conserves H and gives a monotonically varying (increasing) S. Metriplectic systems admit asymptotic equilibria (due to dissipation) in correspondence to extrema of F.
In this paper the MF is applied to some examples of isolated dissipative systems: two “textbook” examples and, more significantly, to visco-resistive magneto-hydrodynamics (MHD).

2. Two “textbook” examples

In order to illustrate how the MF works, two simple systems are considered.
The first one is a particle of mass m dragged by the conservative force of a potential V throughout a viscous medium. A viscous friction force, proportional to the minus velocity of the particle via a coefficient λ, converts its kinetic energy into internal energy U of the medium, with entropy S. The equations of motion of the system read:

$$\dot{\boldsymbol{x}} = \tfrac{\boldsymbol{p}}{m}, \quad \dot{\boldsymbol{p}} = -\nabla V - \lambda \tfrac{\boldsymbol{p}}{m}, \quad \dot{S} = \tfrac{\lambda p^2}{m^2 T}.$$

T is the temperature of the medium, simply defined as the derivative of U with respect to S. If the MB is defined as follows:

$$
\begin{aligned}
&\dot{f} = \langle\langle f, F \rangle\rangle, \quad \langle\langle f, g \rangle\rangle = \{f, g\} + (f, g) \quad / \\
&\{f, g\} = \tfrac{\partial f}{\partial \boldsymbol{x}} \cdot \tfrac{\partial g}{\partial \boldsymbol{p}} - \tfrac{\partial g}{\partial \boldsymbol{x}} \cdot \tfrac{\partial f}{\partial \boldsymbol{p}}, \\
&(f, g) = \Gamma^{ij} \tfrac{\partial f}{\partial \psi^i} \tfrac{\partial g}{\partial \psi^j} \quad / \quad \underline{\psi} = (\boldsymbol{x}, \boldsymbol{p}, S), \\
&\Gamma = \alpha^{-1} \begin{pmatrix} (\nabla V)^2 \mathbf{1}_{3,3} - \nabla V \otimes \nabla V & 0 & 0 \\ 0 & \lambda T \mathbf{1}_{3,3} & -m^{-1} \lambda \boldsymbol{p} \\ 0 & -m^{-1} \lambda \boldsymbol{p}^{\mathrm{T}} & \frac{\lambda p^2}{m^2 T} \end{pmatrix},
\end{aligned}
$$

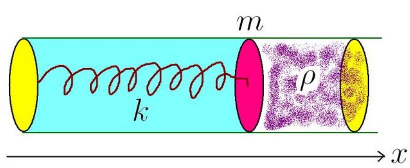
it is easy to show that these ODEs are given by the MB of $\boldsymbol{x}$, $\boldsymbol{p}$ and S, with a free energy F constructed as:

$$
\begin{aligned}
&F(\boldsymbol{x}, \boldsymbol{p}, S) = H(\boldsymbol{x}, \boldsymbol{p}, S) + \alpha S, \\
&H(\boldsymbol{x}, \boldsymbol{p}, S) = \tfrac{p^2}{2m} + V(\boldsymbol{x}) + U(S).
\end{aligned}
$$

The matrix Γ is semi-definite with the same sign as α. The foregoing framework conserves H and increases S, driving the system to the asymptotic equilibrium:

$$
\boldsymbol{p}_{eq} = 0, \quad \nabla V(\boldsymbol{x}_{eq}) = 0, \quad T_{eq} = -\alpha.
$$

At the equilibrium the point particle stops at a stationary point of V once its kinetic energy has been fully dissipated into heat by friction.
The second rather simple example of metriplectic system is a piston of mass m and area A, running along a horizontal guide pushed by a spring of elastic constant k. It works against a viscous gas of pressure P and mass M. The system is depicted in the following Figure.



Piston moved by the spring of elastic constant k and mass m, working against a viscous gas of density ρ.

If ℓ is the rest-length of the spring, then the equations of motion of the system read:

$$\dot{x} = \tfrac{p}{m}, \quad \dot{p} = -PA - k(x - \ell) - \lambda \tfrac{p}{m}, \quad \dot{S} = \tfrac{\lambda p^2}{m^2 T}.$$

These equations of motion may be obtained out of a metriplectic scheme assigned as

$$\begin{aligned}
&\dot{f} = \langle\langle f, F \rangle\rangle, \quad \langle\langle f, g \rangle\rangle = \{f, g\} + (f, g) \quad / \\
&\{f, g\} = \tfrac{\partial f}{\partial x}\tfrac{\partial g}{\partial p} - \tfrac{\partial g}{\partial x}\tfrac{\partial f}{\partial p}, \quad (f, g) = \Gamma^{ij}\tfrac{\partial f}{\partial \psi^i}\tfrac{\partial g}{\partial \psi^j} \quad / \quad \underline{\psi} = (x, p, S), \\
&\Gamma = \alpha^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda T & -m^{-1}\lambda p \\ 0 & -m^{-1}\lambda p & \tfrac{\lambda p^2}{m^2 T} \end{pmatrix},
\end{aligned}$$

provided the following free energy is defined

$$\begin{aligned}
&F(x, p, S) = H(x, p, S) + \alpha S, \\
&H(x, p, S) = \tfrac{p^2}{2m} + \tfrac{k}{2}(x - \ell)^2 + U(\rho(x), S).
\end{aligned}$$

Again, this Γ is semi-definite with the same sign as α. The asymptotic equilibrium of the foregoing F read

$$x_{eq} = \ell - \tfrac{PA}{k}, \quad p_{eq} = 0, \quad T_{eq} = -\alpha$$

(the temperature T is still defined as the derivative of U with respect to S): the piston stops where the spring equilibrates the gas pressure, its kinetic energy all dissipated by friction.

3. Dissipative MHD

Dissipative MHD is expected to describe many plasma processes, wherever its fundamental hypotheses apply to a highly conductive plasma interacting with its own magnetic field [6, 7]. Ideal MHD has already been cast into Hamiltonian formalism [8], here the metriplectic extension of the Poisson algebra, and the free energy extension of the Hamiltonian, is proposed to include dissipative effects [9].

The 3D visco-resistive MHD equations read:

$$\begin{aligned}
\partial_t \boldsymbol{v} &= -(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \tfrac{\nabla p}{\rho} - \tfrac{\nabla B^2}{2\rho} + \tfrac{(\boldsymbol{B}\cdot\nabla)}{\rho}\boldsymbol{B} - \nabla V_{grav} + \tfrac{\nabla\cdot\sigma}{\rho},\\
\partial_t \boldsymbol{B} &= -(\boldsymbol{B}\cdot\nabla)\boldsymbol{v} - (\nabla\cdot\boldsymbol{v})\boldsymbol{B} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{B} + \mu\nabla^2\boldsymbol{B},\\
\partial_t \rho &= -\nabla\cdot(\rho\boldsymbol{v}),\\
\partial_t s &= -(\boldsymbol{v}\cdot\nabla)s + \tfrac{\sigma:(\nabla\otimes\boldsymbol{v})}{\rho T} + \tfrac{\mu(\nabla\times\boldsymbol{B})^2}{\rho T} + \tfrac{\kappa\nabla^2 T}{\rho T}.
\end{aligned}$$

The MHD, defined on a 3D domain **D**, with suitable boundary conditions on ∂**D**, is a *complete system* described by: plasma bulk velocity $\boldsymbol{v}$, magnetic induction $\boldsymbol{B}$, plasma density ρ and plasma mass-specific entropy s. In the foregoing field equations, p is plasma pressure, V_{grav} is an external gravitational potential; σ is plasma stress tensor, containing (linearly) the fluid viscosity coefficients η and ζ (see below), while μ is resistivity. κ is thermal conductivity, and T is temperature of the plasma. The system conserves the total energy:

$$H = \int_{\mathbf{D}} d^3x \left(\tfrac{\rho v^2}{2} + \tfrac{B^2}{2} + \rho V_{grav} + \rho U(\rho, s)\right),$$

which is the Hamiltonian, being U the mass-specific internal energy of the plasma. In the non-dissipative limit σ = 0, μ = 0 and κ = 0, the whole physics is given by H and the following noncanonical Poisson brackets:

$$\begin{aligned}
\{f,g\} = -\int_{\mathbf{D}} d^3x \Big[&\tfrac{\delta f}{\delta\rho}\nabla\cdot\left(\tfrac{\delta g}{\delta\boldsymbol{v}}\right) + \tfrac{\delta g}{\delta\rho}\nabla\cdot\left(\tfrac{\delta f}{\delta\boldsymbol{v}}\right) +\\
&- \tfrac{1}{\rho}(\nabla\times\boldsymbol{v})\cdot\left[\left(\tfrac{\delta f}{\delta\boldsymbol{v}}\right)\times\left(\tfrac{\delta g}{\delta\boldsymbol{v}}\right)\right] + \tfrac{1}{\rho}\left[\left(\tfrac{\delta f}{\delta\boldsymbol{v}}\right)\times\boldsymbol{B}\right]\cdot\left[\nabla\times\left(\tfrac{\delta g}{\delta\mathbf{B}}\right)\right] +\\
&+ \tfrac{\delta f}{\delta\mathbf{B}}\cdot\left[\nabla\times\left(\tfrac{\boldsymbol{B}}{\rho}\times\tfrac{\delta g}{\delta\boldsymbol{v}}\right)\right] + \tfrac{\nabla s}{\rho}\cdot\left(\tfrac{\delta f}{\delta s}\tfrac{\delta g}{\delta\boldsymbol{v}} - \tfrac{\delta g}{\delta s}\tfrac{\delta f}{\delta\boldsymbol{v}}\right)\Big]
\end{aligned}$$

(here $\delta f/\delta\varphi$ is the Fréchet derivative of the functional f with respect to the field φ). When dissipation is considered, the Hamiltonian must be extended to free energy adding a suitable entropic term:

$$\begin{aligned}
&S[\rho,s] = \int_{\mathbf{D}} \rho s\, d^3x, \quad \{f,S\} = 0 \quad \forall \quad f = f[\boldsymbol{v},\boldsymbol{B},\rho,s],\\
&F[\boldsymbol{v},\boldsymbol{B},\rho,s] = H[\boldsymbol{v},\boldsymbol{B},\rho,s] + \alpha S[\rho,s];
\end{aligned}$$

the symmetric bracket to be used to form a complete MB, together with the Poisson bracket defined before, reads:

$$
\begin{aligned}
(f,g) &= \alpha^{-1}\int_{\mathbf{D}} T d^3x \Big[\kappa T \nabla\left(\tfrac{1}{\rho T}\tfrac{\delta f}{\delta s}\right)\cdot\nabla\left(\tfrac{1}{\rho T}\tfrac{\delta g}{\delta s}\right)+ \\
&+\Lambda :: \left[\left(\nabla\otimes\left(\tfrac{1}{T}\tfrac{\delta f}{\delta \boldsymbol{v}}\right)-\tfrac{1}{\rho T}\tfrac{\delta f}{\delta s}\nabla\otimes\boldsymbol{v}\right)\otimes\left(\nabla\otimes\left(\tfrac{1}{T}\tfrac{\delta g}{\delta \boldsymbol{v}}\right)-\tfrac{1}{\rho T}\tfrac{\delta g}{\delta s}\nabla\otimes\boldsymbol{v}\right)\right]+ \\
&+\Theta :: \left[\left(\nabla\otimes\left(\tfrac{\delta f}{\delta \boldsymbol{B}}\right)-\tfrac{1}{\rho T}\tfrac{\delta f}{\delta s}\nabla\otimes\boldsymbol{B}\right)\otimes\left(\nabla\otimes\left(\tfrac{\delta g}{\delta \boldsymbol{B}}\right)-\tfrac{1}{\rho T}\tfrac{\delta g}{\delta s}\nabla\otimes\boldsymbol{B}\right)\right]\Big].
\end{aligned}
$$

Note the strict analogy between the dissipative $\boldsymbol{v}$-terms and $\boldsymbol{B}$-terms, which are so alike because in the equations of motion dissipation terms appear as quadratic in the gradients of $\boldsymbol{v}$ and $\boldsymbol{B}$, respectively through the rank-4 tensors Λ and Θ (quadratic dissipation, see [9]):

$$
\begin{aligned}
&\Lambda_{ikmn} = \eta\left(\delta_{ni}\delta_{mk}+\delta_{nl}\delta_{mi}-\tfrac{2}{3}\delta_{ik}\delta_{mn}\right)+\zeta\delta_{ik}\delta_{mn}\,, \quad \sigma = \Lambda : (\nabla\otimes\boldsymbol{v}), \\
&\Theta_{ikmn} = \mu\varepsilon^{ikj}\varepsilon^{j}{}_{mn}\,.
\end{aligned}
$$

Due to the symmetry properties of Λ and Θ, the symmetric bracket (f,g) just defined is semi-definite with the same sign of α; the functional gradient of H is a null mode of it. Finally, the quantities related to the space-time symmetries, generating the Galileo transformations

$$
\boldsymbol{P} = \int_{\mathbf{D}} \rho\boldsymbol{v} d^3x, \quad \boldsymbol{L} = \int_{\mathbf{D}} \rho(\boldsymbol{x}\times\boldsymbol{v}) d^3x, \quad \boldsymbol{G} = \int_{\mathbf{D}} \rho(\boldsymbol{x}-t\boldsymbol{v}) d^3x
$$

via the Poisson bracket algebra given in [8] and reported above, are conserved by the metriplectic dynamics:

$$
\dot{f} = \{f, H[\boldsymbol{v},\boldsymbol{B},\rho,s]\} + \alpha(f, S[\rho,s]),
$$

provided suitable boundary conditions are assigned to all the fields.

In the above Eulerian description of MHD, the bracket is noncanonical, depends on s, and the entropy S appears as a Casimir of the bracket which, by definition, belongs to the kernel of the co-symplectic form associated to the bracket [10], while in the "textbook" cases the Poisson bracket was canonical and was independent on the entropy-related variable S.

The free energy $F[\boldsymbol{v},\boldsymbol{B},\rho,S]$ constructed before is able to predict the asymptotic equilibrium state:

$$
\boldsymbol{v}_{eq} = 0, \quad \boldsymbol{B}_{eq} = 0, \quad T_{eq} = -\alpha, \quad p_{eq} + \rho_{eq}V_{grav} = \rho_{eq}(Ts-U)_{eq}.
$$

Such an equilibrium configuration has zero bulk velocity and magnetization, while pressure and gravity equilibrates the thermodynamic free energy of the gas.

4. Conclusions

In metriplectic formalism friction forces, acting within isolated systems, are algebrized. The dissipative terms in the equations of motion are given by a suitable symmetric, semi-definite bracket of the variables with the entropy of the degrees of freedom to which friction drains energy.

Two simple "textbook" examples are reported: the point particle moving through a viscous medium; a piston, moved by a spring against a viscous gas in a rigid cylinder. In both the examples the evolution is generated via the metriplectic bracket with the free energy $F = H + \alpha S$, where H is the conserved Hamiltonian and S is the monotonically growing entropy. α appears to coincide with the equilibrium temperature.

The same formalism is then applied to an isolated magnetized plasma, represented by the dissipative (i.e. viscous and resistive) MHD with suitable boundary conditions. A Hamiltonian scheme already exists for the non-dissipative limit; furthermore, the full MF had been introduced for the neutral fluid version. In this paper, we report the extension of the latter formalisms to include the magnetic forces and the dissipation due to Joule Effect [9]. The "macroscopic" level of plasma physics is described by the fluid variable $\boldsymbol{v}$, but a "microscopic" level exists too, encoded effectively in the thermodynamical field s. The energy attributed to the macroscopic degrees of freedom $\boldsymbol{v}$ is passed to the microscopic ones by friction, while the electric dissipation of Joule Effect consumes the energy pertaining to the magnetic degrees of freedom $\boldsymbol{B}$. Notice that the metriplectic formulation for dissipative MHD that we found, does not require div.$\boldsymbol{B} = 0$.

Dissipative MHD is mathematically much more complicated than the two "textbook" examples, nevertheless its essence is rigorously the same: the MF algebraically generates asymptotically stable motions for closed systems. At the equilibrium, mechanical and electromagnetic energies are turned into internal energy of the microscopic degrees of freedom: the asymptotic equilibria found here for the three examples are essentially entropic deaths.

Let's conclude with few more observations.

MF is a deterministic description, but it must be possible to obtain it as an effective representation of a scenario where the superposition of the Hamiltonian and the entropic motion mirrors the Physics of a deterministic Hamiltonian system under the action of noise [8].

The appearance of MF offers potentially great chances because it drives the algebraic Physics out of the realm of Hamiltonian systems: many interesting processes in nature (as the apparent self-organization of space physics systems [12], not to mention biological or learning processes) are not expected to be even conceptually Hamiltonian. It is very stimulating to imagine dealing with algebraic formalisms describing them. MF, however, is not able to compound such processes, because it pertains to *complete*, i.e. closed, systems, while the processes just mentioned take place in open ones. Adapting MF to open systems will then be a necessary step to face this challenge.

Before concluding, let's underline again the dynamical role of entropy in MF: entropy may be interpreted as an information theory quantity [13, 14], and here we find information directly included in the algebraic dynamics. Furthermore: irreversible biophysical processes appear to have something in common with learning processes [15], i.e. processes in which the information is constructed or degraded, and having a formalism where "information" is an essential function appears to offer hopes in this branch.

References

1. L. D. Landau, E. M. Lifshitz, Mechanics. Course of Theoretical Physics.Vol.1, Butterworth-Heinemann, 1982.
2. B. Misra, I. Prigogine, M. Courbage, From deterministic dynamics to probabilistic description, Physica A, vol. 98, 1-26, 1979.
3. P. J. Morrison, Some Observations Regarding Brackets and Dissipation, Center for Pure and Applied Mathematics Report PAM--228, University of California, Berkeley (1984).
4. P. J. Morrison, Thoughts on brackets and dissipation: old and new, Journal of Physics Conference Series, 169, 012006 (2009).
5. P. J. Morrison, A paradigm for joined Hamiltonian and dissipative systems, Physics D, vol. 18, 410-419, 1986.
6. A. Raichoudhuri, The Physics of Fluids and Plasmas – an introduction for astrophysicists, Cambridge University Press, 1998.
7. D. Biskamp, Nonlinear Magnetohydrodynamics, Cambridge University Press, 1993.
8. P. J. Morrison, J. M. Greene, Noncanonical Hamiltonian Density Formulation of Hydrodynamics and Ideal Magnetohydrodynamics, Physical Review Letters, vol. 45, 10 (1980).
9. M. Materassi, E. Tassi, Metriplectic Framwork for the Dissipative Magneto-Hydrodynamics, submitted to Physica D.
10. P. J. Morrison, Hamiltonian description of the ideal fluid, Reviews of Modern Physics, Vol. 70, No. 2, 467-521, April 1998.
11. T. D. Frank, T. D., Nonlinear Fokker-Planck Equations, Springer-Verlag Berlin-Heidelberg, 2005.
12. T. Chang, Self-organized criticality, multi-fractal spectra, sporadic localized reconnections and intermittent turbulence in the magnetotail, Phys. Plasmas, 6, 4137-4145, 1999.
13. E. T. Janes, Information Theory and Statistical Mechanics, Phys. Rev., 106, 4, 620-630, 1957.
14. E. T. Janes, Information Theory and Statistical Mechanics II, Phys. Rev., 108, 2, 171-190, 1957.
15. G. Careri, La fisica della vita (Physics of Life), Sapere, Agosto 2002.