A-nuclei and A-centers of a quasigroup

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Abstract

A-nuclei (groups of regular permutations) of a quasigroup are studied. A quasigroup is A-nuclear if and only if it is group isotope. Any quasigroup with permutation medial or paramedial identity is an abelian group isotope. Definition of A-center of a quasigroup is given. A quasigroup is A-central if and only if it is abelian group isotope. If a quasigroup is central in Belyavskaya-Smith sense, then it is A-central. Conditions when A-nucleus define normal congruence of a quasigroup are established, conditions normality of nuclei of some inverse quasigroups are given. Notice, definition of A-nucleus of a loop and A-center of a loop coincides, in fact, with corresponding standard definition.

Keywords: Quasigroup, left quasigroup, left nucleus, middle nucleus, left congruence, congruence, isostrophy, autotopy, translation.

MSC   20N05.

[1]

Contents

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1   Introduction

1.1   Historical notes

G.N. Garrison [43] was the first who has defined concept of a quasigroup nucleus. A nucleus (middle, left, or right) "measures" how far is a quasigroup from a group. Unfortunately, if a quasigroup $(Q, \cdot)$ has a non-trivial Garrison's nucleus, then this quasigroup is a right loop, or a left loop, either is a loop [70, p. 17]. Therefore many authors tried to diffuse (to generalize) concept of nucleus on "proper" quasigroup case.

Various definitions of groupoid, quasigroup and loop nuclei were given and researched by A. Sade, R.H. Bruck, V.D. Belousov, P.I. Gramma, A.A. Gvaramiya, M.D. Kitaroagă, G.B. Belyavskaya, H.O. Pflugfelder and many other mathematicians [31, 72, 57, 32, 44, 14, 45, 15, 26, 58, 19, 63, 24, 21, 70, 77, 52].

V.D. Belousov discovered connections of quasigroup nuclei with the groups of regular mappings of quasigroups [10, 14, 17, 53, 51, 52, 55].

Belousov studied autotopisms of the form $(\alpha, \varepsilon, \alpha)$ of a quasigroup $(Q, \circ)$. In this paper we study autotopisms of the form $(\alpha, \varepsilon, \gamma)$, i.e. we use Kepka's generalization [53, 54]. By this approach "usual" nuclei of quasigroups are obtained as some orbits by the action of components of A-nuclei on the set Q. The study of A-nuclei of (α, β, γ)- and $(r; s; t)$-inverse quasigroups was initiated in [51, 52].

In the paper we develop Belousov approach to parastrophes of quasigroup nuclei [12] using concept of middle translation [15]. Also we make an attempt to extend some results of A. Drapal and P. Jedlička [38, 39] about normality of nuclei.

1.2   Quasigroups, identity elements, translations

For convenience of readers we start from some definitions which it is possible to find in [14, 17, 70, 89, 77].

Definition 1.1. *A binary groupoid (G, A) is understood to be a non-empty set G together with a binary operation A.*

As usual the product of mappings is their consecutive realization. We shall use the following (left) order of multiplication of maps: if μ, ν are some maps, then $(\mu\nu)(x) = \mu(\nu(x))$.

Definition 1.2. *A groupoid $(Q, \circ)$ is called a right quasigroup (a left quasigroup) if, for all $a, b \in Q$, there exists a unique solution $x \in Q$ to the equation $x \circ a = b$ ($a \circ x = b$), i.e. in this case any right (left) translation of the groupoid $(Q, \circ)$ is a bijective map of the set Q.*

In this case any right (left) translation of the groupoid $(Q, \circ)$ is a bijective map of the set Q.

Definition 1.3. *A groupoid $(Q, \circ)$ is called a quasigroup if for any fixed pair of elements $a, b \in Q$ there exist a unique solution $x \in Q$ to the equation $x \circ a = b$ and a unique solution $y \in Q$ to the equation $a \circ y = b$.*

Definition 1.4. *An element f is a left identity element of a quasigroup $(Q, \cdot)$ means that $f \cdot x = x$ for all $x \in Q$.*
   An element e is a right identity element of a quasigroup $(Q, \cdot)$ means that $x \cdot e = x$ for all $x \in Q$.
   An element s is a middle identity element of a quasigroup $(Q, \cdot)$ means that $x \cdot x = s$ for all $x \in Q$.
   An element e is an identity element of a quasigroup $(Q, \cdot)$, if $x \cdot e = e \cdot x = x$ for all $x \in Q$.

Definition 1.5. *A quasigroup $(Q, \cdot)$ with a left identity element $f \in Q$ is called a left loop.*
   A quasigroup $(Q, \cdot)$ with a right identity element $e \in Q$ is called a right loop.
   A quasigroup $(Q, \cdot)$ with a middle identity element $s \in Q$ is called an unipotent quasigroup.
   A quasigroup $(Q, \cdot)$ with an identity element $e \in Q$ is called a loop.

Define in a quasigroup $(Q, \cdot)$ the following mappings: $f : x \mapsto f(x)$, where $f(x) \cdot x = x$; $e : x \mapsto e(x)$, where $x \cdot e(x) = x$; $s : x \mapsto s(x)$, where $s(x) = x \cdot x$.

Remark 1.6. In a left loop $f(Q) = 1$, in a right loop $e(Q) = 1$, in an unipotent quasigroup $s(Q) = 1$, where 1 is a fixed element of the set Q.

Definition 1.7. *A quasigroup $(Q, \cdot)$ with identity $x \cdot x = x$ is called an idempotent quasigroup.*

Remark 1.8. In an idempotent quasigroup the mappings e, f, s are identity permutations of the set Q. Moreover, if one from these three mappings is identity permutation, then all other also are identity permutations.

In a loop $(Q, \cdot)$, as usual [14], an element b such that $b \cdot a = 1$ is called left inverse element to the element a, $b = {}^{-1}(a)$; an element c such that $a \cdot c = 1$ is called right inverse element to the element a, $c = (a)^{-1}$.

Definition 1.9. *A binary groupoid (Q, A) with a binary operation A such that in the equality $A(x_1, x_2) = x_3$ knowledge of any 2 elements of x_1, x_2, x_3 uniquely specifies the remaining one is called a binary quasigroup [16, 67].*

From Definition 1.9 it follows that with any quasigroup (Q, A) it possible to associate else $(3! - 1) = 5$ quasigroups, so-called parastrophes of quasigroup (Q, A): $A(x_1, x_2) = x_3 \Leftrightarrow A^{(12)}(x_2, x_1) = x_3 \Leftrightarrow A^{(13)}(x_3, x_2) = x_1 \Leftrightarrow A^{(23)}(x_1, x_3) = x_2 \Leftrightarrow A^{(123)}(x_2, x_3) = x_1 \Leftrightarrow A^{(132)}(x_3, x_1) = x_2$.

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We shall denote:

the operation of (12)-parastrophe of a quasigroup $(Q, \cdot)$ by $*$;

the operation of (13)-parastrophe of a quasigroup $(Q, \cdot)$ by $/$;

the operation of (23)-parastrophe of a quasigroup $(Q, \cdot)$ by $\backslash$.

We have defined left and right translations of a groupoid and, therefore, of a quasigroup. But for quasigroups it is possible to define and the third kind of translations, namely the map $P_a : Q \longrightarrow Q$, $x \cdot P_a x = a$ for all $x \in Q$ [15].

In the following table connections between different kinds of translations in different parastrophes of a quasigroup $(Q, \cdot)$ are given [40, 75]. In fact this table is in [15].

Table 1

	ε	(12)	(13)	(23)	(123)	(132)
R	R	L	R^{-1}	P	P^{-1}	L^{-1}
L	L	R	P^{-1}	L^{-1}	R^{-1}	P
P	P	P^{-1}	L^{-1}	R	L	R^{-1}
R^{-1}	R^{-1}	L^{-1}	R	P^{-1}	P	L
L^{-1}	L^{-1}	R^{-1}	P	L	R	P^{-1}
P^{-1}	P^{-1}	P	L	R^{-1}	L^{-1}	R

With any quasigroup $(Q, \cdot)$ we can associate the sets of all left translations ($\mathbb{L}$), right translations ($\mathbb{R}$) and middle translations ($\mathbb{P}$). Denote the groups generated by all left, right and middle translations of a quasigroup $(Q, \cdot)$ as $LM(Q, \cdot)$, $RM(Q, \cdot)$ and $PM(Q, \cdot)$, respectively.

The group generated by all left and right translations of a quasigroup $(Q, \cdot)$ is called (following articles of A.A. Albert [2, 3]) multiplication group of a quasigroup. This group usually is denoted by $M(Q, \cdot)$. By $FM(Q, \cdot)$ we shall denote a group generated by the sets $\mathbb{L}, \mathbb{R}, \mathbb{P}$ of a quasigroup $(Q, \cdot)$.

Connections between different kinds of local identity elements (and of course of identity elements) in different parastrophes of a quasigroup $(Q, \cdot)$ are given in the following table [73, 75].

Table 2

	ε	(12)	(13)	(23)	(123)	(132)
f	f	e	s	f	e	s
e	e	f	e	s	s	f
s	s	s	f	e	f	e

In Table 2, for example, $s^{(123)} = f^{(\cdot)}$.

Lemma 1.10. *Parastrophic image of a loop is a loop, or an unipotent left loop, either an unipotent right loop. Parastrophic image of an idempotent quasigroup is an idempotent quasigroup.*

Proof. It is possible to use Table 2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

In this paper an algebra (or algebraic structure) is a set A together with a collection of operations defined on A. T. Evans [41] defined a binary quasigroup as an algebra $(Q, \cdot, /, \backslash)$ with three binary operations. He has used the following identities:

$$x \cdot (x \backslash y) = y, \tag{1}$$

$$(y/x) \cdot x = y, \tag{2}$$

$$x \backslash (x \cdot y) = y, \tag{3}$$

$$(y \cdot x)/x = y. \tag{4}$$

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Definition 1.11. *An algebra $(Q, \cdot, \backslash, /)$ with identities (1) – (4) is called a quasigroup [41, 30, 33, 14, 17, 70].*

In any quasigroup $(Q, \cdot, \backslash, /)$ the following identities are true [30]:

$$(x/y)\backslash x = y, \tag{5}$$

$$x/(y\backslash x) = y. \tag{6}$$

1.3 Isotopism

Definition 1.12. *A binary groupoid $(G, \circ)$ is an isotope of a binary groupoid $(G, \cdot)$ (in other words $(G, \circ)$ is an isotopic image of $(G, \cdot)$), if there exist permutations μ_1, μ_2, μ_3 of the set G such that $x_1 \circ x_2 = \mu_3^{-1}(\mu_1 x_1 \cdot \mu_2 x_2)$ for all $x_1, x_2 \in G$ [14, 17, 70].*

We can write this fact and in the form $(G, \circ) = (G, \cdot)T$, where $T = (\mu_1, \mu_2, \mu_3)$.

Definition 1.13. *If $(\circ) = (\cdot)$, then the triple (μ_1, μ_2, μ_3) is called an autotopy of groupoid $(Q, \cdot)$.*

The set of all autotopisms of a groupoid $(Q, \cdot)$ forms a group relative to usual component-wise multiplications of autotopisms [14, 17, 70]. We shall denote this group as $Avt(Q, \cdot)$.

The last component of an autotopy of a groupoid is called a *quasiautomorphism*.

Lemma 1.14. *1. If $(Q, \circ) = (Q, \cdot)(\alpha, \varepsilon, \varepsilon)$, then $L_x^\circ = L_{\alpha x}^{\cdot}$, $R_x^\circ = R_x^{\cdot}\alpha$, $P_x^\circ = P_x^{\cdot}\alpha$ for all $x \in Q$.*

2. If $(Q, \circ) = (Q, \cdot)(\varepsilon, \beta, \varepsilon)$, then $L_x^\circ = L_x^{\cdot}\beta$, $R_x^\circ = R_{\beta x}^{\cdot}$, $(P_x^\circ)^{-1} = (P_x^{\cdot})^{-1}\beta$ for all $x \in Q$.

3. If $(Q, \circ) = (Q, \cdot)(\varepsilon, \varepsilon, \gamma)$, then $L_x^\circ = \gamma^{-1}L_x^{\cdot}$, $R_x^\circ = \gamma^{-1}R_x^{\cdot}$, $P_x^\circ = P_{\gamma x}^{\cdot}$ for all $x \in Q$.

Proof. Case 1. We can re-write equality $(Q, \circ) = (Q, \cdot)(\alpha, \varepsilon, \varepsilon)$ in the form $x \circ y = \alpha x \cdot y = z$ for all $x, y \in Q$. Therefore $L_x^\circ y = L_{\alpha x}^{\cdot}y$, $R_y^\circ x = R_y^{\cdot}\alpha x$, $P_z^\circ x = P_z^{\cdot}\alpha x$.

Case 2. We can re-write equality $(Q, \circ) = (Q, \cdot)(\varepsilon, \beta, \varepsilon)$ in the form $x \circ y = x \cdot \beta y = z$ for all $x, y \in Q$. Therefore $L_x^\circ y = L_x^{\cdot}\beta y$, $R_y^\circ x = R_{\beta y}^{\cdot}x$, $(P_z^\circ)^{-1}y = (P_z^{\cdot})^{-1}\beta y$.

Case 3. We can re-write equality $(Q, \circ) = (Q, \cdot)(\varepsilon, \varepsilon, \gamma)$ in the form $x \circ y = \gamma^{-1}(x \cdot y)$ for all $x, y \in Q$. Therefore $L_x^\circ = \gamma^{-1}L_x^{\cdot}$, $R_y^\circ = \gamma^{-1}R_y^{\cdot}$. If $x \circ y = \gamma^{-1}(x \cdot y) = z$, then $P_z^\circ x = y$, $P_{\gamma z}^{\cdot}x = y$, $P_z^\circ = P_{\gamma z}^{\cdot}$. $\square$

Corollary 1.15. *In conditions of Lemma 1.14*

1. (i) *if $\alpha = (R_a^{\cdot})^{-1}$, then $R_a^\circ = \varepsilon$, $(Q, \circ)$ is a right loop.*

    (ii) *if $\alpha = (P_a^{\cdot})^{-1}$, then $P_a^\circ = \varepsilon$, $(Q, \circ)$ is an unipotent quasigroup.*

2. (i) *if $\beta = (L_b^{\cdot})^{-1}$, then $L_b^\circ = \varepsilon$, $(Q, \circ)$ is a left loop.*

    (ii) *if $\beta = P_b^{\cdot}$, then $P_b^\circ = \varepsilon$, $(Q, \circ)$ is an unipotent quasigroup.*

3. (i) *if $\gamma = L_c^{\cdot}$, then $L_c^\circ = \varepsilon$, $(Q, \circ)$ is a left loop.*

    (ii) *if $\gamma = R_c^{\cdot}$, then $R_c^\circ = \varepsilon$, $(Q, \circ)$ is a right loop.*

Proof. The proof follows from Lemma 1.14. $\square$

Lemma 1.16. *If $(Q, \circ) = (Q, \cdot)(\alpha, \beta, \varepsilon)$ and $(Q, \circ)$ is a loop, then there exist elements $a, b \in Q$ such that $\alpha = R_a^{-1}$, $\beta = L_b^{-1}$ [14, 17].*

If $(Q, \circ) = (Q, \cdot)(\varepsilon, \beta, \gamma)$ and $(Q, \circ)$ is a right unipotent loop, then there exist elements $a, b \in Q$ such that $\beta = P_a$, $\gamma = R_b$.

If $(Q, \circ) = (Q, \cdot)(\alpha, \varepsilon, \gamma)$ and $(Q, \circ)$ is a left unipotent loop, then there exist elements $a, b \in Q$ such that $\alpha = P_a^{-1}$, $\gamma = L_b$.

Proof. Let $x \circ y = \alpha x \cdot \beta y$. If $x = 1$, then we have $1 \circ y = y = \alpha 1 \cdot \beta y$. Therefore $L_{\alpha 1} \beta = \varepsilon$, $\beta = L_{\alpha 1}^{-1}$. If we take $y = 1$, then we have $x \circ 1 = x = \alpha x \cdot \beta 1$, $R_{\beta 1} \alpha = \varepsilon$, $\alpha = R_{\beta 1}^{-1}$.

Let $x \circ y = \gamma^{-1}(x \cdot \beta y)$. If $y = 1$, then we have $x = \gamma^{-1}(x \cdot \beta 1)$. Therefore $\gamma x = R_{\beta 1} x$. If $x = y$, then $1 = \gamma^{-1}(x \cdot \beta x)$, $\gamma 1 = x \cdot \beta x$, $P_{\gamma 1} x = \beta x$. Denote $\beta 1$ by b, $\gamma 1$ by a.

Let $x \circ y = \gamma^{-1}(\alpha x \cdot y)$. If $x = 1$, then we have $y = \gamma^{-1}(\alpha 1 \cdot y)$. Therefore $\gamma y = L_{\alpha 1} y$. If $x = y$, then $1 = \gamma^{-1}(\alpha \cdot x)$, $\gamma 1 = \alpha x \cdot x$, $P_{\gamma 1}^{-1} x = \alpha x$. Denote $\alpha 1$ by b, $\gamma 1$ by a. $\square$

Definition 1.17. *Isotopism of the form $(R_a^{-1}, L_b^{-1}, \varepsilon)$ where L_b, R_a are left and right translations of the quasigroup $(Q, \cdot)$ is called LP-isotopism (loop isotopism) [14, 17].*

Theorem 1.18. *Any LP-isotope of a quasigroup $(Q, \cdot)$ is a loop [14, 17].*

If $(Q, \circ) = (Q, \cdot)(\varepsilon, P_a, R_b)$, where $(Q, \cdot)$ is a quasigroup, then $(Q, \circ)$ is a unipotent right loop.

If $(Q, \circ) = (Q, \cdot)(P_a^{-1}, \varepsilon, L_b)$, where $(Q, \cdot)$ is a quasigroup, then $(Q, \circ)$ is a unipotent left loop.

Proof. Case 1. Prove that quasigroup $(Q, \circ)$, where $x \circ y = R_a^{-1} x \cdot L_b^{-1} y$, is a loop. Let $1 = b \cdot a$. If we take $x = 1$, then $1 \circ y = R_a^{-1} ba \cdot L_b^{-1} y = R_a^{-1} R_a b \cdot L_b^{-1} y = b \cdot L_b^{-1} y = L_b L_b^{-1} y = y$.

If we take $y = 1$, then we have $x \circ 1 = R_a^{-1} x \cdot L_b^{-1} ba = R_a^{-1} x \cdot a = R_a R_a^{-1} x = x$. Element 1 is the identity element of the quasigroup $(Q, \circ)$.

Case 2. Prove that quasigroup $(Q, \circ)$, where $x \circ y = R_b^{-1}(x \cdot P_a y)$, is a right unipotent loop. Let $1 = R_b^{-1} a = a/b$. If we take $y = 1$, then $x \circ 1 = R_b^{-1}(x \cdot P_a R_b^{-1} a) = R_b^{-1}(x \cdot b) = x$, since $P_a R_b^{-1} a = b$. Indeed, if $P_a R_b^{-1} a = b$, then $R_b^{-1} a \cdot b = a$, $a = a$. Also we can use identity (2). Therefore element 1 is the right identity element of quasigroup $(Q, \circ)$.

If we take $x = y$, then we have $x \circ x = R_b^{-1}(x \cdot P_a x) = R_b^{-1} a = 1$ since by definition of middle translation $x \cdot P_a x = a$, $(Q, \circ)$ is an unipotent quasigroup.

Case 3. Prove that quasigroup $(Q, \circ)$, where $x \circ y = L_b^{-1}(P_a^{-1} x \cdot y)$, is a left unipotent loop. Let $1 = L_b^{-1} a = b \backslash a$. If we take $x = 1$, then $1 \circ y = L_b^{-1}(P_a^{-1} L_b^{-1} a \cdot y) = L_b^{-1}(b \cdot y) = y$, since $P_a^{-1} L_b^{-1} a = b$. Indeed, if $P_a^{-1} L_b^{-1} a = b$, then $b \cdot L_b^{-1} a = a$, $a = a$. Therefore element 1 is the left identity element of quasigroup $(Q, \circ)$.

If we take $x = y$, then we have $x \circ x = L_b^{-1}(P_a^{-1} x \cdot x) = L_b^{-1} a = 1$ since by definition of middle translation $P_a^{-1} x \cdot x = a$, $(Q, \circ)$ is an unipotent quasigroup. $\square$

Remark 1.19. Instead of direct proof of Theorem 1.18 it is possible to use Corollary 1.15.

There exists well known connection between translations of a quasigroup and translations of its LP-isotopes [84, 47]. In the following lemma we extend this connection.

Lemma 1.20. *If $x \circ y = R_a^{-1} x \cdot L_b^{-1} y$, then $L_x^\circ = L_{R_a^{-1} x} L_b^{-1}$, $R_y^\circ = R_{L_b^{-1} y} R_a^{-1}$, $P_z^\circ = L_b P_z R_a^{-1}$.*

If $x \circ y = R_b^{-1}(x \cdot P_a y)$, then $L_x^\circ = R_b^{-1} L_x P_a$, $R_y^\circ = R_b^{-1} R_{P_a y}$, $P_z^\circ = P_a^{-1} P_{R_b z}$.

If $x \circ y = L_b^{-1}(P_a^{-1} x \cdot y)$, then $L_x^\circ = L_b^{-1} L_{P_a^{-1} x}$, $R_y^\circ = L_b^{-1} R_y P_a^{-1}$, $P_z^\circ = P_{b \cdot z} P_a^{-1}$.

Proof. Case 1. If $x \circ y = R_a^{-1} x \cdot L_b^{-1} y = z$, then $P_z^\circ x = y$, $P_z R_a^{-1} x = L_b^{-1} y$, $L_b P_z R_a^{-1} x = y$, $P_z^\circ x = L_b P_z R_a^{-1} x$.

Case 2. If $x \circ y = R_b^{-1}(x \cdot P_a y) = z$, then $R_y^\circ x = R_b^{-1} R_{P_a y} x$, $P_z^\circ x = y$, $R_b^{-1}(x \cdot P_a y) = z$, $P_{R_b z} x = P_a y$, $P_a^{-1} P_{R_b z} x = y$, $P_z^\circ x = P_a^{-1} P_{R_b z} x$, $L_x^\circ y = R_b^{-1} L_x P_a y$.

Case 3. If $x \circ y = L_b^{-1}(P_a^{-1} x \cdot y) = z$, then $P_z^\circ x = y$, $P_a^{-1} x \cdot y = L_b z$, $P_{L_b z} P_a^{-1} x = y$, $P_z^\circ = P_{b \cdot z} P_a^{-1}$, $L_x^\circ y = L_b^{-1} L_{P_a^{-1} x} y$, $R_y^\circ x = L_b^{-1} R_y P_a^{-1} x$. $\square$

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We shall use the following well known fact.

Lemma 1.21. *If a quasigroup $(Q, \cdot)$ is a group isotope, i.e. $(Q, \cdot) \sim (Q, +)$, where $(Q, +)$ is a group, then any parastrophe of this quasigroup also is a group isotope [90].*

Proof. The proof is based on the fact that any parastrophe of a group is an isotope of this group, i.e. $(Q, +)^\sigma \sim (Q, +)$ [14, Lemma 5.1], [17, p. 53]. If $(Q, \cdot) \sim (Q, +)$, then $(Q, \cdot)^\sigma \sim (Q, +)^\sigma \sim (Q, +)$. $\qquad\square$

1.4 Autotopy. Leakh Theorem

Lemma 1.22. *If quasigroups $(Q, \circ)$ and $(Q, \cdot)$ are isotopic with isotopy T, i.e. $(Q, \circ) = (Q, \cdot)T$, then $Avt(Q, \circ) = T^{-1}Avt(Q, \cdot)T$ ([17], Lemma 1.4).*

Lemma 1.22 allows (up to isomorphism) to reduce the study of autotopy group of a quasigroup to the study autotopy group of an LP-isotope of this quasigroup, i.e. to the study of autotopy group of a loop.

Corollary 1.23. *If H is a subgroup of the group $Avt(Q, \cdot)$, then $T^{-1}HT$ is a subgroup of the group $Avt(Q, \circ)$.*

Proof. The proof follows from Lemma 1.22 and standard algebraic facts [48]. $\qquad\square$

Lemma 1.24. *If T is a quasigroup autotopy, then any its two components define the third component uniquely.*

Proof. If $(\alpha_1, \beta, \gamma)$ and $(\alpha_2, \beta, \gamma)$ are autotopies, then $(\alpha_2^{-1}, \beta^{-1}, \gamma^{-1})$ is an autotopy and $(\alpha_1\alpha_2^{-1}, \beta\beta^{-1}, \gamma\gamma^{-1}) = (\alpha_1\alpha_2^{-1}, \varepsilon, \varepsilon)$ is an autotopy too. We can re-write the last autotopy in such form: $\alpha_1\alpha_2^{-1}x \cdot y = x \cdot y$, then $\alpha_1 = \alpha_2$.

If $(\varepsilon, \varepsilon, \gamma_1\gamma_2)$ is an autotopy, then we have $x \cdot y = \gamma_1\gamma_2^{-1}(x \cdot y)$. If we put in the last equality $y = e(x)$, then we obtain $x = \gamma_1\gamma_2^{-1}x$ for all $x \in Q$, i.e. $\gamma_1 = \gamma_2$. $\qquad\square$

Corollary 1.25. *If two components of a quasigroup autotopy are identity mappings, the the third component also is an identity mapping.*

Proof. The proof follows from Lemma 1.24. $\qquad\square$

There exists [62] more strong result than Lemma 1.24. I.V. Leakh proves this result using geometrical (net theory) approach in more general case than in the following

Theorem 1.26. *Leakh Theorem. Any autotopy $T = (\alpha_1, \alpha_2, \alpha_3)$ of a quasigroup $(Q, \circ)$ is uniquely defined by any autotopy component α_i, $i \in \{1, 2, 3\}$, and by element $b = \alpha_j a$, where a is any fixed element of set Q, $i \neq j$ [62].*

Proof. Case 1. $i = 1, j = 2$. If we have autotopies $(\alpha, \beta_1, \gamma_1)$ and $(\alpha, \beta_2, \gamma_2)$ such that $\beta_1 a = \beta_2 a = b$, then we have $\alpha x \circ \beta_1 a = \gamma_1(x \circ a)$ and $\alpha x \circ \beta_2 a = \gamma_2(x \circ a)$. Since the left sides of the last equalities are equal, then we have $\gamma_1(x \circ a) = \gamma_2(x \circ a)$, $\gamma_1 R_a x = \gamma_2 R_a x$, $\gamma_1 = \gamma_2$ and by Lemma 1.24 $\beta_1 = \beta_2$.

Case 2. $i = 1, j = 3$. Suppose there exist autotopies $(\alpha, \beta_1, \gamma_1)$ and $(\alpha, \beta_2, \gamma_2)$ such that $\gamma_1 a = \gamma_2 a = b$ for some fixed element $a \in Q$. Since $(Q, \circ)$ is a quasigroup, then for any element $x \in Q$ there exists a unique element $x' \in Q$ such that $x \circ x' = a$. Using the concept of middle quasigroup translation we can re-write the last equality in the form $P_a x = x'$ and say that P_a is a permutation of the set Q.

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For all pairs x, x' we have $\alpha x \circ \beta_1 x' = \gamma_1(x \circ x') = b$ and $\alpha x \circ \beta_2 x' = \gamma_2(x \circ x') = b$. Since the right sides of the last equalities are equal we have $\alpha x \circ \beta_1 x' = \alpha x \circ \beta_2 x'$, $\beta_1 x' = \beta_2 x'$ for all $x' \in Q$. The variable x' takes all values from the set Q since P_a is a permutation of the set Q. Therefore $\beta_1 = \beta_2$ and by Lemma 1.24 $\gamma_1 = \gamma_2$.

All other cases are proved in the similar way with Cases 1 and 2. $\square$

Lemma 1.27. *Any autotopy (α, β, γ) of a loop $(Q, \cdot)$ has the form $(R_{\beta 1}^{-1}\gamma, L_{\alpha 1}^{-1}\gamma, \gamma)$, where $\alpha 1 \cdot \beta 1 = 1$.*

Proof. If we put in equality $\alpha x \cdot \beta y = \gamma(xy)$ $x = y = 1$, then $\alpha 1 \cdot \beta 1 = 1$. If $x = 1$, then $\alpha 1 \cdot \beta y = \gamma y$, $\beta = L_{\alpha 1}^{-1}\gamma$. If $y = 1$, then $\alpha x \cdot \beta 1 = \gamma x$, $\alpha = R_{\beta 1}^{-1}\gamma$. $\square$

Theorem 1.28. *The order of autotopy group of a finite quasigroup Q of order n is a divisor of the number $n! \cdot n$.*

Proof. The proof follows from Lemma 1.22 (we can prove loop case), Lemma 1.24 and Lemma 1.27 (we can take the second and third components of loop autotopy). $\square$

Also by the proving of Theorem 1.28 it is possible to use Leakh Theorem (Theorem 1.26).

Example 1.29. The order of autotopy group of the group $Z_2 \times Z_2$ is equal to $4 \cdot 4 \cdot 6 = 4! \cdot 4$, i.e. in this case autotopy group is equal to the upper bound.

Remark 1.30. There exist quasigroups (loops) with identity autotopy group [37]. In this case autotopy group is of minimal order.

1.5 Isostrophism

Isostrophy (synonymous with isostrophism) of a quasigroup is a transformation that is a combination of parastrophy and isotopy, i.e. isostrophe image of a quasigroup (Q, A) is parastrophe image of its isotopic image or, vice versa, isostrophe image of a quasigroup (Q, A) is isotopic image of its parastrophe (Definition 1.33). Therefore, there exists a possibility to define isostrophy by at least in two ways.

If $T = (\alpha_1, \alpha_2, \alpha_3)$ is an isotopy, σ is a parastrophy of a quasigroup (Q, A), then we shall denote by T^σ the triple $(\alpha_{\sigma^{-1}1}, \alpha_{\sigma^{-1}2}, \alpha_{\sigma^{-1}3})$.

In order to give some properties of quasigroup isostrophisms we need the following

Lemma 1.31. $(AT)^\sigma = A^\sigma T^\sigma$, $(T_1 T_2)^\sigma = T_1^\sigma T_2^\sigma$ [14, 16].

Remark 1.32. It is possible to define action of a parastrophy σ on an isotopy $T = (\alpha_1, \alpha_2, \alpha_3)$ also in the following (more standard) way: $T^\sigma = (\alpha_{\sigma 1}, \alpha_{\sigma 2}, \alpha_{\sigma 3})$. In this case Lemma 1.31 also is valid.

Recall, if (Q, A) is a binary groupoid, then $x_3 = A(x_1, x_2)$ (Definition 1.9).

Definition 1.33. *A quasigroup (Q, B) is an isostrophic image of a quasigroup (Q, A) if there exists a collection of permutations $(\sigma, (\alpha_1, \alpha_2, \alpha_3)) = (\sigma, T)$, where $\sigma \in S_3$, $T = (\alpha_1, \alpha_2, \alpha_3)$ and $\alpha_1, \alpha_2, \alpha_3$ are permutations of the set Q such that $B(x_1, x_2) = A(x_1, x_2)(\sigma, T) = (A^\sigma(x_1, x_2))T = \alpha_3^{-1}A(\alpha_1 x_{\sigma^{-1}1}, \alpha_2 x_{\sigma^{-1}2})$ for all $x_1, x_2 \in Q$ [81, 82].*

Remark 1.34. There exist formally other but in some sense equivalent definitions of isostrophy and autostrophy. See, for example, [51, 52].

A collection of permutations $(\sigma, (\alpha_1, \alpha_2, \alpha_3)) = (\sigma, T)$ will be called an *isostrophy* of a quasi-group (Q, A).

Often an isostrophy (σ, T) is called σ-isostrophy T or isostrophy of type σ. We can re-write the equality from Definition 1.33 in the form $A^\sigma T = B$, where $T = (\alpha_1, \alpha_2, \alpha_3)$. It is clear that ε-isostrophy is called an *isotopy*.

Probably R. Artzy was the first who has given algebraic definition of isostrophy [6]. For n-ary quasigroups concept of isostrophy is studied in [16]. Isostrophy has clear geometrical (net) motivation [70, 8]. See also [62].

Let (σ, T) and (τ, S) be some isostrophisms, $T = (\alpha_1, \alpha_2, \alpha_3)$, $S = (\beta_1, \beta_2, \beta_3)$. Then

$$(\sigma, T)(\tau, S) = (\sigma\tau, T^\tau S) \tag{7}$$

We can write the equality $B = A(\sigma\tau, T^\tau S)$ in more details: (the right record of maps)

$$B(x_1, x_2, x_3) = A(x_{1(\tau^{-1}\sigma^{-1})}\alpha_{1\tau^{-1}}\beta_1, x_{2(\tau^{-1}\sigma^{-1})}\alpha_{2\tau^{-1}}\beta_2, x_{3(\tau^{-1}\sigma^{-1})}\alpha_{3\tau^{-1}}\beta_3). \tag{8}$$

If we shall use the left record of maps, then

$$B(x_1, x_2, x_3) = A(\beta_1\alpha_{\tau^{-1}1}x_{\sigma^{-1}(\tau^{-1}1)}, \beta_2\alpha_{\tau^{-1}2}x_{\sigma^{-1}(\tau^{-1}2)}, \beta_3\alpha_{\tau^{-1}3}x_{\sigma^{-1}(\tau^{-1}3)}). \tag{9}$$

Lemma 1.35. *If* $(\sigma, T) = (\sigma, (\alpha_1, \alpha_2, \alpha_3))$ *is an isostrophy, then*

$$(\sigma, T)^{-1} = (\sigma^{-1}, (T^{-1})^{\sigma^{-1}}) = (\sigma^{-1}, (\alpha_{\sigma1}^{-1}, \alpha_{\sigma2}^{-1}, \alpha_{\sigma3}^{-1})) \tag{10}$$

Proof. Indeed, $(\sigma, T)^{-1} = (\sigma^{-1}, (T^{-1})^{\sigma^{-1}})$, $(T^{-1})^{\sigma^{-1}} = (\alpha_1^{-1}, \alpha_2^{-1}, \alpha_3^{-1})^{\sigma^{-1}} = (\alpha_{\sigma1}^{-1}, \alpha_{\sigma2}^{-1}, \alpha_{\sigma3}^{-1})$. Then

$$(\sigma, T)(\sigma^{-1}, (T^{-1})^{\sigma^{-1}}) = (\sigma\sigma^{-1}, T^{\sigma^{-1}}(T^{-1})^{\sigma^{-1}}) = (\varepsilon, \varepsilon),$$
$$(\sigma^{-1}, (T^{-1})^{\sigma^{-1}})(\sigma, T) = (\sigma^{-1}\sigma, T^{-1}T) = (\varepsilon, \varepsilon).$$

$\square$

If $B = A$, then isostrophism is called an *autostrophism (autostrophy)*. Denote by $Aus\,(Q, A)$ the group of all autostrophisms of a quasigroup (Q, A).

We can generalize Lemma 1.22 and similarly prove the following

Lemma 1.36. *If quasigroups* $(Q, \circ)$ *and* $(Q, \cdot)$ *are isostrophic with an isostrophy* T, *i.e.* $(Q, \circ) = (Q, \cdot)T$, *then* $Aus(Q, \circ) = T^{-1}Aus(Q, \cdot)T$.

Proof. The proof repeats the proof of Lemma 1.4 from [17]. Let $S \in Aus(Q, \circ)$. Then $(Q, \cdot)T = (Q, \cdot)TS$, $(Q, \cdot) = (Q, \cdot)TST^{-1}$,

$$TAus(Q, \circ)T^{-1} \subseteq Aus(Q, \cdot). \tag{11}$$

If $(Q, \circ) = (Q, \cdot)T$, then $(Q, \circ)T^{-1} = (Q, \cdot)$, expression (11) takes the form $T^{-1}Aus(Q, \cdot)T \subseteq Aus(Q, \circ)$,

$$TAus(Q, \circ)T^{-1} \supseteq Aus(Q, \cdot). \tag{12}$$

Comparing (11) and (12) we obtain $Aus(Q, \circ) = T^{-1}Aus(Q, \cdot)T$. $\square$

Corollary 1.37. *If quasigroups* $(Q, \circ)$ *and* $(Q, \cdot)$ *are isostrophic with isostrophy* T, *i.e.* $(Q, \circ) = (Q, \cdot)T$, *then* $Avt(Q, \circ) = T^{-1}Avt(Q, \cdot)T$.

Proof. The proof repeats the proof of Lemma 1.36. We only notice, if $S \in Avt(Q, \circ)$, then $T^{-1}ST \in Avt(Q, \cdot)$. $\square$

Corollary 1.38. *If* T *is an autostrophy of a quasigroup* $(Q, \cdot)$, *i.e.* $(Q, \cdot) = (Q, \cdot)T$, *then* $Avt(Q, \cdot) = T^{-1}Avt(Q, \cdot)T$.

Proof. The proof follows from Lemma 1.36. $\square$

1.6   Group action

We shall denote by S_Q the symmetric group of all bijections (permutations) of the set Q.

We recall some definitions from [42, 48].

Definition 1.39. *A group G acts on a set M if for any pair of elements (g, m), $g \in G, m \in M$, an element $(gm) \in M$ is defined. Moreover, $g_1(g_2(m)) = (g_1 g_2)m$ and $em = m$ for all $m \in M$, $g_1, g_2 \in G$. Here e is the identity element of the group G.*

The set $Gm = \{gm \,|\, g \in G\}$ is called an orbit of element m. For every m in M, we define the stabilizer subgroup of m as the set of all elements in G that fix m: $G_m = \{g \,|\, gm = m\}$

The orbits of any two elements of the set M coincide or are not intersected. Then the set M is divided into a set of non-intersected orbits. In other words, if we define on the set M a binary relation $\sim$ as:

$$m_1 \sim m_2 \text{ if and only if there exists } g \in G \text{ such that } m_2 = gm_1,$$

then $\sim$ is an equivalence relation on the set M.

Every orbit is an invariant subset of M on which G acts transitively. The action of G on M is transitive if and only if all elements are equivalent, meaning that there is only one orbit.

A partition θ of the set M on disjoint subsets $\theta(x)$, $x \in M$ is called a partition on blocks relatively the group G, if for any $\theta(a)$ and any $g \in G$ there exists a subset $\theta(b)$ such that $g\theta(a) = \theta(b)$. It is obviously that there exist trivial partition of the set M, namely, partition into one-element blocks and partition into unique block.

If there does not exist a partition of the set M into non-trivial blocks, then the group G is called primitive.

Definition 1.40. *The action of G on M is called:*

1. *faithful (or effective) if for any two distinct $g, h \in G$ there exists an $x \in M$ such that $g(x) \neq h(x)$; or equivalently, if for any $g \neq e \in G$ there exists an $x \in M$ such that $g(x) \neq x$. Intuitively, different elements of G induce different permutations of M;*

2. *free (or semiregular) if for any two distinct $g, h \in G$ and all $x \in M$ we have $g(x) \neq h(x)$; or equivalently, if $g(x) = x$ for some x then $g = e$;*

3. *regular (or simply transitive) if it is both transitive and free; this is equivalent to saying that for any two x, y in M there exists precisely one g in G such that $g(x) = y$. In this case, M is known as a principal homogeneous space for G or as a G-torsor [102].*

Denote the property of a set of permutations $\{p_1, p_2, \ldots, p_m\}$ of an m-element set Q "$p_i p_j^{-1}$ $(i \neq j)$ leaves no variable unchanged"[66] as the τ-property. An m-tuple of permutations T can also have the τ-property. We shall call the m-tuple T as a τ-m-tuple.

In [66], in fact, Mann proves the following

Theorem 1.41. *A set $P = \{p_1, p_2, \ldots, p_m\}$ of m permutations of a finite set Q of order m defines Cayley table of a quasigroup if and only if P has the τ-property.*

A permutation α of a finite non-empty set Q which leaves no elements of the set Q unchanged will be called a *fixed point free permutation*.

Definition 1.42. *A set M of maps of the set Q into itself is called simply transitive (more precise, the set M acts on the set Q simply transitively) if for any pair of elements x, y of the set Q there exists a unique element μ_j of the set M such that $\mu_j(x) = y$.*

In Definition 1.42 we do not suppose that the set M is a group. Notice that concepts from Definition 1.40 are suitable for the set M.

Theorem 1.43. *A set $T = \{p_1, p_2, \ldots, p_n\}$ of n permutations of a finite set Q of order n is simply transitive if and only if the set T has the τ-property [66].*

Lemma 1.44. *If $(Q, \cdot)$ is a quasigroup, then the sets $\mathbb{L}, \mathbb{R}, \mathbb{P}$ of all left, right, middle translations of quasigroup $(Q, \cdot)$ are simply transitive sets of permutations of the set Q.*

Proof. Indeed, if a, b are fixed elements of the set Q, then there exists an unique element $x \in Q$ such that $x \cdot a = b$, there exists an unique element $y \in Q$ such that $a \cdot y = b$, there exists an unique element $z \in Q$ such that $a \cdot b = z$. We establish the following bijections between elements of the sets Q and elements of the sets $\mathbb{L}(Q, \cdot), \mathbb{R}(Q, \cdot), \mathbb{P}(Q, \cdot)$: $\varphi_1 : x \leftrightarrow L_x$, $\varphi_2 : x \leftrightarrow R_x$, $\varphi_3 : x \leftrightarrow P_x$. Therefore there exists an unique element $L_x \in \mathcal{L}(Q, \cdot)$ such that $L_x a = b$, there exists an unique element $R_y \in \mathcal{R}(Q, \cdot)$ such that $R_y a = b$, there exists an unique element $P_z \in \mathcal{P}(Q, \cdot)$ such that $P_z a = b$. $\square$

Definition 1.45. *The centralizer of a set S of a group G (written as $C_G(S)$) is the set of elements of G which commute with any element of S; in other words, $C_G(S) = \{x \in G \mid xa = ax \text{ for all } a \in S\}$ [48, 46].*

Definition 1.46. *The normalizer of a set S in a group G, written as $N_G(S)$, is defined as $N_G(S) = \{x \in G \mid xS = Sx\}$.*

The following theorem is called NC-theorem [101].

Theorem 1.47. *$C(S)$ is always a normal subgroup of $N(S)$ [39, 101].*

Proof. The proof is taken from [101]. If c is in $C(S)$ and n is in $N(S)$, we have to show that $n^{-1}cn$ is in $C(S)$. To that end, pick s in S and let $t = nsn^{-1}$. Then t is in S, so therefore $ct = tc$. Then note that $ns = tn$; and $n^{-1}t = sn^{-1}$. So $(n^{-1}cn)s = (n^{-1}c)tn = n^{-1}(tc)n = (sn^{-1})cn = s(n^{-1}cn)$ which is what we needed. $\square$

2 Garrison's nuclei and A-nuclei

2.1 Definitions of nuclei and A-nuclei

We recall standard Garrison's [43] definition of quasigroup nuclei.

Definition 2.1. *Let $(Q, \circ)$ be a quasigroup. Then $N_l = \{a \in Q \mid (a \circ x) \circ y = a \circ (x \circ y)\}$, $N_r = \{a \in Q \mid (x \circ y) \circ a = x \circ (y \circ a)\}$ and $N_m = \{a \in Q \mid (x \circ a) \circ y = x \circ (a \circ y)\}$ are respectively its left, right and middle nuclei [43, 14, 70].*

Remark 2.2. Garrison names an element of a middle quasigroup nucleus as a *center element* [43].

Definition 2.3. *Let $(Q, \cdot)$ be a quasigroup. Nucleus is given by $N = N_l \cap N_r \cap N_m$ [70].*

The importance of Garrison's quasigroup nuclei is in the fact that N_l, N_r and N_m all are subgroups of a quasigroup $(Q, \cdot)$ [43].

The weakness of Garrrison's definition is in the fact that, if a quasigroup $(Q, \cdot)$ has a non-trivial left nucleus, then $(Q, \cdot)$ is a left loop, i.e. $(Q, \cdot)$ has a left identity element; if a quasigroup $(Q, \cdot)$ has a non-trivial right nucleus, then $(Q, \cdot)$ is a right loop, i.e. $(Q, \cdot)$ has a right identity

element; if a quasigroup $(Q, \cdot)$ has a non-trivial middle nucleus, then $(Q, \cdot)$ is a loop, i.e. $(Q, \cdot)$ has an identity element ([43]; [70], I.3.4. Theorem).

It is well known connection between autotopies and nuclei [10, 12, 14, 53].

Namely the set of autotopies of the form (L_a, ε, L_a) of a quasigroup $(Q, \circ)$ corresponds to left nucleus of $(Q, \circ)$ and vice versa.

Similarly, the set of autotopies of the form (ε, R_a, R_a) of a quasigroup $(Q, \circ)$ corresponds to right nucleus of $(Q, \circ)$, the set of autotopies of the form $(R_a, L_a^{-1}, \varepsilon)$ of a quasigroup $(Q, \circ)$ corresponds to middle nucleus of $(Q, \circ)$.

It is easy to see that from Garrison definition of left nucleus of a loop $(Q, \cdot)$ it follows that $R_{xy}^{-1} R_y R_x a = a$ for all $x, y \in Q$ and all $a \in N_l$. Permutations of the form $R_{xy}^{-1} R_y R_x$ generate right multiplication group of $(Q, \cdot)$. It is clear that any element of left nucleus is invariant relatively to any element of the group $\langle R_{xy}^{-1} R_y R_x \,|\, x, y \in Q \rangle$.

Similarly from Garrison definition of right nucleus of a loop $(Q, \cdot)$ it follows that $L_{xy}^{-1} L_x L_y a = a$ for all $x, y \in Q$ and all $a \in N_l$. Permutations of the form $L_{xy}^{-1} L_x L_y$ generate right multiplication group of $(Q, \cdot)$. It is clear that any element of left nucleus is invariant relatively to any element of the group $\langle L_{xy}^{-1} L_x L_y \,|\, x, y \in Q \rangle$.

For middle nucleus situation is slightly other and of course Garrison was right when called elements of middle loop nucleus as central elements.

P.I. Gramma [44], M.D. Kitaroagă [58], and G.B. Belyavskaya [21, 22, 23, 24] generalized on quasigroup case concepts of nuclei and center using namely this nuclear property. G.B. Belyavskaya obtained in this direction the most general results.

In [52] the following definition is given.

Definition 2.4. *The set of all autotopisms of the form $(\alpha, \varepsilon, \gamma)$ of a quasigroup $(Q, \circ)$, where ε is the identity mapping, is called the left autotopy nucleus (left A-nucleus) of quasigroup $(Q, \circ)$.*

Similarly, the sets of autotopisms of the forms $(\alpha, \beta, \varepsilon)$ and $(\varepsilon, \beta, \gamma)$ form the middle and right A-nuclei of $(Q, \circ)$. We shall denote these three sets of mappings by N_l^A, N_m^A and N_r^A respectively.

Using Definition 2.4 we can say that to the elements of left Kitaroagă nucleus of a quasigroup $(Q, \cdot)$ correspond to autotopies of the form $(L_a L_h^{-1}, \varepsilon, L_a L_h^{-1})$, where $a \in Q$ (in fact $a \in N_l$ in Kitaroagă sense), h is a fixed element of the set Q [58].

Autotopies of the form $(L_a L_h^{-1}, \varepsilon, L_{a \cdot e(h)} L_h^{-1})$, where $a \in Q$ (in fact $a \in N_l$ in Belyavskaya sense) correspond to the elements of left Belyavskaya nucleus [21].

Remark 2.5. Often, by a generalization of some objects or concepts, we not only win in generality, but also lose some important properties of generalized objects or concepts.

The weakness of Definition 2.4 is in the fact that it is not easy to define A-nucleus similarly to quasigroup nucleus as an intersection of left, right and middle quasigroup A-nuclei.

Lemma 2.6. *[52].*

1. *The first components of the autotopisms of any subgroup $K(N_l^A)$ of N_l^A themselves form a group $K_1(N_l^A)$.*

2. *The third components of the autotopisms of any subgroup $K(N_l^A)$ of N_l^A themselves form a group $K_3(N_l^A)$.*

3. *The first components of the autotopisms of any subgroup $K(N_m^A)$ of N_m^A themselves form a group $K_1(N_m^A)$.*

4. *The second components of the autotopisms of any subgroup $K(N_m^A)$ of N_m^A themselves form a group $K_2(N_l^A)$.*

5. *The second components of the autotopisms of any subgroup $K(N_r^A)$ of N_r^A themselves form a group $K_2(N_r^A)$.*

6. *The third components of the autotopisms of any subgroup $K(N_r^A)$ of N_r^A themselves form a group $K_3(N_r^A)$.*

Proof. Case 1. Let $(\alpha_1, \varepsilon, \gamma_1), (\alpha_2, \varepsilon, \gamma_2) \in K(N_l^A)$. Then $\alpha_1 x \circ y = \gamma_1(x \circ y)$ and $\alpha_2 x \circ y = \gamma_2(x \circ y)$ for all $x, y \in Q$. From the first equation, $\alpha_1(\alpha_2 x) \circ y = \gamma_1(\alpha_2 x \circ y) = \gamma_1 \gamma_2(x \circ y)$ by virtue of the second equation. Thus, $(\alpha_1 \alpha_2, \varepsilon, \gamma_1 \gamma_2) \in K(N_l^A)$.

Let $x = \alpha_1^{-1} u$. Then, $\alpha_1 x \circ y = \gamma_1(x \circ y) \Rightarrow u \circ y = \gamma_1(\alpha_1^{-1} u \circ y) \Rightarrow \alpha_1^{-1} u \circ y = \gamma_1^{-1}(u \circ y)$ so $(\alpha_1^{-1}, \varepsilon, \gamma_1^{-1}) \in K(N_l^A)$.

Clearly, $(\varepsilon, \varepsilon, \varepsilon) \in K(N_l^A)$. Hence, ε is a first component and, if α_1 and α_2 are first components so are $\alpha_1 \alpha_2$ and α_1^{-1}.

Other cases are proved in similar way with Case 1. □

From Lemma 2.6 it follows that the first, second and third components of N_l^A, N_m^A and N_r^A each form groups. For brevity, we shall denote these nine groups by $_1N_l^A$, $_2N_l^A$, $_3N_l^A$, $_1N_m^A$, $_2N_m^A$, $_3N_m^A$, $_1N_r^A$, $_2N_r^A$ and $_3N_r^A$.

Next two lemmas demonstrate that A-nuclei have some advantages in comparison with Garrison's nuclei.

Lemma 2.7. *In any quasigroup Q its left, right, middle A-nucleus is normal subgroup of the group $Avt(Q)$.*

Proof. Let $(\mu, \varepsilon, \nu) \in N_l^A$, $(\alpha, \beta, \gamma) \in Avt(Q)$. Then

$$(\alpha^{-1}, \beta^{-1}, \gamma^{-1})(\mu, \varepsilon, \nu)(\alpha, \beta, \gamma) = (\alpha^{-1} \mu \alpha, \varepsilon, \gamma^{-1} \nu \gamma) \in N_l^A$$

The proofs for right and middle nucleus are similar. □

Lemma 2.8. *If quasigroups $(Q, \cdot)$ and $(Q, \circ)$ are isotopic, then these quasigroups have isomorphic autotopy nuclei and isomorphic components of the autotopy nuclei, i.e. $N_l^A(Q, \cdot) \cong N_l^A(Q, \circ)$, $_1N_l^A(Q, \cdot) \cong {}_1N_l^A(Q, \circ)$, $_3N_l^A(Q, \cdot) \cong {}_3N_l^A(Q, \circ)$, and so on.*

Proof. We can use Lemmas 1.22 and 2.6. □

Corollary 2.9. *If a quasigroup $(Q, \cdot)$ is an isotope of a group $(Q, +)$, then all its A-nuclei and components of A-nuclei are isomorphic with the group $(Q, +)$.*

Proof. It is clear that in any group all its A-nuclei and components of A-nuclei are isomorphic with the group $(Q, +)$. Further we can apply Lemma 2.8. □

Lemma 2.10. *Isomorphic components of A-nuclei of a quasigroup $(Q, \cdot)$ act on the set Q in such manner that the numbers and lengths of orbits by these actions are equal.*

Proof. In Lemma 2.8 it is established that A-nuclei and the same components of A-nuclei are isomorphic in pairs. Notice all A-nuclei of a quasigroup $(Q, \cdot)$ are subgroups of the group $S_Q \times S_Q \times S_Q$. Components of these A-nuclei are subgroups of the group S_Q.

If two components of A-nuclei, say B and C are isomorphic, then this isomorphism is an isomorphism of permutation groups which act on the set Q. Isomorphic permutation groups are called *similar* [48, p. 111].

Let $\psi B = C$, where B, C are isomorphic components of A-nuclei, ψ is an isomorphism. Then we can establish a bijection φ of the set Q such that $\psi(g)(\varphi(m)) = \varphi(g(m))$ for all $m \in Q, g \in B$ [48, p. 111]. □

2.2 A-nuclei and isosotrophy

In this section we find connections between components of A-nuclei of a quasigroup $(Q, \cdot)$ and its isostrophic images of the form $(Q, \circ) = (Q, \cdot)((\sigma)(\alpha, \beta, \gamma))$, where $\sigma \in S_3$, $\alpha, \beta, \gamma \in S_Q$.

We omit the symbol of autotopy nuclei (the symbol A) and put on its place the symbols of binary operations "$\circ$" and "$\cdot$" respectively. Denote isostrophy $(\sigma, (\alpha, \beta, \gamma))$ by (σ, T) for all cases.

Lemma 2.11. *If quasigroup $(Q, \circ)$ is isostrophic image of quasigroup $(Q, \cdot)$ with an isostrophy $S = ((12), T)$, i.e. $(Q, \circ) = (Q, \cdot)S$, then $N_l(Q, \circ) = S^{-1}N_r(Q, \cdot)S$, ${}_1N_l(Q, \circ) = \alpha^{-1}{}_2N_r(Q, \cdot)\alpha$, ${}_3N_l(Q, \circ) = \gamma^{-1}{}_3N_r(Q, \cdot)\gamma$.*

Proof. The proof repeats the proof of Lemma 1.36. Let $K \in N_l(Q, \circ)$. Then $(Q, \cdot)S = (Q, \cdot)SK$, $(Q, \cdot) = (Q, \cdot)SKS^{-1}$,

$$SN_l(Q, \circ)S^{-1} \subseteq N_r(Q, \cdot), \tag{13}$$

since

$$((12), (\alpha, \beta, \gamma))\,(\varepsilon, ({}_1N_l^\circ, \varepsilon, {}_3N_l^\circ))\,((12), (\beta^{-1}, \alpha^{-1}, \gamma^{-1})) = (\varepsilon, (\varepsilon, \alpha\,{}_1N_l^\circ\alpha^{-1}, \gamma\,{}_3N_l^\circ\gamma^{-1})).$$

If $(Q, \circ) = (Q, \cdot)S$, then $(Q, \circ)S^{-1} = (Q, \cdot)$, expression (13) takes the form

$$S^{-1}N_r(Q, \cdot)S \subseteq N_l(Q, \circ). \tag{14}$$

Indeed,

$$((12), (\beta^{-1}, \alpha^{-1}, \gamma^{-1}))\,(\varepsilon, (\varepsilon, {}_2N_r, {}_3N_r))\,((12), (\alpha, \beta, \gamma)) = (\varepsilon, (\alpha^{-1}{}_2N_r\alpha,\ \varepsilon,\ \gamma^{-1}{}_3N_r\gamma)).$$

We can rewrite expression (14) in such form

$$N_r(Q, \cdot) \subseteq SN_l(Q, \circ)S^{-1}. \tag{15}$$

Comparing (13) and (15) we obtain $N_r(Q, \cdot) = SN_l(Q, \circ)S^{-1}$. Therefore $\alpha\,{}_1N_l^\circ\alpha^{-1} = {}_2N_r$, ${}_1N_l^\circ = \alpha^{-1}{}_2N_r\alpha$, ${}_3N_l^\circ = \gamma^{-1}{}_3N_r\gamma$. $\qquad\square$

All other analogs of Lemma 2.11 are proved in the similar way.

In Table 3 connections between components of A-nuclei of a quasigroup $(Q, \cdot)$ and its isostrophic images of the form $(Q, \circ) = (Q, \cdot)((\sigma)(\alpha, \beta, \gamma))$, where $\sigma \in S_3$, $\alpha, \beta, \gamma \in S_Q$ are collected.

Table 3

	(ε, T)	$((12), T)$	$((13), T)$	$((23), T)$	$((132), T)$	$((123), T)$
$_1N_l^\circ$	$\alpha^{-1}{}_1N_l\alpha$	$\alpha^{-1}{}_2N_r\alpha$	$\alpha^{-1}{}_3N_l\alpha$	$\alpha^{-1}{}_1N_m\alpha$	$\alpha^{-1}{}_2N_m\alpha$	$\alpha^{-1}{}_3N_r\alpha$
$_3N_l^\circ$	$\gamma^{-1}{}_3N_l\gamma$	$\gamma^{-1}{}_3N_r\gamma$	$\gamma^{-1}{}_1N_l\gamma$	$\gamma^{-1}{}_2N_m\gamma$	$\gamma^{-1}{}_1N_m\gamma$	$\gamma^{-1}{}_2N_r\gamma$
$_2N_r^\circ$	$\beta^{-1}{}_2N_r\beta$	$\beta^{-1}{}_1N_l\beta$	$\beta^{-1}{}_2N_m\beta$	$\beta^{-1}{}_3N_r\beta$	$\beta^{-1}{}_3N_l\beta$	$\beta^{-1}{}_1N_m\beta$
$_3N_r^\circ$	$\gamma^{-1}{}_3N_r\gamma$	$\gamma^{-1}{}_3N_l\gamma$	$\gamma^{-1}{}_1N_m\gamma$	$\gamma^{-1}{}_2N_r\gamma$	$\gamma^{-1}{}_1N_l\gamma$	$\gamma^{-1}{}_2N_m\gamma$
$_1N_m^\circ$	$\alpha^{-1}{}_1N_m\alpha$	$\alpha^{-1}{}_2N_m\alpha$	$\alpha^{-1}{}_3N_r\alpha$	$\alpha^{-1}{}_1N_l\alpha$	$\alpha^{-1}{}_2N_r\alpha$	$\alpha^{-1}{}_3N_l\alpha$
$_2N_m^\circ$	$\beta^{-1}{}_2N_m\beta$	$\beta^{-1}{}_1N_m\beta$	$\beta^{-1}{}_2N_r\beta$	$\beta^{-1}{}_3N_l\beta$	$\beta^{-1}{}_3N_r\beta$	$\beta^{-1}{}_1N_l\beta$

If $(Q, \circ) = (Q, \cdot)((123)(\alpha, \beta, \gamma))$, then from Table 3 we have ${}_2N_r^A(Q, \circ) = \beta^{-1}{}_1N_m^A\beta(Q, \cdot)$.

Theorem 2.12. *Isostrophic quasigroups have isomorphic A-nuclei.*

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Proof. The proof follows from Table 3. □

Remark 2.13. From the proof of Lemma 2.11 it follows that Table 3 can be used for the finding of connections between concrete components of A-nuclei. For example, if $(Q, \circ) = (Q, \cdot)((23)(\alpha, \beta, \gamma))$, then from Table 3 we have ${}_2N_r^A(Q, \circ) = \beta^{-1}{}_3N_r^A\beta(Q, \cdot)$. Therefore, if $\rho \in {}_3N_r^A(Q, \cdot)$, then there exists an element $\mu \in {}_2N_r^A(Q, \circ)$ such that $\mu = \beta^{-1}\rho\beta$ and vice versa, if $\rho \in {}_2N_r^A(Q, \circ)$, then there exists an element $\mu \in {}_3N_r^A(Q, \cdot)$ such that $\rho = \beta^{-1}\mu\beta$.

2.3 Components of A-nuclei and identity elements

There exist some connection between components of A-nuclei and local identity elements in any quasigroup.

Lemma 2.14.   1. If $(Q, \cdot)$ is a quasigroup and $(\alpha, \varepsilon, \gamma) \in Avt(Q, \cdot)$, then $\alpha f(x) \cdot x = \gamma x$, $\alpha x \cdot e(x) = \gamma x$, $\alpha x \cdot x = \gamma s(x)$ for all $x \in Q$.

2. If $(Q, \cdot)$ is a quasigroup and $(\varepsilon, \beta, \gamma) \in Avt(Q, \cdot)$, then $f(x) \cdot \beta x = \gamma x$, $x \cdot \beta e(x) = \gamma x$, $x \cdot \beta x = \gamma s(x)$ for all $x \in Q$.

3. If $(Q, \cdot)$ is a quasigroup and $(\alpha, \beta, \varepsilon) \in Avt(Q, \cdot)$, then $\alpha f(x) \cdot \beta x = x$, $\alpha x \cdot \beta e(x) = x$, $\alpha x \cdot \beta x = s(x)$ for all $x \in Q$.

Proof. Case 1. If we put $x = f(y)$, then we obtain $\alpha f(y) \cdot y = \gamma y$. If we put $y = e(x)$, then we obtain $\alpha(x) \cdot e(x) = \gamma x$. If we put $x = y$, then we obtain $\alpha x \cdot x = \gamma s(x)$.

Cases 2 and 3 are proved similarly. □

Corollary 2.15.   1. If $(Q, \cdot)$ is an idempotent quasigroup and $(\alpha, \varepsilon, \gamma) \in Avt(Q, \cdot)$, then $\alpha x \cdot x = \gamma x$, $\alpha x = P_{\gamma x}^{-1}$ for all $x \in Q$.

2. If $(Q, \cdot)$ is an idempotent quasigroup and $(\varepsilon, \beta, \gamma) \in Avt(Q, \cdot)$, then $x \cdot \beta x = \gamma x$, $\beta x = P_{\gamma x}x$ for all $x \in Q$.

3. If $(Q, \cdot)$ is an idempotent quasigroup and $(\alpha, \beta, \varepsilon) \in Avt(Q, \cdot)$, then $\alpha x \cdot \beta x = x$, $P_x \alpha x = \beta x$ for all $x \in Q$.

Proof. The proof follows from Lemma 2.14 and the fact that in an idempotent quasigroup $(Q, \cdot)$ the maps f, e, s all are equal to identity permutation of the set Q. Further we have $P_{\gamma x}\alpha x = x$, $\alpha x = P_{\gamma x}^{-1}$ (Case 1). From $x \cdot \beta x = \gamma x$ we have $P_{\gamma x}x = \beta x$ in Case 2. From $\alpha x \cdot \beta x = x$ we have $P_x \alpha x = \beta x$ in Case 3. □

Remark 2.16. We can rewrite also:
equality $\alpha x \cdot x = \gamma x$ in the form $x \cdot \alpha^{-1}x = \gamma\alpha^{-1}x$, $\alpha\gamma^{-1}x \cdot \gamma^{-1}x = x$ (Case 1);
equality $x \cdot \beta x = \gamma x$ in the form $\beta^{-1}x \cdot x = \gamma\beta^{-1}x$, $\gamma^{-1}x \cdot \beta\gamma^{-1}x = x$ (Case 2);
equality $\alpha x \cdot \beta x = x$ in the form $x \cdot \beta\alpha^{-1}x = \alpha^{-1}x$, $\alpha\beta^{-1}x \cdot x = \beta^{-1}x$ (Case 3).

Corollary 2.17.   1. Let $(Q, \cdot)$ be a left loop.

   (a) If $(\alpha, \varepsilon, \gamma) \in Avt(Q, \cdot)$, then $\alpha 1 \cdot x = \gamma x$ for all $x \in Q$, i.e. $\gamma = L_{\alpha 1}$.

   (b) If $(\varepsilon, \beta, \gamma) \in Avt(Q, \cdot)$, then $\beta = \gamma$.

   (c) If $(\alpha, \beta, \varepsilon) \in Avt(Q, \cdot)$, then $\alpha 1 \cdot \beta x = x$ for all $x \in Q$, i.e. $\beta = L_{\alpha 1}^{-1}$.

2. Let $(Q, \cdot)$ be a right loop.

(a) If $(\alpha, \varepsilon, \gamma) \in Avt(Q, \cdot)$, then $\alpha = \gamma$.

(b) If $(\varepsilon, \beta, \gamma) \in Avt(Q, \cdot)$, then $x \cdot \beta 1 = \gamma x$ for all $x \in Q$, i.e. $\gamma = R_{\beta 1}$.

(c) If $(\alpha, \beta, \varepsilon) \in Avt(Q, \cdot)$, then $\alpha x \cdot \beta 1 = x$ for all $x \in Q$, i.e. $\alpha = R_{\beta 1}^{-1}$.

3. Let $(Q, \cdot)$ be an unipotent quasigroup.

(a) If $(\alpha, \varepsilon, \gamma) \in Avt(Q, \cdot)$, then $\alpha x \cdot x = \gamma 1$ for all $x \in Q$, i.e. $\alpha = P_{\gamma 1}^{-1}$.

(b) If $(\varepsilon, \beta, \gamma) \in Avt(Q, \cdot)$, then $x \cdot \beta x = \gamma 1$ for all $x \in Q$, i.e. $\beta = P_{\gamma 1}$.

(c) If $(\alpha, \beta, \varepsilon) \in Avt(Q, \cdot)$, then $\alpha = \beta$.

Proof. The proof follows from Lemma 2.14 and Remarks 1.6 and 1.8.

In Case 3(c) we have. If $(Q, \cdot)$ is an unipotent quasigroup and $(\alpha, \beta, \varepsilon) \in Avt(Q, \cdot)$, then $\alpha x \cdot \beta x = 1$ for all $x \in Q$, i.e. $\beta = P_1 \alpha$. But in unipotent quasigroup $P_1 = \varepsilon$. $\square$

Theorem 2.18.   *1. Let $(Q, \cdot)$ be a loop.*

(a) If $(\alpha, \varepsilon, \gamma) \in Avt(Q, \cdot)$, then $\alpha = \gamma = L_{\alpha 1}$.

(b) If $(\varepsilon, \beta, \gamma) \in Avt(Q, \cdot)$, then $\beta = \gamma = R_{\beta 1}$.

(c) If $(\alpha, \beta, \varepsilon) \in Avt(Q, \cdot)$, then $\alpha = R_{\beta 1}$, $\beta = L_{\beta 1}^{-1}$.

2. Let $(Q, \cdot)$ be an unipotent left loop.

(a) If $(\alpha, \varepsilon, \gamma) \in Avt(Q, \cdot)$, then $\alpha = P_{\alpha 1}$, $\gamma = L_{\alpha 1}$.

(b) If $(\varepsilon, \beta, \gamma) \in Avt(Q, \cdot)$, then $\beta = \gamma = P_{\gamma 1}$.

(c) If $(\alpha, \beta, \varepsilon) \in Avt(Q, \cdot)$, then $\alpha = \beta = L_{\alpha 1}^{-1}$.

3. Let $(Q, \cdot)$ be an unipotent right loop.

(a) If $(\alpha, \varepsilon, \gamma) \in Avt(Q, \cdot)$, then $\alpha = \gamma = P_{\gamma 1}^{-1}$.

(b) If $(\varepsilon, \beta, \gamma) \in Avt(Q, \cdot)$, then $\beta = P_{\beta 1}^{-1}$, $\gamma = R_{\beta 1}$.

(c) If $(\alpha, \beta, \varepsilon) \in Avt(Q, \cdot)$, then $\alpha = \beta = R_{\beta 1}^{-1}$.

Proof. Case 1(c) is well known. See for example [52]. From Corollary 2.17 it follows that in loop case $\alpha = R_{\beta 1}^{-1}$ and $\beta = L_{\alpha 1}^{-1}$ for any autotopy of the form $(\alpha, \beta, \varepsilon)$. If $R_a^{-1} \in {}_1 N_m^A(Q, \cdot)$, then $R_a \in {}_1 N_m^A(Q, \cdot)$, since ${}_1 N_m^A(Q, \cdot)$ is a group. Therefore in loop case any autotopy of the kind $(\alpha, \beta, \varepsilon)$ takes the following form

$$(R_{\beta 1}, L_{\alpha 1}, \varepsilon) \tag{16}$$

Taking into consideration equality (16) and the fact that ${}_1 N_m^A(Q, \cdot)$ is a group we have: if $R_a, R_b \in {}_1 N_m^A(Q, \cdot)$, then $R_a R_b = R_c$. Thus $R_a R_b 1 = R_c 1$, $c = b \cdot a$.

We find now the form of inverse element to the element $R_a \in {}_1 N_m^A(Q, \cdot)$. Notice, if $R_a^{-1} = R_b$ for some $b \in Q$, then $R_a R_b = R_{b \cdot a} = \varepsilon = R_1$. Then $b \cdot a = 1$ and since any right (left, middle) quasigroup translation is defined in an unique way by its index element we have $b = {}^{-1}a$. From the other side $R_b R_a = R_{a \cdot b} = \varepsilon = R_1$, $b = a^{-1}$. Therefore in this situation $a^{-1} = {}^{-1}a$ for any suitable element $a \in Q$, $R_a^{-1} = R_{({}^{-1}a)} = R_{a^{-1}}$.

Similarly we can obtain that $L_a^{-1} = L_{({}^{-1}a)} = L_{a^{-1}}$ for any $L_a \in {}_2 N_m^A(Q, \cdot)$, where $(Q, \cdot)$ is a loop.

In loop $(Q, \cdot)$ from equality $\alpha x \cdot \beta y = x \cdot y$ by $x = y = 1$ we have $\alpha 1 \cdot \beta 1 = 1$. Then $\beta 1 = (\alpha 1)^{-1}$, $\alpha 1 = {}^{-1}(\beta 1)$. Moreover, we have that $\beta 1 = {}^{-1}(\alpha 1)$, $\alpha 1 = (\beta 1)^{-1}$, $R_{\beta 1} = R_{\alpha 1}^{-1}$.

Finally we obtain that any element of middle loop A-nucleus has the form

$$(R_b, L_b^{-1}, \varepsilon). \tag{17}$$

It is easy to see that $(R_{\alpha 1}, L_{\alpha 1}^{-1}, \varepsilon) \in N_m^A \Leftrightarrow \alpha 1 \in N_m$.

Case 2(a). From Corollary 2.17 it follows that in unipotent left loop $\alpha = P_{\gamma 1}^{-1} = P_{\alpha 1 \cdot 1}^{-1}$, $\gamma = L_{\alpha 1}$.

In unipotent loop any left A-nucleus autotopy $(\alpha, \varepsilon, \gamma)$ takes the following form

$$(P_{\gamma 1}^{-1}, \varepsilon, L_{\alpha 1}) \tag{18}$$

It is clear, if $(P_{\gamma 1}^{-1}, \varepsilon, L_{\alpha 1}) \in N_l^A$, then any element of the group N_l^A has the form $(P_{\gamma 1}, \varepsilon, L_{\alpha 1}^{-1})$. We find now the form of inverse element to the element $P_a \in {}_1 N_l^A(Q, \cdot)$.

At first we have: if $P_a, P_b \in {}_1 N_l^A(Q, \cdot)$, then $P_a P_b = P_c$ for some element $c \in Q$. Thus $P_a P_b 1 = P_c 1$. If $P_c 1 = d$, then $1 \cdot d = c$, $c = d$. Therefore $P_a b = c$, $b \cdot c = a$, $c = b \backslash a$.

Notice, if $P_a^{-1} = P_b$ for some $b \in Q$, then $P_a P_b = P_{b \backslash a} = \varepsilon = P_1$ since $(Q, \cdot)$ is unipotent quasigroup. Since any right (left, middle) quasigroup translation is defined in an unique way by its index element, then $b \backslash a = 1$, $a = b \cdot 1$, $b = a / 1$. From the other side $P_b P_a = P_{a \backslash b} = \varepsilon = P_1$, $a \backslash b = 1$, $b = a \cdot 1$. Thus, if $(Q, \cdot)$ is a left unipotent loop, then $a / 1 = a \cdot 1$ for any element $a \in Q$ such that $P_a \in {}_1 N_l^A(Q, \cdot)$. Then $P_a^{-1} = P_{a/1} = P_{a \cdot 1}$, $P_a = P_{a \cdot 1}^{-1}$, $\alpha = P_{\alpha 1 \cdot 1}^{-1} = P_{\alpha 1}$. And we obtain that any element of left A-nucleus of an unipotent left loop has the form

$$(P_{\alpha 1}, \varepsilon, L_{\alpha 1}) \tag{19}$$

Case 3(b). From Corollary 2.17 it follows that in unipotent right loop $\beta = P_{1 \cdot \beta 1}$, $\gamma = R_{\beta 1}$. Further we can do as in Case 2(a) but we shall use parastrophic ideas.

It is easy to see that any unipotent right loop $(Q, \circ)$ is (12)-parastrophic image of a unipotent left loop $(Q, \cdot)$ (Table 2). In this case ${}_1 N_l^A(Q, \circ) = {}_2 N_r^A(Q, \cdot)$, ${}_3 N_l^A(Q, \circ) = {}_3 N_r^A(Q, \cdot)$ (Table 3).

From formula (19) it follows that any element of unipotent right loop right A-nucleus has the form

$$(\varepsilon, P_{\beta 1}, L_{\beta 1}) \tag{20}$$

Finally, the use of Table 1 gives us that in terms of translations of unipotent right loop $(Q, \cdot)$ we have that any element of right A-nucleus of an unipotent right loop $(Q, \circ)$ has the form

$$(\varepsilon, P_{\beta 1}^{-1}, R_{\beta 1}) \tag{21}$$

$\square$

Remark 2.19. In fact from Theorem 2.18 it follows that in loop case Belousov's concept of regular permutations coincides with the concepts of left, right, and middle A-nucleus [14, 17, 70].

2.4 A-nuclei of loops by isostrophy

Taking into consideration the importance of loops in the class of all quasigroups we give some information on A-nuclear components of isostrophic images of a loop. From Table 2 it follows the following

Remark 2.20. The (12)-parastrophe of a loop $(Q, \cdot)$ is a loop, (13)-parastrophe of a loop $(Q, \cdot)$ is an unipotent right loop, (23)-parastrophe of a loop $(Q, \cdot)$ is an unipotent left loop, (123)-parastrophe of a loop $(Q, \cdot)$ is an unipotent left loop, (132)-parastrophe of a loop $(Q, \cdot)$ is an unipotent right loop.

Taking into consideration Theorem 2.18, Tables 1 and 3 we can give more detailed connections between components of A-nuclei of a loop $(Q, \cdot)$ and its isostrophic images of the form $(Q, \circ) = (Q, \cdot)((\sigma)(\alpha, \beta, \gamma))$, where $\sigma \in S_3$, $\alpha, \beta, \gamma \in S_Q$.

Example 2.21. If $(Q, \circ) = (Q, \cdot)((123)(\alpha, \beta, \gamma))$, then from Table 3 we have $_2N_r^A(Q, \circ) = \beta^{-1}{}_1N_m^A\beta(Q, \cdot)$. By Theorem 2.18 any element of the group $_1N_m^A(Q, \cdot)$ is some right translation R_a of the loop $(Q, \cdot)$.

If, additionally, $\beta = \varepsilon$, then using Table 1 and Remark 2.20 we can say that in the unipotent left loop $(Q, \circ)$ any element of the group $_2N_r^A(Q, \circ)$ is middle translation P_a^{-1}.

In Table 4 $(Q, \circ) = (Q, \cdot)((\sigma)(\alpha, \beta, \gamma))$, where $(Q, \cdot)$ is a loop. We suppose that elements of left, right and middle loop nucleus have the following forms (L_a, ε, L_a), (ε, R_b, R_b), and $(R_c, L_c^{-1}, \varepsilon)$, respectively.

Table 4

	(ε, T)	$((12), T)$	$((13), T)$	$((23), T)$	$((132), T)$	$((123), T)$
$_1N_l^\circ$	$\alpha^{-1}L_a\alpha$	$\alpha^{-1}R_b\alpha$	$\alpha^{-1}L_a\alpha$	$\alpha^{-1}R_c\alpha$	$\alpha^{-1}(L_c)^{-1}\alpha$	$\alpha^{-1}R_b\alpha$
$_3N_l^\circ$	$\gamma^{-1}L_a\gamma$	$\gamma^{-1}R_b\gamma$	$\gamma^{-1}L_a\gamma$	$\gamma^{-1}L_c^{-1}\gamma$	$\gamma^{-1}R_c\gamma$	$\gamma^{-1}R_b\gamma$
$_2N_r^\circ$	$\beta^{-1}R_b\beta$	$\beta^{-1}L_a\beta$	$\beta^{-1}(L_c^{-1})\beta$	$\beta^{-1}R_b\beta$	$\beta^{-1}L_a\beta$	$\beta^{-1}R_c\beta$
$_3N_r^\circ$	$\gamma^{-1}R_b\gamma$	$\gamma^{-1}L_a\gamma$	$\gamma^{-1}R_c\gamma$	$\gamma^{-1}R_b\gamma$	$\gamma^{-1}L_a\gamma$	$\gamma^{-1}(L_c^{-1})\gamma$
$_1N_m^\circ$	$\alpha^{-1}R_c\alpha$	$\alpha^{-1}(L_c^{-1})\alpha$	$\alpha^{-1}R_b\alpha$	$\alpha^{-1}L_a\alpha$	$\alpha^{-1}R_b\alpha$	$\alpha^{-1}L_a\alpha$
$_2N_m^\circ$	$\beta^{-1}(L_c^{-1})\beta$	$\beta^{-1}R_c\beta$	$\beta^{-1}R_b\beta$	$\beta^{-1}L_a\beta$	$\beta^{-1}R_b\beta$	$\beta^{-1}L_a\beta$

2.5 Isomorphisms of A-nuclei

Lemma 2.22. *In any quasigroup $(Q, \circ)$ we have*

$$N_l^A = \{(\alpha, \varepsilon, R_c\alpha R_c^{-1}) \mid \text{ for all } c \in Q\} = \{(R_c^{-1}\gamma R_c, \varepsilon, \gamma) \mid \text{ for all } c \in Q\} \tag{22}$$

$$N_r^A = \{(\varepsilon, \beta, L_c\beta L_c^{-1}) \mid \text{ for all } c \in Q\} = \{(\varepsilon, L_c^{-1}\gamma L_c, \gamma) \mid \text{ for all } c \in Q\} \tag{23}$$

$$N_m^A = \{(\alpha, P_c\alpha P_c^{-1}, \varepsilon) \mid \text{ for all } c \in Q\} = \{(P_c^{-1}\beta P_c, \beta, \varepsilon) \mid \text{ for all } c \in Q\} \tag{24}$$

Proof. Re-write left A-nuclear equality $\alpha x \circ y = \gamma(x \circ y)$ in the following form $R_y\alpha x = \gamma R_y x$, $\gamma = R_y\alpha R_y^{-1}$ for all $y \in Q$.

Re-write right A-nuclear equality $x \circ \beta y = \gamma(x \circ y)$ in the following form $L_x\beta y = \gamma L_x y$, $\gamma = L_x\beta L_x^{-1}$ for all $x \in Q$.

In middle A-nuclear equality $\alpha x \circ \beta y = x \circ y$ we put $y = x'$, where $x \circ x' = a$ and a is a fixed element of the set Q, i.e. $x' = P_a x$, where P_a is a permutation of the set Q.

Then $\alpha x \circ \beta x' = x \circ x' = a$ for all $x \in Q$, $P_a\alpha x = \beta x' = \beta P_a x$. Therefore $\beta = P_a\alpha P_a^{-1}$ for all $a \in Q$. □

Lemma 2.23. *If K is a subgroup of the group $_1N_l^A$, then $K \cong H \subseteq {}_3N_l^A$, moreover $H = R_x K R_x^{-1}$ for any $x \in Q$.*

If K is a subgroup of the group $_2N_r^A$, then $K \cong H \subseteq {}_3N_r^A$, moreover $H = L_x K L_x^{-1}$ for any $x \in Q$.

If K is a subgroup of the group $_1N_m^A$, then $K \cong H \subseteq {}_2N_m^A$, moreover $H = P_x K P_x^{-1}$ for any $x \in Q$.

Proof. The proof follows from Lemma 2.22. □

Lemma 2.24. *In any quasigroup $(Q, \circ)$ the groups N_l^A, $_1N_l^A$ and $_3N_l^A$; N_m^A, $_1N_m^A$ and $_2N_m^A$; N_r^A, $_2N_r^A$ and $_3N_r^A$ are isomorphic in pairs.*

Proof. The proof follows from Lemma 2.23. □

Theorem 2.25. *In any quasigroup:*

1. *$_1N_l^A = {}_3N_l^A$ if and only if $RM \subseteq N_{S_Q}(_1N_l^A)$;*

2. *$_1N_l^A = {}_3N_l^A$ if and only if $RM \subseteq N_{S_Q}(_3N_l^A)$;*

3. *$_2N_r^A = {}_3N_r^A$ if and only if $LM \subseteq N_{S_Q}(_2N_r^A)$;*

4. *$_2N_r^A = {}_3N_r^A$ if and only if $LM \subseteq N_{S_Q}(_3N_r^A)$;*

5. *$_1N_m^A = {}_2N_m^A$ if and only if $PM \subseteq N_{S_Q}(_1N_m^A)$;*

6. *$_1N_m^A = {}_2N_m^A$ if and only if $PM \subseteq N_{S_Q}(_2N_m^A)$.*

Proof. Case 1. If $_1N_l^A = {}_3N_l^A$, then from Lemma 2.23 it follows that $R_c \in N_{S_Q}(_1N_l^A)$ for any $c \in Q$. Since $N_{S_Q}(_1N_l^A)$ is a group, further we have that $RM \subseteq N_{S_Q}(_1N_l^A)$.

Converse. Let $\alpha \in {}_1N_l^A$. Then by Lemma 2.23 $R_c\alpha R_c^{-1} \in {}_3N_l^A$. But $RM \subseteq N_{S_Q}(_1N_l^A)$. Then $R_c\alpha R_c^{-1} = \beta \in {}_1N_l^A$. Thus $_3N_l^A \subseteq {}_1N_l^A$.

Let $\gamma \in {}_3N_l^A$. Then by Lemma 2.23 $R_c^{-1}\gamma R_c \in {}_1N_l^A$. But $RM \subseteq N_{S_Q}(_1N_l^A)$. Then $R_cR_c^{-1}\gamma R_cR_c^{-1} = \gamma \in {}_1N_l^A$. Thus $_3N_l^A \subseteq {}_1N_l^A$. Therefore $_1N_l^A = {}_3N_l^A$.

Cases 2–6 are proved in the similar way. □

Moreover we can specify Theorem 2.25.

Theorem 2.26.   1. *In the group N_l^A of a quasigroup $(Q, \circ)$ the first and third components of any element $T \in N_l^A$ coincide if and only if (i) $RM(Q, \circ) \subseteq C_{S_Q}(_1N_l^A)$.*

2. *In the group N_l^A of a quasigroup $(Q, \circ)$ the first and third components of any element $T \in N_l^A$ coincide if and only if $RM(Q, \circ) \subseteq C_{S_Q}(_3N_l^A)$.*

3. *In the group N_r^A of a quasigroup $(Q, \circ)$ the second and third components of any element $T \in N_r^A$ coincide if and only if $LM(Q, \circ) \subseteq C_{S_Q}(_2N_r^A)$.*

4. *In the group N_r^A of a quasigroup $(Q, \circ)$ the second and third components of any element $T \in N_r^A$ coincide if and only if $LM(Q, \circ) \subseteq C_{S_Q}(_3N_r^A)$.*

5. *In the group N_m^A of a quasigroup $(Q, \circ)$ the first and second components of any element $T \in N_m^A$ coincide if and only if $PM(Q, \circ) \subseteq C_{S_Q}(_1N_m^A)$.*

6. *In the group N_m^A of a quasigroup $(Q, \circ)$ the first and second components of any element $T \in N_m^A$ coincide if and only if $PM(Q, \circ) \subseteq C_{S_Q}(_2N_m^A)$.*

Proof.
The proof follows from Lemma 2.23. □

Corollary 2.27.   1. *If $(Q, \cdot)$ is a right loop, then first and third components of any element $T \in N_l^A$ coincide and $_1N_l^A = {}_3N_l^A = C_{S_Q}(\mathbb{R}) = C_{S_Q}(RM)$.*

2. *If $(Q, \cdot)$ is a left loop, then second and third components of any element $T \in N_r^A$ coincide and $_2N_r^A = {}_3N_r^A = C_{S_Q}(\mathbb{L}) = C_{S_Q}(LM)$.*

3. If $(Q, \cdot)$ is an unipotent quasigroup, then the first and second components of any element $T \in N_m^A$ coincide and $_1N_m^A = {_2N_m^A} = C_{S_Q}(\mathbb{P}) = C_{S_Q}(PM)$.

4. If $(Q, \cdot)$ is a loop, then $_1N_l^A = {_3N_l^A} = C_{S_Q}(\mathbb{R}) = C_{S_Q}(RM) = C_M(RM) = C_{FM}(RM)$; $_2N_r^A = {_3N_r^A} = C_{S_Q}(\mathbb{L}) = C_{S_Q}(LM) = C_M(LM) = C_{FM}(LM)$.

5. If $(Q, \cdot)$ is an unipotent left loop, then $_2N_r^A = {_3N_r^A} = C_{S_Q}(\mathbb{L}) = C_{S_Q}(LM) = C_{PLM}(LM)$ $= C_{FM}(LM)$; $_1N_m^A = {_2N_m^A} = C_{S_Q}(\mathbb{P}) = C_{S_Q}(PM) = C_{PLM}(PM) = C_{FM}(PM)$.

6. If $(Q, \cdot)$ is an unipotent right loop, then $_1N_l^A = {_3N_l^A} = C_{S_Q}(\mathbb{R}) = C_{S_Q}(RM) = C_{PRM}(RM)$ $= C_{FM}(RM)$; $_1N_m^A = {_2N_m^A} = C_{S_Q}(\mathbb{P}) = C_{S_Q}(PM) = C_{PRM}(PM) = C_{FM}(PM)$.

Proof. Case 1. The proof follows from Lemma 2.23. In a right loop $(Q, \circ)$ there is right identity element 1. Then $R_c = \varepsilon$, if $c = 1$. Thus $\gamma = R_c \alpha R_c^{-1} = \alpha$. See also Corollaries 2.17 and 2.18. From equality $R_c \alpha = \alpha R_c$ that is true for all $c \in Q$ and for all $\alpha \in {_1N_l^A}$ we conclude that $_1N_l^A \subseteq C_{S_Q}(\mathbb{R})$.

Prove that $C_{S_Q}(\mathbb{R}) \subseteq {_1N_l^A}$. We rewrite the proof from ([39], Lemma 2.6). Any element ψ of the group $C_{S_Q}(\mathbb{R})$ fulfils equality $\psi R_y x = R_y \psi x$, i.e. $\psi(xy) = \psi(x) \cdot y$, i.e. $(\psi, \varepsilon, \psi) \in N_l^A$. Therefore $C_{S_Q}(\mathbb{R}) \subseteq {_1N_l^A}$, $_1N_l^A = C_{S_Q}(\mathbb{R})$. Equality $C_{S_Q}(\mathbb{R}) = C_{S_Q}(RM)$ is true since $\langle \mathbb{R} \rangle = RM$.

Case 2 is mirror of Case 1 and we omit its proof.

Case 3. From equality $P_c \alpha = \alpha P_c$ (Lemma 2.23) that is true for all $c \in Q$ and for all $\alpha \in {_1N_m^A}$ we conclude that $_1N_m^A \subseteq C_{S_Q}(\mathbb{P})$.

Any element ψ of the group $C_{S_Q}(\mathbb{P})$ fulfils equality $\psi P_y x = P_y \psi x = z$. If $\psi P_y x = z$, then $x \cdot \psi^{-1} z = y$. If $P_y \psi x = z$, then $\psi x \cdot z = y$. Then $x \cdot \psi^{-1} z = \psi x \cdot z$, $(\psi, \psi, \varepsilon) \in N_m^A$. Therefore $C_{S_Q}(\mathbb{P}) \subseteq {_1N_m^A}$. Thus $_1N_m^A = C_{S_Q}(\mathbb{P})$. Equality $C_{S_Q}(\mathbb{P}) = C_{S_Q}(PM)$ is true since $\langle \mathbb{P} \rangle = PM$.

Case 4. Equality $C_{S_Q}(\mathbb{R}) = C_{S_Q}(RM)$ follows from the fact that $\langle \mathbb{R} \rangle = RM$.

Equality $C_{S_Q}(RM) = C_M(RM)$ follows from the fact that $_1N_l^A \subseteq M$ and $RM \subseteq M$ since in loop case any element of the group N_l^A has the form (L_c, ε, L_c) (Corollary 2.18).

Equality $C_M(RM) = C_{FM}(RM)$ follows from the fact that $C_M(RM) = C_{S_Q}(RM)$, $M \subseteq FM \subseteq S_Q$, $_1N_l^A \subseteq FM$ and $RM \subseteq FM$.

The proofs of Cases 5 and 6 are similar to the proof of Case 4. $\qquad\square$

Remark 2.28. 1. From equality $R_c \alpha = \alpha R_c$ (Corollary 2.27, Case 1) that is true for all $c \in Q$ and for all $\alpha \in {_1N_l^A}$ we can do some additional conclusions: $\mathbb{R} \subseteq C_{S_Q}(\alpha)$; $\mathbb{R} \subseteq C_{S_Q}({_1N_l^A})$; $RM(Q) \subseteq C_{S_Q}(\alpha)$; $RM(Q) \subseteq C_{S_Q}({_1N_l^A})$.

2. From equality $L_c \beta = \beta L_c$ (Corollary 2.27, Case 2) that is true for all $c \in Q$ and for all $\beta \in {_2N_r^A}$ we can do some additional conclusions, namely: $\mathbb{L} \subseteq C_{S_Q}(\beta)$; $\mathbb{L} \subseteq C_{S_Q}({_2N_r^A})$; $LM(Q) \subseteq C_{S_Q}(\beta)$; $LM(Q) \subseteq C_{S_Q}({_2N_r^A})$.

3. From equality $P_c \alpha = \alpha P_c$ (Corollary 2.27, Case 3) that is true for all $c \in Q$ and for all $\alpha \in {_1N_m^A}$ we can do some additional conclusions, namely: $\mathbb{P} \subseteq C_{S_Q}(\alpha)$; $\mathbb{P} \subseteq C_{S_Q}({_1N_m^A})$; $PM(Q) \subseteq C_{S_Q}(\alpha)$; $PM(Q) \subseteq C_{S_Q}({_1N_m^A})$.

2.6 A-nuclei by some isotopisms

Mainly we shall use Lemma 1.22 [17, Lemma 1.4]: if quasigroups $(Q, \circ)$ and $(Q, \cdot)$ are isotopic with isotopy T, i.e. $(Q, \circ) = (Q, \cdot)T$, then $Avt(Q, \circ) = T^{-1} Avt(Q, \cdot)T$.

Lemma 2.29. For any quasigroup $(Q, \cdot)$ there exists its isotopic image $(Q, \circ)$ such that:

1. *any autotopy of the form $(\alpha, \varepsilon, \gamma)$ of quasigroup $(Q, \cdot)$ is transformed in autotopy of the form $(\gamma, \varepsilon, \gamma)$ of quasigroup $(Q, \circ)$;*

2. *any autotopy of the form $(\alpha, \varepsilon, \gamma)$ of quasigroup $(Q, \cdot)$ is transformed in autotopy of the form $(\alpha, \varepsilon, \alpha)$ of quasigroup $(Q, \circ)$;*

3. *any autotopy of the form $(\varepsilon, \beta, \gamma)$ of quasigroup $(Q, \cdot)$ is transformed in autotopy of the form $(\varepsilon, \beta, \beta)$ of quasigroup $(Q, \circ)$;*

4. *any autotopy of the form $(\varepsilon, \beta, \gamma)$ of quasigroup $(Q, \cdot)$ is transformed in autotopy of the form $(\varepsilon, \gamma, \gamma)$ of quasigroup $(Q, \circ)$;*

5. *any autotopy of the form $(\alpha, \beta, \varepsilon)$ of quasigroup $(Q, \cdot)$ is transformed in autotopy of the form $(\alpha, \alpha, \varepsilon)$ of quasigroup $(Q, \circ)$;*

6. *any autotopy of the form $(\alpha, \beta, \varepsilon)$ of quasigroup $(Q, \cdot)$ is transformed in autotopy of the form $(\beta, \beta, \varepsilon)$ of quasigroup $(Q, \circ)$.*

Proof. Case 1. If $(Q, \circ) = (Q, \cdot)(R_a^{-1}, \varepsilon, \varepsilon)$, where element a is a fixed element of the set Q, then $(Q, \circ)$ is a right loop (Corollary 1.15).

By Lemma 1.22, if quasigroups $(Q, \circ)$ and $(Q, \cdot)$ are isotopic with isotopy, then $Avt(Q, \circ) = T^{-1}Avt(Q, \cdot)T$. Then autotopy of the form $(\alpha, \varepsilon, \gamma)$ of $(Q, \cdot)$ passes in autotopy of the form $(R_a \alpha R_a^{-1}, \varepsilon, \gamma)$.

In any right loop any element of the group $N_l^A(Q, \circ)$ has equal the first and the third components, i.e. it has the form $(\gamma, \varepsilon, \gamma)$ (Corollary 2.17).

Case 2. We can take the following isotopy $(Q, \circ) = (Q, \cdot)(\varepsilon, \varepsilon, R_a)$. In this case $(Q, \circ)$ is a right loop.

Case 3. We can take the following isotopy $(Q, \circ) = (Q, \cdot)(\varepsilon, \varepsilon, L_a)$. In this case $(Q, \circ)$ is a left loop.

Case 4. We can take the following isotopy $(Q, \circ) = (Q, \cdot)(\varepsilon, L_a^{-1}, \varepsilon)$. In this case $(Q, \circ)$ is a left loop.

Case 5. We can take the following isotopy $(Q, \circ) = (Q, \cdot)(\varepsilon, P_a, \varepsilon)$. In this case $(Q, \circ)$ is an unipotent quasigroup.

Case 6. We can take the following isotopy $(Q, \circ) = (Q, \cdot)(P_a^{-1}, \varepsilon, \varepsilon)$. In this case $(Q, \circ)$ is an unipotent quasigroup. $\square$

Corollary 2.30. *For any quasigroup $(Q, \cdot)$ there exists its isotopic image $(Q, \circ)$ such that:*

1. $_1N_l^A(Q, \circ) = {}_3N_l^A(Q, \circ) = {}_3N_l^A(Q, \cdot)$;

2. $_1N_l^A(Q, \circ) = {}_3N_l^A(Q, \circ) = {}_1N_l^A(Q, \cdot)$.

3. $_2N_r^A(Q, \circ) = {}_3N_r^A(Q, \circ) = {}_2N_r^A(Q, \cdot)$;

4. $_2N_r^A(Q, \circ) = {}_3N_r^A(Q, \circ) = {}_3N_r^A(Q, \cdot)$;

5. $_1N_m^A(Q, \circ) = {}_2N_m^A(Q, \circ) = {}_1N_m^A(Q, \cdot)$;

6. $_1N_m^A(Q, \circ) = {}_2N_m^A(Q, \circ) = {}_2N_m^A(Q, \cdot)$.

Proof. The proof follows from Lemma 2.29. $\square$

Isotopy of the form $(R_x^{-1}, L_y^{-1}, \varepsilon)$ is called LP-isotopy (Definition 1.17).

Lemma 2.31. 1. *Let $(Q, \cdot)$ be a loop. If $(Q, \circ) = (Q, \cdot)(R_a^{-1}, L_b^{-1}, \varepsilon)$, then*

(i) $N_l^A(Q, \circ) = N_l^A(Q, \cdot)$,

(ii) $N_r^A(Q, \circ) = N_r^A(Q, \cdot)$.

2. Let $(Q, \cdot)$ be an unipotent right loop. If $(Q, \circ) = (Q, \cdot)(\varepsilon, P_a, R_b)$, then

(i) $N_l^A(Q, \circ) = N_l^A(Q, \cdot)$,

(ii) $N_m^A(Q, \circ) = N_m^A(Q, \cdot)$.

3. Let $(Q, \cdot)$ be an unipotent left loop. If $(Q, \circ) = (Q, \cdot)(P_a^{-1}, \varepsilon, L_b)$, then

(i) $N_r^A(Q, \circ) = N_r^A(Q, \cdot)$,

(ii) $N_m^A(Q, \circ) = N_m^A(Q, \cdot)$.

Proof. In this case $(Q, \circ)$ is a loop.

Case 1, (i). Let $(L_c, \varepsilon, L_c) \in N_l^A(Q, \cdot)$. By the isotopy any element of the group $N_l^A(Q, \circ)$ has the form $(R_a L_c R_a^{-1}, \varepsilon, L_c)$. Since $(Q, \circ)$ (or $(Q, \cdot)$) is a loop, then $R_a L_c R_a^{-1} = L_c$, $N_l^A(Q, \cdot) \subseteq N_l^A(Q, \circ)$. Inverse inclusion is proved in the similar way. Therefore $N_l^A(Q, \cdot) = N_l^A(Q, \circ)$.

Case 1, (ii) is proved in the similar way with Case 1.

Case 2 is "(13)-parastrophe image" of Case 1. In this case $(Q, \circ)$ is an unipotent right loop (Theorem 1.18). The forms of components of left nucleus and middle nucleus of unipotent right loop are given in Theorem 2.18.

Further proof of Cases 2, (i) and 2, (ii) are similar with the proof of Case 1, (i) and we omit it.

Case 3 is "(23)-parastrophe image" of Case 1. $\square$

Notice, also it is possible to see on Case 3 as on "(12)-parastrophe image" of Case 2.

2.7 Quasigroup bundle and nuclei

Definition 2.32. *Let $(Q, \circ)$ be a quasigroup.*

Denote the set of all elements of the form $L_c L_d^{-1}$, where L_c, L_d are left translations of $(Q, \circ)$, by $\mathfrak{M}_{\mathcal{L}}$.

Denote the set of all elements of the form $R_c R_d^{-1}$, where R_c, R_d are right translations of $(Q, \circ)$, by $\mathfrak{M}_{\mathcal{R}}$.

Denote the set of all elements of the form $P_c P_d^{-1}$, where P_c, P_d are middle translations of $(Q, \circ)$, by $\mathfrak{M}_{\mathcal{P}}$.

Further denote the following sets $\mathfrak{M}_{\mathcal{L}}^* = \{L_c^{-1} L_d \mid a, b \in Q\}$, $\mathfrak{M}_{\mathcal{R}}^* = \{R_c^{-1} R_d \mid a, b \in Q\}$, $\mathfrak{M}_{\mathcal{P}}^* = \{P_c^{-1} P_d \mid a, b \in Q\}$.

It is clear that in any quasigroup $\mathfrak{M}_{\mathcal{L}} \subseteq LM$, $\mathfrak{M}_{\mathcal{L}}^* \subseteq LM$, $\mathfrak{M}_{\mathcal{R}} \subseteq RM$, $\mathfrak{M}_{\mathcal{R}}^* \subseteq RM$, $\mathfrak{M}_{\mathcal{P}} \subseteq PM$, $\mathfrak{M}_{\mathcal{P}}^* \subseteq PM$.

Notice that $(L_c L_d^{-1})^{-1} = L_d L_c^{-1}$. Therefore $(\mathfrak{M}_{\mathcal{L}})^{-1} = \mathfrak{M}_{\mathcal{L}}$, $(\mathfrak{M}_{\mathcal{R}})^{-1} = \mathfrak{M}_{\mathcal{R}}$ and so on.

Remark 2.33. It is clear that $\mathfrak{M}_{\mathcal{L}} \subseteq LM(Q, \cdot)$. Then from Definition 1.45 it follows that $C_{S_Q}(\mathfrak{M}_{\mathcal{L}}) \supseteq C_{S_Q} LM(Q, \cdot)$ and so on.

In [84] the set of all LP-isotopes of a fixed quasigroup $(Q, \cdot)$ is called a bundle of a quasigroup $(Q, \cdot)$.

Remark 2.34. If $|Q| = n$, then by Lemma 1.20 any of the sets $\mathfrak{M}_{\mathcal{L}}$, $\mathfrak{M}_{\mathcal{R}}$, $\mathfrak{M}_{\mathcal{P}}$, $\mathfrak{M}_{\mathcal{L}}^*$, $\mathfrak{M}_{\mathcal{R}}^*$, and $\mathfrak{M}_{\mathcal{P}}^*$ can contain n (not necessary different) simply transitive subsets of order n, that correspond to the sets of left, right and middle translations of corresponding loops, unipotent left loops and unipotent right loops of a quasigroup $(Q, \cdot)$.

Theorem 2.35. *In any quasigroup $(Q, \cdot)$:*

$$
\begin{array}{llll}
(1) & {}_1N_l^A(Q, \cdot) \subseteq C_{S_Q}(\mathfrak{M}_\mathcal{R}^*) & (2) & {}_3N_l^A(Q, \cdot) \subseteq C_{S_Q}(\mathfrak{M}_\mathcal{R}) \\
(3) & {}_2N_r^A(Q, \cdot) \subseteq C_{S_Q}(\mathfrak{M}_\mathcal{L}^*) & (4) & {}_3N_r^A(Q, \cdot) \subseteq C_{S_Q}(\mathfrak{M}_\mathcal{L}) \\
(5) & {}_1N_m^A(Q, \cdot) \subseteq C_{S_Q}(\mathfrak{M}_\mathcal{P}^*) & (6) & {}_2N_m^A(Q, \cdot) \subseteq C_{S_Q}(\mathfrak{M}_\mathcal{P})
\end{array}
$$

Proof. Case 1. From Lemma 1.24 it follows that in any autotopy $(\alpha, \varepsilon, \gamma)$ of a quasigroup $(Q, \circ)$ permutations α and ε uniquely determine permutation γ. Therefore from Lemma 2.23 in this case $R_c\alpha R_c^{-1} = R_d\alpha R_d^{-1}$, $\alpha = R_c^{-1}R_d\alpha R_d^{-1}R_c = R_c^{-1}R_d\alpha(R_c^{-1}R_d)^{-1}$ for all $c, d \in Q$.

Case 2 is proved in the similar way.

Case 3. From Lemma 1.24 it follows that in any autotopy $(\varepsilon, \beta, \gamma)$ of a quasigroup $(Q, \circ)$ permutations ε and β uniquely determine permutation γ. Therefore from Lemma 2.23 in this case $L_c\beta L_c^{-1} = L_d\beta L_d^{-1}$, $\beta = L_c^{-1}L_d\beta L_d^{-1}L_c = L_c^{-1}L_d\beta(L_c^{-1}L_d)^{-1}$ for all $c, d \in Q$.

Case 4 is proved similarly.

Case 5. From Lemma 1.24 it follows that in any autotopy $(\alpha, \beta, \varepsilon)$ of a quasigroup $(Q, \circ)$ permutations β and ε uniquely determine permutation α. Therefore from Lemma 2.23 in this case $P_c\alpha P_c^{-1} = P_d\alpha P_d^{-1}$, $\alpha = P_c^{-1}P_d\alpha P_d^{-1}P_c = P_c^{-1}P_d\alpha(P_c^{-1}P_d)^{-1}$ for all $c, d \in Q$.

Case 6 is proved similarly. $\square$

Corollary 2.36. *1. In any right loop $(Q, \cdot)$*

(a) ${}_1N_l^A(Q, \cdot) \subseteq C_{S_Q}RM(Q, \cdot)$,

(b) ${}_3N_l^A(Q, \cdot) \subseteq C_{S_Q}RM(Q, \cdot)$.

2. In any left loop $(Q, \cdot)$

(a) ${}_2N_r^A(Q, \cdot) \subseteq C_{S_Q}LM(Q, \cdot)$,

(b) ${}_3N_r^A(Q, \cdot) \subseteq C_{S_Q}LM(Q, \cdot)$.

3. In any unipotent quasigroup $(Q, \cdot)$

(a) ${}_1N_m^A(Q, \cdot) \subseteq C_{S_Q}PM(Q, \cdot)$,

(b) ${}_2N_m^A(Q, \cdot) \subseteq C_{S_Q}PM(Q, \cdot)$.

Proof. Case 1, (a). Since any right loop $(Q, \cdot)$ has right identity element 1, then R_1 is identity permutation of the set Q. Therefore the set $\mathfrak{M}_\mathcal{R}^*$ contains any right translation of this right loop $(Q, \cdot)$. It is clear that in this case $C_{S_Q}\mathbb{R} \supseteq C_{S_Q}(\mathfrak{M}_\mathcal{R}) \supseteq C_{S_Q}RM$. But $C_{S_Q}\mathbb{R} = C_{S_Q}RM$ since $\langle \mathbb{R} \rangle = RM$. Then $C_{S_Q}(\mathfrak{M}_\mathcal{R}^*) = C_{S_Q}RM$.

All other cases are proved in the similar way. $\square$

Corollary 2.37. *Let $x \circ y = R_a^{-1}x \cdot y$ for all $x, y \in Q$. Then ${}_1N_l^A(Q, \cdot) \subseteq C_{S_Q}(\mathbb{R}(Q, \circ))$; ${}_3N_l^A(Q, \cdot) \subseteq C_{S_Q}(\mathbb{R}(Q, \circ))$.*

Let $x \circ y = x \cdot L_b^{-1}y$ for all $x, y \in Q$. Then ${}_2N_r^A(Q, \cdot) \subseteq C_{S_Q}(\mathbb{L}(Q, \circ))$; ${}_3N_r^A(Q, \cdot) \subseteq C_{S_Q}(\mathbb{L}(Q, \circ))$.

2.8 Action of A-nuclei

Theorem 2.38. *Any of the groups ${}_1N_l^A$, ${}_3N_l^A$, ${}_1N_m^A$, ${}_2N_m^A$, ${}_2N_r^A$ and ${}_3N_r^A$ of a quasigroup $(Q, \cdot)$ acts the set Q free (semiregular) and in such manner that stabilizer of any element $x \in Q$ by this action is the identity group, i.e. $|{}_1N_l^A{}_x| = |{}_3N_l^A{}_x| = |{}_2N_r^A{}_x| = |{}_3N_r^A{}_x| = |{}_1N_m^A{}_x| = |{}_3N_m^A{}_x| = 1$.*

Proof. By Theorem 1.26 any autotopy (α, β, γ) $((\alpha, \beta, \gamma) \equiv (\alpha_1, \alpha_2, \alpha_3))$ is uniquely defined by any autotopy component α_i, $i \in \{1, 2, 3\}$, and by element $b = \alpha_j a$, $i \neq j$. By definition of $_1N_l^A$ any element α of the group $_1N_l^A$ is the first component of an autotopy of the form $(\alpha, \varepsilon, R_c \alpha R_c^{-1})$ (Lemma 2.23).

In order to apply Theorem 1.26 to the group N_l^A we take $i = 2$ (in this case $\beta = \varepsilon$ for any $T \in N_l^A$). And in fact any left nuclear autotopy T is defined by the image αa of a fixed element $a \in Q$ by action of the permutation $\alpha \in {_1N_l^A}$.

Therefore, if $\alpha_1, \alpha_2 \in {_1N_l^A}$, then $\alpha_1 x \neq \alpha_2 x$ for any $x \in Q$, i.e. $_1N_l^A$ acts on the set Q free (fixed point free), or equivalently, if $\alpha(x) = x$ for some $x \in Q$ then $\alpha = \varepsilon$.

If $\alpha a = \beta a$ for some $\alpha, \beta \in {_1N_l^A}$, $a \in Q$, then $\beta^{-1}\alpha x = x$, $\beta^{-1}\alpha = \varepsilon$. Therefore $|{_1N_l^A}x| = |{_3N_l^A}x| = |{_2N_r^A}x| = |{_3N_r^A}x| = |{_1N_m^A}x| = |{_3N_m^A}x| = 1$. $\qquad\square$

Corollary 2.39. *If H is a subgroup of the group $_1N_l^A$ ($_3N_l^A$, $_1N_m^A$, $_2N_m^A$, $_2N_r^A$ and $_3N_r^A$), then the group H acts on the set Q free and in such manner that stabilizer of any element $x \in Q$ by this action is the identity group.*

Proof. The proof follows from Theorem 2.38. $\qquad\square$

Corollary 2.40. *In any finite quasigroup $(Q, \cdot)$ the orders of orbits of the group $_1N_l^A$ ($_3N_l^A$, $_1N_m^A$, $_2N_m^A$, $_2N_r^A$ and $_3N_r^A$) by its action on quasigroup $(Q, \cdot)$ are equal with the order of the group $_1N_l^A$ ($_3N_l^A$, $_1N_m^A$, $_2N_m^A$, $_2N_r^A$ and $_3N_r^A$), respectively.*

Proof. By Theorem 2.38 $|{_1N_l^A}x| = 1$ for any x in Q. $\qquad\square$

Remark 2.41. For infinite case we can re-formulate Corollary 2.40 in the language of bijections. For example, there exists a bijection between any orbit of the group $_1N_l^A$ by its action on quasigroup $(Q, \cdot)$ and the group $_1N_l^A$. And so on.

Corollary 2.42. *In any finite quasigroup $(Q, \cdot)$ the order of any subgroup of the group N_l^A (N_m^A, N_r^A, $_1N_l^A$, $_3N_l^A$, $_1N_m^A$, $_2N_m^A$, $_2N_r^A$ and $_3N_r^A$) divide the order of the set Q.*

Proof. We can use Corollaries 2.39, 2.40 and Lagrange Theorem [48] about order of subgroup of any finite group. $\qquad\square$

Corollary 2.43. *In any finite quasigroup $(Q, \cdot)$ we have $|N_l^A| \leqslant |Q|$, $|N_m^A| \leqslant |Q|$, $|N_r^A| \leqslant |Q|$, $|{_1N_l^A}| \leqslant |Q|$, $|{_3N_l^A}| \leqslant |Q|$, $|{_1N_m^A}| \leqslant |Q|$, $|{_2N_m^A}| \leqslant |Q|$, $|{_2N_r^A}| \leqslant |Q|$ and $|{_3N_r^A}| \leqslant |Q|$.*

Proof. The proof follows from Corollary 2.42. $\qquad\square$

Lemma 2.44. *If the order of a quasigroup $(Q, \cdot)$ is equal to the order of the group N_l^A, then the orders of all groups N_m^A, N_r^A, $_1N_l^A$, $_3N_l^A$, $_1N_m^A$, $_2N_m^A$, $_2N_r^A$ and $_3N_r^A$ are equal to $|Q|$.*

Proof. It is clear that the order of a quasigroup $(Q, \cdot)$ is invariant relative to the parastrophy. I.e. $|(Q, \cdot)| = |(Q, \cdot)^\sigma|$, for any $\sigma \in S_3$. From Table 3 it follows that $\left(N_l^A\right)^{(12)} = N_r^A$, $\left(N_l^A\right)^{(23)} = N_m^A$. Therefore $|N_l^A| = |N_r^A| = |N_m^A| = |Q|$. Further we can apply Lemma 2.23. $\qquad\square$

Remark 2.45. Analogs of Lemma 2.44 are true for any group from the following list: N_m^A, N_r^A, $_1N_l^A$, $_3N_l^A$, $_1N_m^A$, $_2N_m^A$, $_2N_r^A$ and $_3N_r^A$.

Corollary 2.46.   1. *If $(Q, \cdot)$ is finite right loop, then $|C_{S_Q}(FM)| \leq |C_{S_Q}(M)| \leq |C_{S_Q}(RM)| \leq |Q|$.*

2. *If $(Q, \cdot)$ is finite left loop, then $|C_{S_Q}(FM)| \leq |C_{S_Q}(M)| \leq |C_{S_Q}(LM)| \leq |Q|$.*

3. If $(Q, \cdot)$ is finite unipotent quasigroup, then $|C_{S_Q}(FM)| \leq |C_{S_Q}(PM)| \leq |Q|$.

Proof. Case 1. Inclusions $FM \supseteq M \supseteq RM$ follow from definitions of these groups. Therefore $|C_{S_Q}(FM)| \leq |C_{S_Q}(M)| \leq |C_{S_Q}(RM)|$. Inequality $|C_{S_Q}(RM)| \leq |Q|$ follows from Corollaries 2.27 and 2.43.

Cases 2 and 3 are proved similarly. $\square$

Example 2.47. We give an example of a loop with $N_m = \{0, 1\}$. Notice this nucleus is not a normal subloop of loop $(Q, *)$. Indeed, $5\{0, 1\} = \{5, 3\}$, $\{0, 1\}5 = \{5, 4\}$.

Then $_1N_m^A = \{\varepsilon, R_1\}$, where $R_1 = (0\,1)(2\,4)(3\,5)$, $_2N_m^A = \{\varepsilon, L_1^{-1}\}$, where $L_1 = L_1^{-1} = (0\,1)(2\,3)(4\,5)$. The group $_1N_m^A$ by action on the set Q has the following set of orbits $\{\{0, 1\}, \{2, 4\}, \{3, 5\}\}$, and the group $_2N_m^A$ has the following set $\{\{0, 1\}, \{2, 3\}, \{4, 5\}\}$.

*	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	3	2	5	4
2	2	4	1	5	3	0
3	3	5	4	0	2	1
4	4	2	5	1	0	3
5	5	3	0	4	1	2

2.9 A-nuclear quasigroups

We start from theorem which is a little generalization of Belousov Regular Theorem [14, Theorem 2.2]. In [53, 54] Kepka studied regular mappings of groupoids including n-ary case.

Theorem 2.48. 1. If an orbit K by the action of a subgroup H of the group $_1N_l^A$ on a quasigroup $(Q, \cdot)$ is a subquasigroup of $(Q, \cdot)$, then $(K, \cdot)$ is an isotope of the group H.

2. If an orbit K by the action of a subgroup H of the group $_3N_l^A$ on a quasigroup $(Q, \cdot)$ is a subquasigroup of $(Q, \cdot)$, then $(K, \cdot)$ is an isotope of the group H.

3. If an orbit K by the action of a subgroup H of the group $_2N_r^A$ on a quasigroup $(Q, \cdot)$ is a subquasigroup of $(Q, \cdot)$, then $(K, \cdot)$ is an isotope of the group H.

4. If an orbit K by the action of a subgroup H of the group $_3N_r^A$ on a quasigroup $(Q, \cdot)$ is a subquasigroup of $(Q, \cdot)$, then $(K, \cdot)$ is an isotope of the group H.

5. If an orbit K by the action of a subgroup H of the group $_1N_m^A$ on a quasigroup $(Q, \cdot)$ is a subquasigroup of $(Q, \cdot)$, then $(K, \cdot)$ is an isotope of the group H.

6. If an orbit K by the action of a subgroup H of the group $_2N_m^A$ on a quasigroup $(Q, \cdot)$ is a subquasigroup of $(Q, \cdot)$, then $(K, \cdot)$ is an isotope of the group H.

Proof. Sometimes in the proof we shall denote by the symbol $*$ operation of composition of permutations (of bijections) of the set Q.

Case 1. Let $k \in K$. By Lemma 2.29 for any quasigroup $(Q, \cdot)$ there exists its isotopic image $(Q, \star) = (Q, \cdot)(\varepsilon, \varepsilon, R_k)$ such that: $_1N_l^A(Q, \star) = {}_3N_l^A(Q, \star) = {}_1N_l^A(Q, \cdot)$. Notice by isotopy of such form ($k \in K$) subquasigroup $(K, \cdot)$ passes in subquasigroup $(K, \star)$ of quasigroup $(Q, \star)$.

Moreover in right loop $(Q, \star)$ any autotopy of the form $(\alpha, \varepsilon, \gamma)$ of quasigroup $(Q, \cdot)$ takes the form $(\alpha, \varepsilon, \alpha)$. Therefore $H \subseteq {}_1N_l^A(Q, \star)$.

We follow [14, Theorem 2.2]. In this part of the proof we "are" in right loop $(Q, \star)$.

Any element $l \in K$ it is possible to present in the form $l = \delta k$, where $\delta \in H \subseteq {}_1N_l^A$. Notice $k = \varepsilon k$. If $l = \delta k$, $r = \lambda k$, then $\delta k \star \lambda k \in K$ since $(K, \star)$ is a subquasigroup of quasigroup $(Q, \star)$. Thus there exists $\mu \in H$ such that $\delta k \star \lambda k = \mu k$ since $Hk = K$.

On the set H we can define operation $\circ$ in the following way: $\delta \circ \lambda = \mu$ if and only if $\delta k \star \lambda k = \mu k$.

Prove that

$$(H, \circ) \cong (K, \star) \tag{25}$$

Define the map φ in the following way: $\varphi : \lambda \mapsto \lambda k$, $\varphi(\delta \circ \lambda) = \varphi(\delta) \star \varphi(\lambda) = \delta k \star \lambda k$. The map φ is bijective, since action of the group of permutations $H \subseteq {}_1N_l^A$ on the set K is regular (i.e. it is simply transitive). Therefore $(H, \circ) \cong (K, \star)$.

Let $\alpha, \lambda, \mu \in H$. Notice, corresponding permutation to the permutation α from the set ${}_3N_l^A(Q, \star)$ also is permutation α (Lemma 2.29). From definition of the set K we have $\alpha K = K$. Indeed $\alpha K = \alpha(Hk) = (\alpha H)k = Hk = K$.

Then restriction of the action of the triple $(\alpha, \varepsilon, \alpha) \in N_l^A(Q, \star)$ on subquasigroup $(K, \star) \subseteq (Q, \star)$ is an autotopy of subquasigroup $(K, \star)$. We have $\alpha((\lambda \circ \mu)k) = \alpha(\lambda k \star \mu k) = (\alpha \lambda)k \star \mu k = ((\alpha \lambda) \circ \mu)k$.

Therefore we obtain

$$\alpha(\lambda \circ \mu) = (\alpha \lambda) \circ \mu \tag{26}$$

If we put in equality (26) $\lambda = \varepsilon$, then we obtain $\alpha(\varepsilon \circ \mu) = \alpha \circ \mu$.

Since $(H, \circ)$ is a quasigroup, $\varepsilon \in H$, then the mapping L_ε°, $L_\varepsilon^\circ \mu = \varepsilon \circ \mu$ for all $\mu \in H$ is a permutation of the set H. Then

$$\alpha * L_\varepsilon^\circ \mu = \alpha \circ \mu \tag{27}$$

where $*$ is operation of the group H, i.e. it is composition of bijections of the set Q.

Then from equality (27) we conclude that quasigroup $(H, \circ)$ is isotopic to the group $(H, *)$, i.e. $(H, \circ) \sim (H, *)$, $(H, \circ) = (H, *)(\varepsilon, L_\varepsilon^\circ, \varepsilon)$.

Further we have

$$(H, *) \sim (H, \circ) \cong (K, \star) \sim (K, \cdot) \tag{28}$$

Case 2. The proof of Case 2 is similar to the proof of Case 1 and we omit it.

Case 3. In the proof of Case 3 instead of equality (26) we obtain the following equality

$$\beta * (\lambda \circ \mu) = \lambda \circ (\beta * \mu) \tag{29}$$

where $\beta, \lambda, \mu \in H \subseteq {}_2N_r^A$. If we put $\mu = \varepsilon$, then we have $\beta * \lambda' = \lambda \circ \beta$. Notice, since $(H, *)$ is a group, then its (12)-parastrophe $(H, \overset{(12)}{*})$ also is a group. Therefore quasigroup $(H, \circ)$ is isotopic to a group.

Case 4. The proof of this case is similar to the proof of Case 3.

Case 5. In this case equality (26) is transformed in the following equality

$$(\alpha * \lambda) \circ (\alpha * \mu) = \lambda \circ \mu \tag{30}$$

In this case $H \subseteq {}_1N_m^A$. If we put in equality (30) $\mu = \varepsilon$, then we have $(\alpha * \lambda) \circ \alpha = \lambda'$, and using operation of left division of quasigroup $(H, \circ)$ further we obtain $\lambda'/\alpha = \alpha * \lambda$.

If we take (12)-parastrophe of the operation $/$, then we obtain $\alpha \overset{(12)}{/} \lambda' = \alpha * \lambda$. In fact the operation $\overset{(12)}{/}$ is (132)-parastrophe of the operation $\circ$.

Therefore quasigroup $(H, \overset{(132)}{\circ})$ is a group isotope. Then quasigroup $(H, \circ)$ also is group isotope as a parastrophe of a group isotope (Lemma 1.21).

Case 6. The proof of this case is similar to the proof of Case 5. $\qquad \square$

From Theorem 2.48 we obtain the following

Theorem 2.49. *If an orbit K by the action of the group $_1N_l^A$ ($_3N_l^A$, $_2N_r^A$, $_3N_r^A$, $_1N_m^A$, $_2N_m^A$) on a quasigroup $(Q, \cdot)$ is a subquasigroup of $(Q, \cdot)$, then $(K, \cdot)$ is an isotope of the group $_1N_l^A$ ($_3N_l^A$, $_2N_r^A$, $_3N_r^A$, $_1N_m^A$, $_2N_m^A$), respectively.*

Proof. This theorem is partial case of Theorem 2.48. In Case 1 $H = {}_1N_l^A$ and so on. $\square$

Theorem 2.50. *The group $_1N_l^A$ ($_3N_l^A$, $_2N_r^A$, $_3N_r^A$, $_1N_m^A$, $_2N_m^A$) of a quasigroup $(Q, \cdot)$ acts on the set Q simply transitively if and only if quasigroup $(Q, \cdot)$ is a group isotope.*

Proof. If the group $_1N_l^A$ of a quasigroup $(Q, \cdot)$ acts on the set Q simply transitively, then by Theorem 2.49 $(Q, \cdot) \sim {}_1N_l^A$.

Let $(Q, \cdot) \sim (Q, +)$. Prove, that in this case the group $_1N_l^A(Q, \cdot)$ acts on the set Q simply transitively. Any element of left A-nucleus of the group $(Q, +)$ has the form $(L_a^+, \varepsilon, L_a^+)$ for all $a \in Q$. The group $_1N_l^A(Q, +)$ acts on the set Q simply transitively.

If $(Q, \cdot) = (Q, +)(\alpha, \beta, \gamma)$, then $_1N_l^A(Q, \cdot) = \alpha^{-1} {}_1N_l^A(Q, +)\alpha$ (Lemma 1.22).

Let $a, b \in Q$. Prove that there exists permutation $\psi \in {}_1N_l^A(Q, \cdot)$ such that $\psi a = b$. We can write permutation ψ in the form $\alpha^{-1}L_x^+\alpha$. Since the group $_1N_l^A(Q, +)$ is transitive on the set Q, we can take element x such that $L_x^+\alpha a = \alpha b$. Then $\psi a = \alpha^{-1}L_x^+\alpha a = \alpha^{-1}\alpha b = b$.

The fact that action of the group $_1N_l^A$ on the set Q is semiregular follows from Theorem 2.38.

Other cases are proved in the similar way. $\square$

Definition 2.51. *A-nuclear quasigroup is a quasigroup with transitive action of at least one from its components of A-nuclei.*

Definition 2.51 is a generalization of corresponding definition from [14].

Corollary 2.52. *If at least one component of a quasigroup A-nucleus is transitive, then all components of quasigroup A-nuclei are transitive.*

Proof. The proof follows from Theorem 2.50. $\square$

We can reformulate Theorem 2.50 in the following form.

Theorem 2.53. *A quasigroup is A-nuclear if and only if it is group isotope.*

We give slightly other proof of Lemma 2.44.

Lemma 2.54. *If the order of a finite quasigroup $(Q, \cdot)$ is equal to the order of the group $_1N_l^A$, then the orders of groups N_m^A, N_r^A, N_l^A, $_3N_l^A$, $_1N_m^A$, $_2N_m^A$, $_2N_r^A$ and $_3N_r^A$ are equal to $|Q|$.*

Proof. By Theorem 2.38 the group N_l^A acts on the set Q free (semiregular). From condition of the lemma it follows that the group N_l^A acts on the set Q regular (simply transitively). Further we can apply Theorem 2.50. $\square$

2.10   Identities with permutation and group isotopes

Conditions when a quasigroup is a group isotope were studied in classical article of V.D. Belousov [13]. Functional equations on quasigroups are studied in [11, 60, 94, 59, 61]. Linearity, one-sided linearity, anti-linearity and one sided anti-linearity of group and abelian group isotopes is studied in the articles of V.D. Belousov [13], T. Kepka and P. Nemec [68, 56], G.B. Belyavskaya and A.Kh. Tabarov [28, 27, 29, 24, 97, 98], F.M. Sokhatskii [93], J.D.H. Smith [88] and many other mathematicians.

Definition 2.55. *Let (Q, Ω) be an algebra. We shall name an identity of algebra (Q, Ω) with incorporated in this identity fixed permutations of the set Q as identity with permutations (permutation identity) [52].*

Identity with permutations can be obtained from "usual" quasigroup identity by rewriting these identities using quasigroup translations. Notice any permutation of the set Q can be viewed as special kind of unary operation.

Permutation identities in explicit or implicit form are in works of V.D. Belousov, G.B. Belyavskaya, A.A. Gvaramiya, A.D. Keedwell, A. Krapez, F.N. Sokhatsky, A.Kh. Tabarov and many other mathematicians that study quasigroup identities.

We give the following based on Theorem 2.53 procedure to answer the following question: is a quasigroup with an identity a group isotope?

Procedure 2.56. 1. *If we can write a permutation identity of a quasigroup $(Q, \cdot)$ in the form $Ax \cdot y = \Gamma(x \cdot y)$, where A, Γ are the sets of permutations of the set Q, and can prove that the set A or the set Γ acts on the set Q transitively, then quasigroup $(Q, \cdot)$ is a group isotope.*

2. *If we can write a permutation identity of a quasigroup $(Q, \cdot)$ in the form $x \cdot By = \Gamma(x \cdot y)$, where B, Γ are the sets of permutations of the set Q, and can prove that the set B or the set Γ acts on the set Q transitively, then quasigroup $(Q, \cdot)$ is a group isotope.*

3. *If we can write a permutation identity of a quasigroup $(Q, \cdot)$ in the form $Ax \cdot By = x \cdot y$, where A, B are the sets of permutations of the set Q, and can prove that the set A or the set B acts on the set Q transitively, then quasigroup $(Q, \cdot)$ is a group isotope.*

A procedure similar to Procedure 2.56 is given in [90, 91, 92].

Remark 2.57. Procedure 2.56 shows that it is possible to generalize Definition 2.55 changing the words "fixed permutations of the set Q" by the words "fixed sets of permutations of the set Q".

We give Belousov criteria, when a group isotope is an abelian group isotope.

Lemma 2.58. *Belousov criteria. If in a group $(Q, +)$ the equality $\alpha x + \beta y = \gamma y + \delta x$ holds for all $x, y \in Q$, where $\alpha, \beta, \gamma, \delta$ are some fixed permutations of Q, then $(Q, +)$ is an abelian group [13].*

Proof. From equality $\alpha x + \beta y = \gamma y + \delta x$ we have

$$\alpha \delta^{-1} x + \beta \gamma^{-1} y = y + x \tag{31}$$

If we put in equality (31) $x = 0$, then $\beta \gamma^{-1} = L_k^+$, where $k = -\alpha \delta^{-1} 0$.

If we put in equality (31) $y = 0$, then $\alpha \delta^{-1} = R_d^+$, where $d = -\beta \gamma^{-1} 0$.

We can rewrite equality (31) in the form

$$R_d^+ x + L_k^+ y = y + x \tag{32}$$

If we put $x = y = 0$ in equality (32), then $d + k = 0$ and equality (32) takes the form $x + y = y + x$. □

There exists also the following corollary from results of F.N. Sokhatskii ([95], Theorem 6.7.2). Notice up to isomorphism any isotope is principal.

Corollary 2.59. *If in a principal group isotope $(Q, \cdot)$ of a group $(Q, +)$ the equality $\alpha x \cdot \beta y = \gamma y \cdot \delta x$ holds for all $x, y \in Q$, where $\alpha, \beta, \gamma, \delta$ are some fixed permutations of Q, then $(Q, +)$ is an abelian group.*

Proof. If $x \cdot y = \xi x + \chi y$, then we can re-write the equality $\alpha x \cdot \beta y = \gamma y \cdot \delta x$ in the form $\xi \alpha x + \chi \beta y = \xi \gamma y + \chi \delta x$. Now we can apply Belousov criteria (Lemma 2.58). $\square$

Lemma 2.60. *A quasigroup $(Q, \cdot)$ with identity*

$$\alpha_1 x \cdot \alpha_2 (y \cdot z) = y \cdot \alpha_3 (x \cdot \alpha_4 z), \tag{33}$$

where $\alpha_1, \ldots, \alpha_4$ are permutations of the set Q, is abelian group isotope.

Proof. We can re-write identity (33) in the form $L_{\alpha_1 x} \alpha_2 (y \cdot z) = y \cdot \alpha_3 L_x \alpha_4 z$. Since $(Q, \cdot)$ is a quasigroup, then for any fixed elements $a, b \in Q$ the exists an element t such that $ta = b$, i.e. $L_t a = b$. Then the set of translations $\{L_x \mid x \in Q\}$ acts on the set Q transitively. Therefore the set of permutations of the form $\alpha_3 L_x \alpha_4$ also acts transitively on Q. By Theorem 2.53 quasigroup $(Q, \cdot)$ is group isotope.

We can re-write identity (33) in the form $\alpha_1 x \cdot \alpha_2 R_z y = y \cdot \alpha_3 R_{\alpha_4 z} x$. From Corollary 2.59 if follows that quasigroup $(Q, \cdot)$ is abelian group isotope. $\square$

Definition 2.61. *Let $(Q, \cdot)$ be a groupoid. Identity of the form*

$$\alpha_1 (\alpha_2 x \cdot \alpha_3 y) \cdot \alpha_4 (\alpha_5 u \cdot \alpha_6 v) = \alpha_7 (xu) \cdot \alpha_8 (yv) \tag{34}$$

where $\alpha_1, \ldots, \alpha_8$ are fixed permutations of the set Q, we shall name permutation medial identity.
   Identity of the form

$$\alpha_1 (\alpha_2 x \cdot \alpha_3 y) \cdot \alpha_4 (\alpha_5 u \cdot \alpha_6 v) = \alpha_7 (vy) \cdot \alpha_8 (ux) \tag{35}$$

where $\alpha_1, \ldots, \alpha_8$ are fixed permutations of the set Q, we shall name permutation paramedial identity.

Theorem 2.62.   *1. Permutation medial quasigroup $(Q, \cdot)$ is an abelian group isotope.*

   2. Permutation paramedial quasigroup $(Q, \cdot)$ is an abelian group isotope.

Proof. Case 1. Using language of translations we can rewrite identity (34) in the form $\beta_1 x \cdot \beta_2 v = \beta_3 x \cdot \beta_4 v$, where $\beta_1 = \alpha_1 R_{\alpha_3 y} \alpha_2$, $\beta_2 = \alpha_4 L_{\alpha_5 u} \alpha_6$, $\beta_3 = \alpha_7 R_u$, $\beta_4 = \alpha_8 L_y$.

Then $\beta_1 \beta_3^{-1} \in {}_1 N_m^A$ and the set of permutations of the form $\beta_1 \beta_3^{-1}$ acts on the set Q transitively (we can take $u = a$, where the element a is a fixed element of the set Q). Therefore by Theorem 2.53 medial quasigroup is a group isotope.

We can write identity (34) in the form $\gamma_1 y \cdot \gamma_2 u = \gamma_3 u \cdot \gamma_4 y$, where $\gamma_1 = \alpha_1 L_{\alpha_2 x} \alpha_3$, $\gamma_2 = \alpha_4 R_{\alpha_6 v} \alpha_5$, $\gamma_3 = \alpha_7 L_x$, $\gamma_4 = \alpha_8 R_v$. From Corollary 2.59 it follows that any permutation medial quasigroup is an isotope of abelian group.
   Case 2 is proved similarly to Case 1. $\square$

Corollary 2.63. *Medial quasigroup $(Q, \cdot)$ is an abelian group isotope [14, p.33]. Paramedial quasigroup $(Q, \circ)$ is an abelian group isotope [68].*

Proof. The proof follows from Theorem 2.62. $\square$

Any permutation identity of the form

$$\alpha_1(\alpha_2(\alpha_3 x \cdot \alpha_4 y) \cdot \alpha_5 z) = \alpha_6 x \cdot \alpha_7(\alpha_8 y \cdot \alpha_9 z) \tag{36}$$

on a quasigroup $(Q, \cdot)$, where all α_i are some fixed permutations of the set Q, it is possible to reduce to the following identity [25]

$$\alpha_1(\alpha_2(x \cdot y) \cdot z) = \alpha_3 x \cdot (\alpha_4 y \cdot \alpha_5 z) \tag{37}$$

In [25] using famous Four quasigroups theorem [1, 16] it is proved: if a quasigroup $(Q, \cdot)$ satisfies identity (37) then this quasigroup is a group isotope and vice versa.

We can rewrite equality (37) in the form

$$\alpha_1 \alpha_2 R_z(x \cdot y) = \alpha_3 x \cdot R_{\alpha_5 z} \alpha_4 y \tag{38}$$

If $\alpha_3 = \varepsilon$, then by Procedure 2.56 Case 2 quasigroup $(Q, \cdot)$ with equality $\alpha_1 \alpha_2 R_z(x \cdot y) = x \cdot R_{\alpha_5 z} \alpha_4 y$ is a group isotope.

3 A-centers of a quasigroup

3.1 Classical definitions of center of a loop and a quasigroup

R.H. Bruck defined a center of a loop $(Q, \cdot)$ in the following way [32].

Definition 3.1. *Let $(Q, \cdot)$ be a loop. Then center Z of loop $(Q, \cdot)$ is the following set $Z(Q, \cdot) = N \cap C$, where $C = \{a \in Q \mid a \cdot x = x \cdot a \ \forall x \in Q\}$ [32, 70].*

We follow [70] in the denoting of loop center by the letter Z.

Notice, that in a loop (even in a groupoid) $Z = N_l \cap N_r \cap C = N_l \cap N_m \cap C = N_m \cap N_r \cap C$ [77].

It is well known that in loop case $Z(Q)$ is normal abelian subgroup of the loop Q [32].

J.D.H. Smith [86, 87, 34, 89] has given definition of center of quasigroup in the language of universal algebra (Congruence Theory). J.D.H. Smith defined central congruence in a quasigroup. Center of a quasigroup is a coset class of central congruence. Notice that any congruence of quasigroup Q defines a subquasigroup of Q^2 [86]. G.B. Belyavskaya and J.D.H. Smith definitions are close.

A quasigroup $(Q, \cdot)$ is a T-quasigroup if and only if there exists an abelian group $(Q, +)$, its automorphisms φ and ψ and a fixed element $a \in Q$ such that $x \cdot y = \varphi x + \psi y + a$ for all $x, y \in Q$ [68].

A quasigroup is central, if it coincides with its center.

Theorem 3.2. *(Belyavskaya Theorem). A quasigroup is central (in Belyavskaya and Smith sense) if and only if it is a T-quasigroup [23].*

An overview of various definitions of quasigroup center is in [77].

3.2 A-nuclei and autotopy group

We recall that isotopic quasigroups have isomorphic autotopy groups [14] (see Lemma 1.22 of this paper).

It is well known that any quasigroup $(Q, \cdot)$ that is an isotope of a group Q has the following structure of its autotopy group [62, 76]

$$Avt(Q, \cdot) \cong (Q \times Q) \rtimes Aut(Q)$$

Theorem 3.3. *For any loop Q we have*

$$(N_l^A \times N_r^A) \lambda Aut(Q) \cong H \subseteq Avt(Q)$$

$$(N_l^A \times N_m^A) \lambda Aut(Q) \cong H \subseteq Avt(Q)$$

$$(N_m^A \times N_r^A) \lambda Aut(Q) \cong H \subseteq Avt(Q)$$

Proof. Case 1. It is clear that the groups N_l^A, N_r^A and $Aut(Q)$ are subgroup of the group $Avt(Q)$.

Further proof is quit standard for group theory [48]. Let $H = N_l^A \cdot N_r^A \cdot Aut(Q)$. Here the operation "·" is the operation of multiplication of triplets of the group $Avt(Q)$.

We demonstrate that $N_l^A \cap N_r^A = (\varepsilon, \varepsilon, \varepsilon)$.

Any element of the group $N_l^A \cap N_r^A$ has the form $(\varepsilon, \varepsilon, \gamma)$. By Corollary 1.25 $\gamma = \varepsilon$ in this case.

It is easy to see that $N_l^A \trianglelefteq Avt(Q)$. Then $N_l^A \trianglelefteq H$. Similarly $N_r^A \trianglelefteq H$.

Prove that $N_l^A \cdot N_r^A = N_r^A \cdot N_l^A$. Let $(\delta, \varepsilon, \lambda) \in N_l^A$, $(\varepsilon, \mu, \psi) \in N_r^A$. Any element of the set $N_l^A \cdot N_r^A$ has the form $(\delta, \mu, \lambda\psi)$ and any element of the set $N_r^A \cdot N_l^A$ has the form $(\delta, \mu, \psi\lambda)$. By Lemma 1.24 any two autotopy components define the third component in the unique way, therefore $\psi\lambda = \lambda\psi$ in our case. Thus $N_l^A \cdot N_r^A = N_l^A \times N_r^A$.

Prove that $N_l^A \times N_r^A \trianglelefteq Avt(Q)$. It is easy to see that $N_l^A \trianglelefteq Avt(Q)$, $N_r^A \trianglelefteq Avt(Q)$. Indeed, in loop case we have

$$(\alpha^{-1}, \beta^{-1}, \gamma^{-1})(\delta_1, \varepsilon, \lambda_1)(\alpha, \beta, \gamma) = (\delta_2, \varepsilon, \lambda_2)$$

where $(\alpha, \beta, \gamma) \in Avt(Q)$, $(\delta_1, \varepsilon, \lambda_1), (\delta_2, \varepsilon, \lambda_2) \in N_l^A$.

Similarly

$$(\alpha^{-1}, \beta^{-1}, \gamma^{-1})(\varepsilon, \mu_1, \psi_1)(\alpha, \beta, \gamma) = (\varepsilon, \mu_2, \psi_2)$$

Further we have

$$
\begin{aligned}
&(\alpha^{-1}, \beta^{-1}, \gamma^{-1})(\delta, \mu, \lambda\psi)(\alpha, \beta, \gamma) = \\
&= (\alpha^{-1}, \beta^{-1}, \gamma^{-1})(\delta, \varepsilon, \lambda)(\varepsilon, \mu, \psi)(\alpha, \beta, \gamma) = \\
&= (\alpha^{-1}, \beta^{-1}, \gamma^{-1})(\delta, \varepsilon, \lambda)(\alpha, \beta, \gamma)(\alpha^{-1}, \beta^{-1}, \gamma^{-1})(\varepsilon, \mu, \psi)(\alpha, \beta, \gamma) = \\
&= (\delta_1, \varepsilon, \lambda_1)(\varepsilon, \mu_1, \psi_1) = (\delta_1, \mu_1, \lambda_1\psi_1) \in N_l^A \times N_r^A
\end{aligned}
\tag{39}
$$

Therefore $N_l^A \times N_r^A \trianglelefteq Avt(Q)$ in any quasigroup Q.

Prove that $(N_l^A \times N_r^A) \cap Aut(Q) = (\varepsilon, \varepsilon, \varepsilon)$. In this place we shall use the fact that Q is a loop. In a loop any element of the group $(N_l^A \times N_r^A)$ has the form $(L_a, R_b, R_b L_a)$. Since any automorphism is an autotopy with equal components we have $L_a = R_b L_a$, $R_b = \varepsilon$, $R_b = R_b L_a$, $L_a = \varepsilon$.

Therefore we have that $H \cong (N_l^A \times N_r^A) \lambda Aut(Q)$.

Cases 2 and 3 are proved similarly to Case 1. $\qquad\square$

Remark 3.4. The fact $N_l^A \cap N_r^A = (\varepsilon, \varepsilon, \varepsilon)$ demonstrates that there is difference between Garrison left nucleus N_l and left A-nucleus N_l^A, between Garrison right nucleus N_r and right A-nucleus N_r^A, and so on.

Remark 3.5. Theorem 3.3 is true for any quasigroup $(Q, \cdot)$ but by isotopy $((Q, \cdot) \sim (Q, +))$, in general, the group $Aut(Q, +)$ passes in a subgroup of the group $Avt(Q, \cdot)$ [76].

Corollary 3.6. *For any quasigroup Q we have*

$$(N_l^A \times N_r^A) \trianglelefteq Avt(Q)$$

$$(N_l^A \times N_m^A) \trianglelefteq Avt(Q)$$

$$(N_m^A \times N_r^A) \trianglelefteq Avt(Q)$$

Proof. The proof follows from the proof of Theorem 3.3 (equality (39)). $\qquad \square$

Corollary 3.7. *For any quasigroup Q we have:*

1. *if $\mu \in {}_1 N_l^A$, $\nu \in {}_1 N_m^A$, then $\mu\nu = \nu\mu$;*

2. *if $\mu \in {}_2 N_r^A$, $\nu \in {}_2 N_m^A$, then $\mu\nu = \nu\mu$;*

3. *if $\mu \in {}_3 N_l^A$, $\nu \in {}_3 N_r^A$, then $\mu\nu = \nu\mu$.*

Proof. The proof follows from Theorem 3.3. $\qquad \square$

Corollary 3.8. *In any quasigroup*

1. ${}_1 N_l^A \cdot {}_1 N_m^A = {}_1 N_m^A \cdot {}_1 N_l^A$;

2. ${}_2 N_r^A \cdot {}_2 N_m^A = {}_2 N_m^A \cdot {}_2 N_r^A$;

3. ${}_3 N_r^A \cdot {}_3 N_l^A = {}_3 N_l^A \cdot {}_3 N_r^A$.

Proof. The proof follows from Corollary 3.7. $\qquad \square$

Example 3.9. Loop $(Q, \cdot)$ has identity group $Aut(Q)$ [85], the identity groups $(N_l^A \times N_r^A) \rtimes Aut(Q)$, $(N_l^A \times N_m^A) \rtimes Aut(Q), (N_r^A \times N_r^A) \rtimes Aut(Q)$. Its autotopy group $Avt(Q)$ is isomorphic to the alternating group A_4 [46].

·	1	2	3	4	5
1	1	2	3	4	5
2	2	3	1	5	4
3	3	4	5	1	2
4	4	5	2	3	1
5	5	1	4	2	3

We recall $|A_4| = 12$ and this number is divisor of the number $5! \cdot 5 = 600$.

3.3 A-centers of a loop

Definition 3.10. *Let Q be a loop.*

Autotopy of the form $\{(L_a, \varepsilon, L_a) \,|\, a \in Z(Q)\}$ we shall name left central autotopy. Group of all left central autotopies we shall denote by Z_l^A.

Autotopy of the form $\{(\varepsilon, L_a, L_a) \,|\, a \in Z(Q)\}$ we shall name right central autotopy. Group of all right central autotopies we shall denote by Z_r^A.

Autotopy of the form $\{(L_a, L_a^{-1}, \varepsilon) \,|\, a \in Z(Q)\}$ we shall name middle central autotopy. Group of all middle central autotopies we shall denote by Z_m^A.

Lemma 3.11. *In any loop the groups Z_l^A, Z_r^A, Z_m^A, ${}_1 Z_l^A$, ${}_3 Z_l^A$, ${}_2 Z_r^A$, ${}_3 Z_r^A$, ${}_1 Z_m^A$, ${}_2 Z_m^A$, and Z are isomorphic. In more details $Z \cong Z_l^A \cong Z_r^A \cong Z_m^A \cong {}_1 Z_l^A = {}_3 Z_l^A = {}_2 Z_r^A = {}_3 Z_r^A = {}_1 Z_m^A = {}_2 Z_m^A$.*

Proof. The proof follows from definition of center of a loop (Definition 3.1) and Definition 3.10. The map $\xi : a \mapsto L_a$, where $a \in Z$, gives necessary isomorphism of the group Z and ${}_1 Z_l^A$. And so on. $\quad\square$

Lemma 3.12. *If Q is a loop, then ${}_1 Z_l^A 1 = {}_3 Z_l^A = {}_2 Z_r^A 1 = {}_3 Z_r^A = {}_1 Z_m^A = {}_2 Z_m^A 1 = Z$.*

Proof. It is possible to use Definition 3.10. $\quad\square$

Corollary 3.13. *In any loop Q the orbit ${}_1 Z_l^A 1$ $({}_3 Z_l^A,\ {}_2 Z_r^A 1,\ {}_3 Z_r^A,\ {}_1 Z_m^A,\ {}_2 Z_m^A 1)$ coincides with the set Q if and only if Q is an abelian group.*

Theorem 3.14. *Let Q be a loop. Then*

1. $(N_l^A \times N_r^A) \cap N_m^A = \{(L_a, L_a^{-1}, \varepsilon) \mid a \in Z(Q)\} = Z_m^A$;

2. $(N_l^A \times N_m^A) \cap N_r^A = \{(\varepsilon, L_a, L_a) \mid a \in Z(Q)\} = Z_r^A$;

3. $(N_r^A \times N_m^A) \cap N_l^A = \{(L_a, \varepsilon, L_a) \mid a \in Z(Q)\} = Z_l^A$.

4. $Z_l^A \trianglelefteq Avt(Q), Z_r^A \trianglelefteq Avt(Q), Z_m^A \trianglelefteq Avt(Q)$.

5. *The groups Z_l^A, Z_r^A and Z_m^A are abelian subgroups of the group $Avt(Q)$.*

Proof. Case 1. Any element of the group $N_l^A \times N_r^A$ has the form

$$(L_a, R_b, L_a R_b) \tag{40}$$

any element of the group N_m^A has the form

$$(R_c, L_c^{-1}, \varepsilon) \tag{41}$$

(Theorem 2.18). Then $R_b L_a = \varepsilon$, $R_b = L_a^{-1}$. Expression (40) takes the form $(L_a, L_a^{-1}, L_a L_a^{-1}) = (L_a, L_a^{-1}, \varepsilon)$. Comparing it with expression (41) we have that $L_a = R_a$, i.e. $ax = xa$ for any element $x \in Q$. In addition taking into consideration that $(L_a, L_a^{-1}, \varepsilon) \in N_m^A, (L_a, \varepsilon, L_a) \in N_l^A, (\varepsilon, L_a^{-1}, L_a^{-1}) \in N_r^A$, we obtain that the element a is central, i.e. $a \in Z(Q)$.

Therefore $(N_l^A \times N_r^A) \cap N_m^A \subseteq \{(L_a, L_a^{-1}, \varepsilon) \mid a \in Z(Q)\}$.

Converse. It is easy to see that any element from the center $Z(Q)$ of a loop Q "generate" autotopies of left, right, middle A-nucleus of the loop Q, i.e. $\{(L_a, L_a^{-1}, \varepsilon) \mid a \in Z(Q)\} \subseteq (N_l^A \times N_r^A) \cap N_m^A$.

And finally we have $(N_l^A \times N_r^A) \cap N_m^A = \{(L_a, L_a^{-1}, \varepsilon) \mid a \in Z(Q)\}$.

Case 2. The proof of Case 2 is similar to the proof of Case 1 and we omit some details. Any element of the group $N_l^A \times N_m^A$ has the form

$$(L_a R_b, L_b^{-1}, L_a) \tag{42}$$

any element of the group N_r^A has the form

$$(\varepsilon, R_c, R_c) \tag{43}$$

(Theorem 2.18). Since expressions (42) and (43) in the intersection should be componentwise equal, we have the following system of equations

$$\begin{cases} L_a R_b = \varepsilon \\ L_b^{-1} = R_c \\ L_a = R_c \end{cases}$$

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We remember that in any loop for any nuclear translation we have $L_a^{-1} = L_{a^{-1}}$, $R_a^{-1} = R_{a^{-1}}$ (Theorem 2.18).

From equality $L_b^{-1} = L_a$ we have that $a = b^{-1}$ and $b = a^{-1}$. Then we from equality $L_a R_b = \varepsilon$ it follows that $L_b = R_b$ for all elements of the group $(N_l^A \times N_m^A) \cap N_r^A$.

Expression (42) takes the form $(L_a L_a^{-1}, L_a, L_a) = (\varepsilon, L_a, L_a)$. Expression (41) takes the same form.

Taking into consideration that $L_a = R_a$, $(L_a^{-1}, L_a, \varepsilon) \in N_m^A$, $(L_a, \varepsilon, L_a) \in N_l^A$, $(\varepsilon, L_a, L_a) \in N_r^A$, we obtain that the element a is central, i.e. $a \in Z(Q)$.

Therefore $(N_l^A \times N_r^A) \cap N_m^A \subseteq \{(L_a, L_a^{-1}, \varepsilon) \,|\, a \in Z(Q)\}$.

Converse inclusion is proved similarly with the proof in Case 1.

Finally we have $(N_l^A \times N_m^A) \cap N_r^A = \{(\varepsilon, L_a, L_a) \,|\, a \in Z(Q)\}$.

Case 3. The proof of Case 3 is similar to the proof of Cases 1, 2. Any element of the group $N_r^A \times N_m^A$ has the form

$$(R_b, R_a L_b^{-1}, R_a) \tag{44}$$

any element of the group N_l^A has the form

$$(L_c, \varepsilon, L_c) \tag{45}$$

(Theorem 2.18). Since expressions (44) and (45) in the intersection should be equal componentwise, we have the following system of equations

$$\begin{cases} R_b = L_c \\ R_a L_b^{-1} = \varepsilon \\ R_a = L_c \end{cases}$$

From equality $R_b = R_a$ we have that $a = b$. Then we from equality $R_a L_b^{-1} = \varepsilon$ it follows that $L_b = R_b$ for all elements of the group $(N_r^A \times N_m^A) \cap N_l^A$.

Expression (44) takes the form $(L_a, L_a L_a^{-1}, L_a) = (L_a, \varepsilon, L_a)$.

Taking into consideration that $L_a = R_a$, $(L_a, L_a^{-1}, \varepsilon) \in N_m^A$, $(L_a, \varepsilon, L_a) \in N_l^A$, $(\varepsilon, L_a, L_a) \in N_r^A$, we obtain that the element a is central, i.e. $a \in Z(Q)$.

Therefore $(N_r^A \times N_m^A) \cap N_l^A = \{(L_a, \varepsilon, L_a) \,|\, a \in Z(Q)\}$.

Case 4. The proof follows from Corollary 3.6 and the fact that intersection of normal subgroups of a group is normal subgroup.

Case 5. The proof follows from Lemma 3.11. Indeed, in this case, if we denote the loop operation by $+$, then we have $L_a L_b = L_{a+b} = L_{b+a} = L_b L_a$. $\qquad \square$

Theorem 3.15. *Let Q be a loop, $K = N_l^A \cdot N_r^A \cdot N_m^A$. Then*

(i) $K/Z_l^A \cong (N_r^A \times N_m^A)/Z_l^A \times N_l^A/Z_l^A$;

(ii) $K/Z_r^A \cong (N_l^A \times N_m^A)/Z_r^A \times N_r^A/Z_r^A$;

(iii) $K/Z_m^A \cong (N_l^A \times N_r^A)/Z_m^A \times N_m^A/Z_m^A$.

Proof. Case (i). The groups N_l^A, N_r^A and N_m^A are normal subgroups of the group $Avt(Q)$. From Theorem 3.3 we have that the groups N_l^A, N_r^A and N_m^A are intersected in pairs by identity group.

From Lemma 2.7 and Corollary 3.6 we have that $N_l^A \trianglelefteq Avt(Q)$, $(N_r^A \times N_m^A) \trianglelefteq Avt(Q)$. Then $N_l^A \trianglelefteq K$, $(N_r^A \times N_m^A) \trianglelefteq K$.

From Theorem 3.14 and Theorem 4.2.1 [48, p. 47] we have that groups $(N_r^A \times N_m^A)/Z_l^A$ and N_l^A/Z_l^A are normal subgroups of the group K/Z_l^A that are intersected by the identity subgroup. Therefore $K/Z_l^A \cong (N_r^A \times N_m^A)/Z_l^A \times N_l^A/Z_l^A$.

Cases (ii) and (iii) are proved in the similar way. $\qquad \square$

3.4 A-centers of a quasigroup

Definition 3.16. *Let Q be a quasigroup. We shall name*

(i) the group $(N_r^A \times N_m^A) \cap N_l^A$ as left A-center of Q and shall denote by Z_l^A;

(ii) the group $(N_l^A \times N_m^A) \cap N_r^A$ as right A-center of Q and shall denote by Z_r^A;

(iii) the group $(N_l^A \times N_r^A) \cap N_m^A$ as middle A-center of Q and shall denote by Z_m^A.

Lemma 3.17. *Left, right, and middle A-centers and their components of isotopic quasigroups $(Q, \circ)$ and $(Q, \cdot)$ are isomorphic, i.e., if $(Q, \circ) \sim (Q, \cdot)$, then*

1. $Z_l^A(Q, \cdot) \cong Z_l^A(Q, \circ)$; 2. ${}_1Z_l^A(Q, \cdot) \cong {}_1Z_l^A(Q, \circ)$; 3. ${}_3Z_l^A(Q, \cdot) \cong {}_3Z_l^A(Q, \circ)$;
4. $Z_r^A(Q, \cdot) \cong Z_r^A(Q, \circ)$; 5. ${}_2Z_r^A(Q, \cdot) \cong {}_2Z_r^A(Q, \circ)$; 6. ${}_3Z_r^A(Q, \cdot) \cong {}_3Z_r^A(Q, \circ)$;
7. $Z_l^A(Q, \cdot) \cong Z_l^A(Q, \circ)$; 8. ${}_1Z_m^A(Q, \cdot) \cong {}_1Z_m^A(Q, \circ)$; 9. ${}_2Z_m^A(Q, \cdot) \cong {}_2Z_m^A(Q, \circ)$;

Proof. Case 1. Suppose that $(Q, \circ) = (Q, \cdot)T$, where T is an isotopy. By Lemma 1.22 $Avt(Q, \circ) = T^{-1}Avt(Q, \cdot)T$.

Notice, $T^{-1}Z_l^A(Q, \cdot)T = T^{-1}((N_r^A(Q, \cdot) \times N_m^A(Q, \cdot)) \cap N_l^A(Q, \cdot))T$. For short below we shall omit denotation of quasigroups $(Q, \cdot)$ and $(Q, \circ)$.

Denote $(N_r^A \times N_m^A)$ by A, N_l^A by B, the conjugation isomorphism by the letter φ. Prove that $\varphi(A \cap B) = \varphi A \cap \varphi B$.

Let $\varphi x \in \varphi(A \cap B)$. Then $x \in A \cap B$ since φ is a bijective map. If $x \in A \cap B$, then $x \in A$, $x \in B$, $\varphi(x) \in \varphi(A)$, $\varphi(x) \in \varphi(B)$, $\varphi(x) \in \varphi(A) \cap \varphi(B)$, $\varphi(A \cap B) \subseteq \varphi A \cap \varphi B$.

Let $\varphi x \in \varphi A \cap \varphi B$. Then $\varphi x \in \varphi A$, $\varphi x \in \varphi B$. Since φ is a bijective map, then $x \in A$, $x \in B$, $x \in A \cap B$, $\varphi x \in \varphi(A \cap B)$, $\varphi A \cap \varphi B \subseteq \varphi(A \cap B)$.

Finally, $\varphi A \cap \varphi B = \varphi(A \cap B)$, i.e. $T^{-1}((N_r^A \times N_m^A) \cap N_l^A)T = T^{-1}(N_r^A \times N_m^A)T \cap T^{-1}N_l^AT$.

The fact, that $T^{-1}(N_r^A \times N_m^A)T = T^{-1}N_r^AT \times T^{-1}N_m^AT$ is well known. Indeed, in the group of all triplets $S_Q \times S_Q \times S_Q$ isomorphic image of the direct product of two subgroups is the direct product of their isomorphic images. Also it is possible to prove this fact using equality (39).

We have the following chain of equalities

$$T^{-1}Z_l^A(Q, \cdot)T = T^{-1}((N_r^A \times N_m^A) \cap N_l^A)T =$$
$$T^{-1}(N_r^A \times N_m^A)T \cap T^{-1}N_l^AT = ((T^{-1}N_r^AT) \times (T^{-1}N_m^AT)) \cap T^{-1}N_l^AT =$$
$$(N_r^A(Q, \circ) \times N_m^A(Q, \circ)) \cap N_l^A(Q, \circ) = Z_l^A(Q, \circ)$$

Therefore $Z_l^A(Q, \cdot) \cong Z_l^A(Q, \circ)$.

Cases 2, 3 follows from Case 1. Cases 4 and 7 are proved in the similar way with Case 1. Cases 5, 6 and 8, 9 follow from Cases 2 and 3, respectively. □

Corollary 3.18. *In any quasigroup $(Q, \cdot)$ the groups Z_l^A, Z_r^A, Z_m^A, ${}_1Z_l^A$, ${}_3Z_l^A$, ${}_2Z_r^A$, ${}_3Z_r^A$, ${}_1Z_m^A$, ${}_2Z_m^A$ are isomorphic abelian groups, i.e.*

$$Z_l^A(Q, \cdot) \cong Z_r^A(Q, \cdot) \cong Z_m^A(Q, \cdot) \cong$$
$${}_1Z_l^A(Q, \cdot) \cong {}_3Z_l^A(Q, \cdot) \cong {}_2Z_r^A(Q, \cdot) \cong$$
$${}_3Z_r^A(Q, \cdot) \cong {}_1Z_m^A(Q, \cdot) \cong {}_2Z_m^A(Q, \cdot)$$

Proof. Suppose that quasigroup $(Q, \cdot)$ is isomorphic to a loop $(Q, \circ)$. Then by Lemma 3.17 $Z_l^A(Q, \cdot) \cong Z_l^A(Q, \circ)$; $Z_r^A(Q, \cdot) \cong Z_r^A(Q, \circ)$; $Z_m^A(Q, \cdot) \cong Z_m^A(Q, \circ)$ and so on.

By Lemma 3.11 the groups $Z_l^A(Q, \circ)$, ${}_1Z_l^A(Q, \circ)$, ${}_3Z_l^A(Q, \circ)$, $Z_r^A(Q, \circ)$, ${}_2Z_r^A(Q, \circ)$, ${}_3Z_r^A(Q, \circ)$, $Z_m^A(Q, \circ)$, ${}_1Z_m^A(Q, \circ)$, ${}_2Z_m^A(Q, \circ)$ are isomorphic abelian groups. Therefore the same is true for A-centers of any quasigroup. □

Theorem 3.19. *If an orbit K by the action of a subgroup H of the group $_1Z_l^A(Q, \cdot)$ ($_3Z_l^A(Q, \cdot)$, $_2Z_r^A(Q, \cdot)$, $_3Z_r^A(Q, \cdot)$, $_1Z_m^A(Q, \cdot)$, $_2Z_m^A(Q, \cdot)$) on the set Q is a subquasigroup of quasigroup $(Q, \cdot)$, then $(K, \cdot)$ is an isotope of abelian group H.*

Proof. From Corollary 3.18 it follows that the group $_1Z_l^A$ is abelian. Then any subgroup H of group $_1Z_l^A$ also is abelian. The fact that $(K, \cdot)$ is group isotope follows from definition of left A-center Z_l^A and Theorem 2.48.

Other cases are proved in the similar way. $\qquad\qquad\square$

Corollary 3.20. *If an orbit K by the action of group $_1Z_l^A(Q, \cdot)$ ($_3Z_l^A(Q, \cdot)$, $_2Z_r^A(Q, \cdot)$, $_3Z_r^A(Q, \cdot)$, $_1Z_m^A(Q, \cdot)$, $_2Z_m^A(Q, \cdot)$) on the set Q is a subquasigroup of quasigroup $(Q, \cdot)$, then $(K, \cdot)$ is an isotope of abelian group $_1Z_l^A(Q, \cdot)$ ($_3Z_l^A(Q, \cdot)$, $_2Z_r^A(Q, \cdot)$, $_3Z_r^A(Q, \cdot)$, $_1Z_m^A(Q, \cdot)$, $_2Z_m^A(Q, \cdot)$, respectively).*

Proof. This corollary is partial case of Theorem 3.19. For example in Case 1 $H = {}_1Z_l^A(Q, \cdot)$. $\quad\square$

Theorem 3.21. *The group $_1Z_l^A$ ($_3Z_l^A$, $_2Z_r^A$, $_3Z_r^A$, $_1Z_m^A$, $_2Z_m^A$) of a quasigroup $(Q, \cdot)$ acts on the set Q simply transitively if and only if quasigroup $(Q, \cdot)$ is abelian group isotope.*

Proof. If the group $_1Z_l^A$ of a quasigroup $(Q, \cdot)$ acts on the set Q transitively, then the orbit by the action of group $_1Z_l^A$ coincides with the set Q, and by Corollary 3.20 $(Q, \cdot) \sim {}_1Z_l^A$.

Let $(Q, \cdot) \sim (Q, +)$, where $(Q, +)$ is abelian group. Prove, that in this case the group $_1Z_l^A(Q, \cdot)$ acts on the set Q simply transitively. Any element of left A-center of the group $(Q, +)$ has the form $(L_a^+, \varepsilon, L_a^+)$ for all $a \in Q$.

It is clear that group $_1Z_l^A(Q, +)$ acts on the set Q simply transitively.

If $(Q, \cdot) = (Q, +)(\alpha, \beta, \gamma)$, then $_1Z_l^A(Q, \cdot) = \alpha^{-1}\, {}_1Z_l^A(Q, +)\alpha$ (Corollary 1.23).

Prove that action of the group $_1Z_l^A(Q, \cdot)$ on the set Q is transitive. Let $a, b \in Q$. Prove that there exists permutation $\psi \in {}_1Z_l^A(Q, \cdot)$ such that $\psi a = b$. We can write permutation ψ in the form $\alpha^{-1}L_x^+\alpha$. Since the group $_1Z_l^A(Q, +)$ is transitive on the set Q, we can take element x such that $L_x^+\alpha a = \alpha b$. Then $\psi a = \alpha^{-1}L_x^+\alpha a = \alpha^{-1}\alpha b = b$.

The fact that action of the group $_1Z_l^A$ on the set Q has identity stabilizers for any element $x \in Q$ follows from Theorem 2.38 since in any abelian group $_1N_l^A = {}_1Z_l^A$.

Other cases are proved in the similar way. $\qquad\qquad\square$

Corollary 3.22. *If at least one component of quasigroup A-center is transitive, then all components of A-centers are transitive.*

Proof. The proof follows from Theorem 3.21. $\qquad\qquad\square$

Definition 3.23. *A-central quasigroup is a quasigroup Q with transitive action of at least one from its components of A-centers on Q.*

Using this definition we formulate main theorem of this paper

Theorem 3.24. *A quasigroup is A-central if and only if it is abelian group isotope.*

Proof. We can use Theorem 3.21. $\qquad\qquad\square$

Corollary 3.25. *Any central quasigroup in G.B. Belyavskaya and J.D.H. Smith sense is A-central quasigroup.*

Proof. The proof follows from Theorems 3.2 and 2.53. Any T-quasigroup is an isotope of abelian group. $\qquad\qquad\square$

4 Normality of A-nuclei and A-centers

4.1 Normality of equivalences in quasigroups

A binary relation on a set Q is any subset of the set $Q \times Q$.

Definition 4.1. *A relation θ on a set S satisfying the reflexive ($x\theta x$), symmetric ($x\theta y \Leftrightarrow y\theta x$), and transitive ($x\theta y, y\theta z \Longrightarrow x\theta z$) properties is an equivalence relation on S. Each cell $\bar{a} = \theta(a)$ in the natural partition given by an equivalence relation is an equivalence class [42].*

Definition 4.2. *Let A be an algebra of type $\mathbb{F}$ and let θ an equivalence. Then θ is a congruence on A if θ satisfies the following compatibility property: For each n-ary function symbol $f \in \mathbb{F}$ and elements $a_i, b_i \in A$, if $a_i\theta b_i$ holds for $1 \leq i \leq n$ then $f^A(a_1, \ldots a_n)\theta f^A(b_1, \ldots b_n)$ holds [33].*

"The compatibility property is an obvious condition for introducing an algebraic structure on the set of equivalence classes A/θ an algebraic structure which is inherited from the algebra A" [33].

Notice there exists a quasigroup $(Q, \cdot)$ in the sense of Definition 1.3 ("existential quasigroup") and its congruence θ in the sense of Definition 4.2 such that Q/θ ("an algebraic structure which is inherited from the algebra A") is a division groupoid which, in general, is not a quasigroup [9, 78]. Therefore for quasigroups congruence and homomorphism theory has a little non-standard character from the point of view of universal-algebraic approach [33].

Definition 4.3. *An equivalence θ is a left congruence of a groupoid $(Q, \circ)$, if the following implication is true for all $x, y, z \in Q$: $x\,\theta\,y \Longrightarrow (z \circ x)\,\theta\,(z \circ y)$. In other words equivalence θ is stable relative to any left translation of $(Q, \circ)$.*

An equivalence θ is a right congruence of a groupoid $(Q, \circ)$, if the following implication is true for all $x, y, z \in Q$: $x\,\theta\,y \Longrightarrow (x \circ z)\,\theta\,(y \circ z)$. In other words equivalence θ is stable relative to any right translation of $(Q, \circ)$ [36].

Definition 4.4. *An equivalence θ of a groupoid $(Q, \circ)$ is called cancellative from the left, if the following implication is true for all $x, y, z \in Q$: $(z \circ x)\,\theta\,(z \circ y) \Longrightarrow x\,\theta\,y$.*

An equivalence θ of a groupoid $(Q, \circ)$ is called cancellative from the right, if the following implication is true for all $x, y, z \in Q$: $(z \circ x)\,\theta\,(z \circ y) \Longrightarrow x\,\theta\,y$, $(x \circ z)\,\theta\,(y \circ z) \Longrightarrow x\,\theta\,y$ [14, 17].

An equivalence θ of a groupoid $(Q, \circ)$ is called normal from the right (right normal), if it is stable and cancellative from the right.

An equivalence θ of a groupoid $(Q, \circ)$ is called normal from the left (left normal), if it is stable and cancellative from the left.

An equivalence θ of a groupoid $(Q, \circ)$ is called normal, if it is left and right normal.

Remark 4.5. In [79] there is slightly other approach to normality of groupoid congruences.

Definition 4.6. *If θ is a binary relation on a set Q, α is a permutation of the set Q and from $x\theta y$ it follows $\alpha x\theta\alpha y$ and $\alpha^{-1}x\theta\alpha^{-1}y$ for all $(x, y) \in \theta$, then we shall say that the permutation α is an admissible permutation relative to the binary relation θ [14].*

Moreover, we shall say that a binary relation θ admits a permutation α.

One from the most important properties of e-quasigroup $(Q, \cdot, \backslash, /)$ is the following property.

Lemma 4.7. *Any congruence of a quasigroup $(Q, \cdot, \backslash, /)$ is a normal congruence of quasigroup $(Q, \cdot)$; any normal congruence of a quasigroup $(Q, \cdot)$ is a congruence of quasigroup $(Q, \cdot, \backslash, /)$ [30, 65, 14, 17].*

Lemma 4.8. *Any normal quasigroup congruence of a quasigroup $(Q, \cdot)$ is admissible relative to any left, right and middle quasigroup translation [15].*

Proof. The fact that any normal quasigroup congruence is admissible relative to any left and right quasigroup translation follows from Definitions 4.3 and 4.4.

Let θ be a normal congruence of a quasigroup $(Q, \cdot)$. Prove the following implication

$$a\theta b \rightarrow P_c a \,\theta\, P_c b \tag{46}$$

If $P_c a = k$, then $a \cdot k = c$, $k = a\backslash c$, $k = R_c^{\backslash} a$. Similarly if $P_c b = m$, then $b \cdot m = c$, $m = b\backslash c$, $m = R_c^{\backslash} b$. Since θ is a congruence of quasigroup $(Q, \cdot, \backslash, /)$ (Lemma 4.7), then implication (46) is true.

Implication

$$a\theta b \rightarrow P_c^{-1} a \,\theta\, P_c^{-1} b \tag{47}$$

is proved in the similar way. If $P_c^{-1} a = k$, then $k \cdot a = c$, $k = c/a$, $k = L_c^{/} a$. Similarly if $P_c^{-1} b = m$, then $m \cdot b = c$, $m = c/b$, $m = L_c^{/} b$. Since θ is a congruence of quasigroup $(Q, \cdot, \backslash, /)$ (Lemma 4.7), then implication (47) is true. $\qquad\square$

Corollary 4.9. *If θ is a normal quasigroup congruence of a quasigroup Q, then θ is a normal congruence of any parastrophe of Q [15].*

Proof. The proof follows from Lemma 4.8 and Table 1. $\qquad\square$

From Corollary 4.9 it follows that any element of the group $FM(Q, \cdot)$ of a quasigroup $(Q, \cdot)$ is admissible relative to any normal congruence of the quasigroup $(Q, \cdot)$.

In the following lemma we give information on behavior of a left quasigroup congruence relative to quasigroup parastrophy.

Lemma 4.10. *The following propositions are equivalent.*

1. *An equivalence θ is stable relative to translation L_c of a quasigroup $(Q, \cdot)$.*

2. *An equivalence θ is stable relative to translation R_c of a quasigroup $(Q, *)$.*

3. *An equivalence θ is stable relative to translation P_c^{-1} of a quasigroup $(Q, /)$.*

4. *An equivalence θ is stable relative to translation L_c^{-1} of a quasigroup $(Q, \backslash)$.*

5. *An equivalence θ is stable relative to translation R_c^{-1} of a quasigroup $(Q, \backslash\backslash)$.*

6. *An equivalence θ is stable relative to translation P_c of a quasigroup $(Q, //)$.*

Proof. From Table 1 it follows that $L_c^* = R_c$, $L_c^{/} = P_c^{-1}$. And so on. $\qquad\square$

Remark 4.11. It is easy to see that similar equivalences (Lemma 4.10) are true for the other five kinds of translations and its combinations of a quasigroup $(Q, \cdot)$.

Lemma 4.12.   *1. If an equivalence θ of a quasigroup $(Q, \cdot)$ is admissible relative to any left and right translation of this quasigroup, then θ is a normal congruence.*

2. *If an equivalence θ of a quasigroup $(Q, \cdot)$ is admissible relative to any left and middle translation of this quasigroup, then θ is a normal congruence.*

3. *If an equivalence θ of a quasigroup $(Q, \cdot)$ is admissible relative to any right and middle translation of this quasigroup, then θ is a normal congruence.*

Proof. Case 1. Definition 4.4.

Case 2. Equivalence θ is a normal congruence of quasigroup $(Q, \backslash)$ (Table 1, Definition 4.4) and we apply Corollary 4.9.

Case 3. Equivalence θ is a normal congruence of quasigroup $(Q, /)$ (Table 1, Definition 4.4) and we apply Corollary 4.9. $\qquad\square$

4.2 Other conditions of normality of equivalences

We find additional conditions when an equivalence is left, right or "middle" quasigroup congruence. We use H. Thurston [100, 99] and A.I. Mal'tsev [64] approaches. Results similar to the results from this section are in [78, 83].

The set of all left and right translations of a quasigroup $(Q, \cdot)$ will be denoted by $\mathbb{T}(Q, \cdot)$. If φ and ψ are binary relations on Q, then their product is defined in the following way: $(a, b) \in \varphi \circ \psi$ if there is an element $c \in Q$ such that $(a, c) \in \varphi$ and $(c, b) \in \psi$. If φ is a binary relation on Q, then $\varphi^{-1} = \{(y, x) \mid (x, y) \in \varphi\}$. The operation of the product of binary relations is associative [35, 69, 86, 71].

Remark 4.13. Translations of a quasigroup can be considered as binary relations: $(x, y) \in L_a$, if and only if $y = a \cdot x$; $(x, y) \in R_b$, if and only if $y = x \cdot b$; $(x, y) \in P_c$, if and only if $y = x \backslash c$ [74, 75, 78].

Remark 4.14. To coordinate the multiplication of translations with their multiplication as binary relations, we use the following multiplication of translations: if α, β are translations, x is an element of the set Q, then $(\alpha\beta)(x) = \beta(\alpha(x))$, i.e. $(\alpha\beta)x = \beta\alpha\, x$.

Proposition 4.15. *1. An equivalence θ is a left congruence of a quasigroup $(Q, \cdot)$ if and only if $\theta\omega \subseteq \omega\theta$ for all $\omega \in \mathbb{L}$.*

2. An equivalence θ is a right congruence of a quasigroup $(Q, \cdot)$ if and only if $\theta\omega \subseteq \omega\theta$ for all $\omega \in \mathbb{R}$.

3. An equivalence θ is a "middle" congruence of a quasigroup $(Q, \cdot)$ if and only if $\theta\omega \subseteq \omega\theta$ for all $\omega \in \mathbb{P}$.

4. An equivalence θ is a congruence of a quasigroup $(Q, \cdot)$ if and only if $\theta\omega \subseteq \omega\theta$ for all $\omega \in \mathbb{T}$ [74, 75, 78].

Proof. Case 1. Let θ be an equivalence, $\omega = L_a$. It is clear that $(x, z) \in \theta L_a$ is equivalent to that there exists an element $y \in Q$ such that $(x, y) \in \theta$ and $(y, z) \in L_a$. But if $(y, z) \in L_a$, $z = ay$, then $y = L_a^{-1}z$. Therefore, from the relation $(x, z) \in \theta L_a$ it follows that $(x, L_a^{-1}z) \in \theta$.

Let us prove that from $(x, L_a^{-1}z) \in \theta$ it follows $(x, z) \in \theta L_a$. We have $(x, L_a^{-1}z) \in \theta$ and $(L_a^{-1}z, z) \in L_a$, $(x, z) \in \theta L_a$. Thus $(x, z) \in \theta L_a$ is equivalent to $(x, L_a^{-1}z) \in \theta$.

Similarly, $(x, z) \in L_a\theta$ is equivalent to $(ax, z) \in \theta$. Now we can say that the inclusion $\theta\omega \subseteq \omega\theta$ by $\omega = L_a$ is equivalent to the following implication:

$$(x, L_a^{-1}z) \in \theta \Longrightarrow (ax, z) \in \theta$$

for all suitable $a, x, z \in Q$.

If we replace in the last implication z with $L_a z$, we shall obtain the following implication:

$$(x, z) \in \theta \Longrightarrow (ax, az) \in \theta$$

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for all $a \in Q$.

Thus, the inclusion $\theta L_a \subseteq L_a \theta$ is equivalent to the stability of the relation θ from the left relative to an element a. Since the element a is an arbitrary element of the set Q, we have that the inclusion $\theta \omega \subseteq \omega \theta$ by $\omega \in \mathbb{L}$ is equivalent to the stability of the relation θ from the left.

Case 2. Similarly, the inclusion $\theta \omega \subseteq \omega \theta$ for any $\omega \in \mathbb{R}$ is equivalent to the stability from the right of relation θ.

Case 3. Applying Case 1 to the quasigroup $(Q, //) = (Q, \cdot)^{(132)}$ we obtain that an equivalence θ is a left congruence of a quasigroup $(Q, \cdot)^{(132)}$ if and only if $\theta \omega \subseteq \omega \theta$ for all $\omega \in \mathbb{L}^{(132)}$.

Using parastrophic equivalence of translations (Table 1 or Lemma 4.10) we conclude that an equivalence θ is stable relative to any middle translation of a quasigroup $(Q, \cdot)$ if and only if $\theta \omega \subseteq \omega \theta$ for all $\omega \in \mathbb{P}$.

In other words an equivalence θ is a "middle" congruence of a quasigroup $(Q, \cdot)$ if and only if $\theta \omega \subseteq \omega \theta$ for all $\omega \in \mathbb{P}$.

Case 4. Uniting Cases 1 and 2 we obtain required equivalence. $\qquad \square$

Let us remark Proposition 4.15 can be deduced from results of H. Thurston [99].

Proposition 4.16.   *1. An equivalence θ is left-cancellative congruence of a quasigroup $(Q, \cdot)$ if and only if $\omega \theta \subseteq \theta \omega$ for all $\omega \in \mathbb{L}$.*

   2. An equivalence θ is right-cancellative congruence of a quasigroup $(Q, \cdot)$ if and only if $\omega \theta \subseteq \theta \omega$ for all $\omega \in \mathbb{R}$.

   3. An equivalence θ is middle-cancellative congruence of a quasigroup $(Q, \cdot)$ if and only if $\omega \theta \subseteq \theta \omega$ for all $\omega \in \mathbb{P}$.

Proof.   As it is proved in Proposition 4.15, the inclusion $\theta L_a \subseteq L_a \theta$ is equivalent to the implication $x \theta y \Longrightarrow ax \theta ay$.

Let us check up that the inclusion $L_a \theta \subseteq \theta L_a$ is equivalent to the implication

$$ax \theta ay \Rightarrow x \theta y.$$

Indeed, as it is proved in Proposition 4.15, $(x, z) \in \theta L_a$ is equivalent with $(x, L_a^{-1} z) \in \theta$. Similarly, $(x, z) \in L_a \theta$ is equivalent with $(ax, z) \in \theta$. The inclusion $\omega \theta \subseteq \theta \omega$ by $\omega = L_a$ has the form $L_a \theta \subseteq \theta L_a$ and it is equivalent to the following implication:

$$(ax, z) \in \theta \Longrightarrow (x, L_a^{-1} z) \in \theta$$

for all $a, x, z \in Q$. If we change in the last implication the element z by the element $L_a z$, we shall obtain that the inclusion $\theta L_a \supseteq L_a \theta$ is equivalent to the implication $ax \theta ay \Rightarrow x \theta y$. Therefore, the equivalence θ is cancellative from the left.

Similarly, the inclusion $R_b \theta \subseteq \theta R_b$ is equivalent to the implication:

$$(xa, za) \in \theta \Longrightarrow (x, z) \in \theta.$$

Case 3 is proved similarly. It is possible to use identity (6). $\qquad \square$

Below by the symbol $\langle \mathbb{L}, \mathbb{P} \rangle$ we shall denote the group generated by all left and middle translations of a quasigroup.

Corollary 4.17.   *1. An equivalence θ is left normal congruence of a quasigroup $(Q, \cdot)$ if and only if $\omega \theta = \theta \omega$ for all $\omega \in \mathbb{L}$.*

   2. An equivalence θ is right normal congruence of a quasigroup $(Q, \cdot)$ if and only if $\omega \theta = \theta \omega$ for all $\omega \in \mathbb{R}$.

3. An equivalence θ is middle normal congruence of a quasigroup $(Q, \cdot)$ if and only if $\omega\theta = \theta\omega$ for all $\omega \in \mathbb{P}$.

4. An equivalence θ is normal congruence of a quasigroup $(Q, \cdot)$ if and only if $\omega\theta = \theta\omega$ for all $\omega \in \mathbb{L} \cup \mathbb{P}$.

5. An equivalence θ is normal congruence of a quasigroup $(Q, \cdot)$ if and only if $\omega\theta = \theta\omega$ for all $\omega \in \mathbb{R} \cup \mathbb{P}$.

6. An equivalence θ is normal congruence of a quasigroup $(Q, \cdot)$ if and only if $\omega\theta = \theta\omega$ for all $\omega \in \langle \mathbb{L}, \mathbb{R} \rangle = M(Q, \cdot)$.

7. An equivalence θ is normal congruence of a quasigroup $(Q, \cdot)$ if and only if $\omega\theta = \theta\omega$ for all $\omega \in \langle \mathbb{L}, \mathbb{P} \rangle$.

8. An equivalence θ is normal congruence of a quasigroup $(Q, \cdot)$ if and only if $\omega\theta = \theta\omega$ for all $\omega \in \langle \mathbb{R}, \mathbb{P} \rangle$.

Proof. The proof follows from Propositions 4.15, 4.16 and Lemma 4.12.

In the proving of Cases 6–8 we can use the following fact: if $\omega\theta = \theta\omega$, then $\omega^{-1}\theta = \theta\omega^{-1}$. $\quad\square$

The following proposition is almost obvious corollary of Theorem 5 from [64].

Proposition 4.18.   1. An equivalence θ is a left congruence of a quasigroup Q if and only if $\omega\theta(x) \subseteq \theta(\omega x)$ for all $x \in Q$, $\omega \in \mathbb{L}$.

2. An equivalence θ is a right congruence of a quasigroup Q if and only if $\omega\theta(x) \subseteq \theta(\omega x)$ for all $x \in Q$, $\omega \in \mathbb{R}$.

3. An equivalence θ is a "middle" congruence of a quasigroup Q if and only if $\omega\theta(x) \subseteq \theta(\omega x)$ for all $x \in Q$, $\omega \in \mathbb{P}$.

4. An equivalence θ is left-cancelative congruence of a quasigroup Q if and only if $\theta(\omega x) \subseteq \omega\theta(x)$ for all $x \in Q$, $\omega \in \mathbb{L}$.

5. An equivalence θ is right-cancelative congruence of a quasigroup Q if and only if $\theta(\omega x) \subseteq \omega\theta(x)$ for all $x \in Q$, $\omega \in \mathbb{R}$.

6. An equivalence θ is middle-cancelative congruence (i.e. $P_x a\, \theta\, P_x b \Rightarrow a\theta b$) of a quasigroup Q if and only if $\theta(\omega x) \subseteq \omega\theta(x)$ for all $x \in Q$, $\omega \in \mathbb{P}$.

Proof. Case 1. Let θ be an equivalence relation and for all $\omega \in \mathbb{L}$, $\omega\theta(x) \subseteq \theta(\omega x)$. We shall prove that from $a\theta b$ follows $ca\theta cb$ for all $c \in Q$.

By definition of the equivalence θ, $a\theta b$ is equivalent to $a \in \theta(b)$. Then $ca \in c\theta(b) \subseteq \theta(cb)$, $ca\theta cb$.

Converse. Let θ be a left congruence. We shall prove that $c\theta(a) \subseteq \theta(ca)$ for all $c, a \in Q$. Let $x \in c\,\theta(a)$. Then $x = cy$, where $y \in \theta(a)$, that is $y\theta a$. Then, since θ is a left congruence, we obtain $cy\theta ca$. Therefore $x = cy \in \theta(ca)$. Thus, $L_c\theta \subseteq \theta(ca)$.

Case 2 is proved similarly.

Case 3. We can use approach similar to the approach used by the proof of Case 4 of Proposition 4.15.

Cases 4–6 are proved in the similar way with Cases 1–3. $\quad\square$

Proposition 4.19.   1. An equivalence θ of a quasigroup Q is left normal if and only if $\theta(\omega x) = \omega\theta(x)$ for all $\omega \in \mathbb{L}$, $x \in Q$;

2. A equivalence θ of a quasigroup Q is right normal if and only if $\theta(\omega x) = \omega\theta(x)$ for all $\omega \in \mathbb{R}$, $x \in Q$.

3. An equivalence θ of a quasigroup Q is middle normal if and only if $\theta(\omega x) = \omega\theta(x)$ for all $\omega \in \mathbb{P}$, $x \in Q$.

Proof. The proof follows from Proposition 4.18. $\qquad\square$

Proposition 4.20.   1. An equivalence θ is normal congruence of a quasigroup Q if and only if $\theta(\omega x) = \omega\theta(x)$ for all $\omega \in \mathbb{L} \cup \mathbb{R}$, $x \in Q$;

2. An equivalence θ is normal congruence of a quasigroup Q if and only if $\theta(\omega x) = \omega\theta(x)$ for all $\omega \in \mathbb{L} \cup \mathbb{P}$, $x \in Q$;

3. An equivalence θ is normal congruence of a quasigroup Q if and only if $\theta(\omega x) = \omega\theta(x)$ for all $\omega \in \mathbb{R} \cup \mathbb{P}$, $x \in Q$.

4. An equivalence θ is normal congruence of a quasigroup Q if and only if $\theta(\omega x) = \omega\theta(x)$ for all $\omega \in \langle \mathbb{L}, \mathbb{R} \rangle$, $x \in Q$;

5. An equivalence θ is normal congruence of a quasigroup Q if and only if $\theta(\omega x) = \omega\theta(x)$ for all $\omega \in \langle \mathbb{L}, \mathbb{P} \rangle$, $x \in Q$;

6. An equivalence θ is normal congruence of a quasigroup Q if and only if $\theta(\omega x) = \omega\theta(x)$ for all $\omega \in \langle \mathbb{R}, \mathbb{P} \rangle$, $x \in Q$.

Proof. The proof follows from Proposition 4.19 and Lemma 4.12.

In the proving of Cases 4–6 we can use the following fact. If $\theta(\omega x) = \omega\theta(x)$, then $\theta(\omega\omega^{-1}x) = \omega\theta(\omega^{-1}x) = \omega\omega^{-1}\theta(x)$. Thus $\omega\theta(\omega^{-1}x) = \omega\omega^{-1}\theta(x)$, $\theta(\omega^{-1}x) = \omega^{-1}\theta(x)$. $\qquad\square$

4.3   A-nuclei and congruences

Definition 4.21. *We define the following binary relation on a quasigroup $(Q, \cdot)$ that correspond to a subgroup H of the group $_1N_l^A$: $a\left(_1\theta_l^A\right)b$ if and only if there exists a permutation $\alpha \in H \subseteq {_1N_l^A}$ such that $b = \alpha a$.*

For a subgroup H of the group $_3N_l^A$ ($_1N_m^A$, $_2N_m^A$, $_2N_r^A$ and $_3N_r^A$) binary relation $_3\theta_l^A$ ($_1\theta_m^A$, $_2\theta_m^A$, $_2\theta_r^A$ and $_3\theta_r^A$, respectively) is defined in the similar way.

Lemma 4.22. *The binary relation $_1\theta_l^A$ is an equivalence of the set Q.*

Proof. Since binary relation $_1\theta_l^A$ is defined using orbits by the action on the set Q of a subgroup of the group $_1N_l^A$, we conclude that this binary relation is an equivalences. It is clear that any orbit defines an equivalence class. Notice, all these equivalence classes are of equal order (Corollary 2.39). $\qquad\square$

Analogs of Lemma 4.22 are true for any subgroup of the groups $_3\theta_l^A$, $_1\theta_m^A$, $_2\theta_m^A$, $_2\theta_r^A$ and $_3\theta_r^A$. These equivalences have additional properties.

Lemma 4.23. *Let $(Q, \cdot)$ be a quasigroup.*

1. *Let $_1\theta_l^A$ be an equivalence that is defined by a subgroup K of the group $_1N_l^A$ and equivalence $_3\theta_l^A$ is defined by isomorphic to K subgroup $R_x K R_x^{-1}$ of the group $_3N_l^A$ (Lemma 2.23). Then we have the following implications:*

(a) if $a(_1\theta_l^A)b$, then $(a \cdot c)(_3\theta_l^A)(b \cdot c)$ for all $a, b, c \in Q$;

(b) if $(a \cdot c)(_1\theta_l^A)(b \cdot c)$, then $a(_3\theta_l^A)b$ for all $a, b, c \in Q$;

2. Let $_3\theta_l^A$ be an equivalence that is defined by a subgroup K of the group $_3N_l^A$ and equivalence $_1\theta_l^A$ is defined by isomorphic to K subgroup $R_x^{-1}KR_x$ of the group $_1N_l^A$ (Lemma 2.23). Then we have the following implications:

(a) if $a(_3\theta_l^A)b$, then $(a \cdot c)(_1\theta_l^A)(b \cdot c)$ for all $a, b, c \in Q$;

(b) if $(a \cdot c)(_3\theta_l^A)(b \cdot c)$, then $a(_1\theta_l^A)b$ for all $a, b, c \in Q$;

3. Let $_2\theta_r^A$ be an equivalence that is defined by a subgroup K of the group $_2N_r^A$ and equivalence $_3\theta_r^A$ is defined by isomorphic to K subgroup $L_xKL_x^{-1}$ of the group $_3N_r^A$ (Lemma 2.23). Then we have the following implications:

(a) if $a(_2\theta_r^A)b$, then $(c \cdot a)(_3\theta_r^A)(c \cdot b)$ for all $a, b, c \in Q$;

(b) if $(c \cdot a)(_2\theta_r^A)(c \cdot b)$, then $a(_3\theta_r^A)b$ for all $a, b, c \in Q$;

4. Let $_3\theta_r^A$ be an equivalence that is defined by a subgroup K of the group $_3N_r^A$ and equivalence $_2\theta_r^A$ is defined by isomorphic to K subgroup $L_x^{-1}KL_x$ of the group $_3N_r^A$ (Lemma 2.23). Then we have the following implications:

(a) if $a(_3\theta_r^A)b$, then $(c \cdot a)(_2\theta_r^A)(c \cdot b)$ for all $a, b, c \in Q$;

(b) if $(c \cdot a)(_3\theta_r^A)(c \cdot b)$, then $a(_2\theta_r^A)b$ for all $a, b, c \in Q$;

5. Let $_1\theta_m^A$ be an equivalence that is defined by a subgroup K of the group $_1N_m^A$ and equivalence $_2\theta_m^A$ is defined by isomorphic to K subgroup $P_xKP_x^{-1}$ of the group $_2N_m^A$ (Lemma 2.23). Then we have the following implications:

(a) if $a(_1\theta_m^A)b$, then $P_ca(_2\theta_m^A)P_cb$ for all $a, b, c \in Q$;

(b) if $P_ca(_1\theta_m^A)P_cb$, then $a(_2\theta_m^A)b$ for all $a, b, c \in Q$;

6. Let $_2\theta_m^A$ be an equivalence that is defined by a subgroup K of the group $_2N_m^A$ and equivalence $_1\theta_m^A$ is defined by isomorphic to K subgroup $P_x^{-1}KP_x$ of the group $_1N_m^A$ (Lemma 2.23). Then we have the following implications:

(a) if $a(_2\theta_m^A)b$, then $P_c^{-1}a(_1\theta_m^A)P_c^{-1}b$ for all $a, b, c \in Q$;

(b) if $P_c^{-1}a(_2\theta_m^A)P_c^{-1}b$, then $a(_1\theta_m^A)b$ for all $a, b, c \in Q$.

Proof. Case 1, (a). Expression $a(_1\theta_l^A)b$ means that there exists a permutation $\alpha \in K \subseteq {}_1N_l^A$ such that $b = \alpha a$. Expression $(a \cdot c)(_3\theta_l^A)(b \cdot c)$ means that there exists a permutation $\gamma \in R_xKR_x^{-1} \subseteq {}_3N_l^A$ such that $b \cdot c = \gamma(a \cdot c)$. We can take $\gamma = R_cKR_c^{-1}$, i.e. can take $x = c$. Then $b \cdot c = \alpha a \cdot c$.

Then we obtain the following implication $b = \alpha a \implies b \cdot c = \alpha a \cdot c$ for all $a, b, c \in Q$. It is clear that this implication is true for all $a, b, c \in Q$, since $(Q, \cdot)$ is a quasigroup. Therefore, implication

$$\text{if } a(_1\theta_l^A)b, \text{ then } (a \cdot c)(_3\theta_l^A)(b \cdot c) \text{ for all } a, b, c \in Q$$

also is true.

Case 1, (b). Expression $(a \cdot c)(_1\theta_l^A)(b \cdot c)$ means that there exists a permutation $\alpha \in K \subseteq {}_1N_l^A$ such that $b \cdot c = \alpha(a \cdot c)$, i.e. $b = R_c^{-1}\alpha R_c a$. Expression $a(_3\theta_l^A)b$ means that there exists

a permutation $\gamma \in H = R_x K R_x^{-1} \subseteq {}_3 N_l^A$ such that $b = \gamma a$. By Lemma 2.23 we can take $\gamma = R_c^{-1} \alpha R_c$.

Then we obtain the following implication

$$b = R_c^{-1} \alpha R_c a \Longrightarrow b = R_c^{-1} \alpha R_c a \text{ for all } a, b, c \in Q$$

It is clear that this implication is true.

Cases 2–4 are proved in the similar way with Case 1.

Case 5, (a). Expression $a({}_1\theta_m^A)b$ means that there exists a permutation $\alpha \in K \subseteq {}_1 N_m^A$ such that $b = \alpha a$. Expression $P_c a({}_2\theta_m^A)P_c b$ means that there exists a permutation $\beta \in H \subseteq {}_2 N_m^A$ such that $P_c b = \beta P_c a$. By Lemma 2.23 we can take $\beta = P_c \alpha P_c^{-1}$. Then $P_c b = \beta P_c a = P_c \alpha P_c^{-1} P_c a = P_c \alpha a$.

Then we obtain the following implication $b = \alpha a \Longrightarrow P_c b = P_c \alpha a$ for all $a, b, c \in Q$. It is clear that this implication is true for all $a, b, c \in Q$.

Case 5, (b). Expression $P_c a({}_1\theta_m^A)P_c b$ means that there exists a permutation $\alpha \in H \subseteq {}_1 N_m^A$ such that $P_c a = \alpha P_c b$. By Lemma 2.23 we can take $\alpha = P_c \beta P_c^{-1}$, where $\beta \in {}_2 N_m^A$. Thus $P_c a = P_c \beta P_c^{-1} P_c b$, $a = \beta b$, $a({}_2\theta_m^A)b$.

Case 6 is proved in the similar way with Case 5. $\square$

Cases 1 and 2 show that the pair of equivalences ${}_1\theta_l^A$ and ${}_3\theta_l^A$ is normal from the right reciprocally.

Cases 3 and 4 show that the pair of equivalences ${}_2\theta_r^A$ and ${}_3\theta_r^A$ is normal from the left reciprocally.

Cases 5 and 6 show that the pair of equivalences ${}_1\theta_m^A$ and ${}_2\theta_m^A$ is stable relative to middle quasigroup translations and its inverse.

Corollary 4.24. *If in a quasigroup $(Q, \cdot)$:*

${}_1\theta_l^A = {}_3\theta_l^A = \theta_l^A$, *then the equivalence ${}_1\theta_l^A$ (${}_3\theta_l^A$) is normal from the right;*

${}_2\theta_r^A = {}_3\theta_r^A = \theta_r^A$, *then the equivalence ${}_2\theta_r^A$ (${}_3\theta_r^A$) is normal from the left;*

${}_1\theta_m^A = {}_2\theta_m^A = \theta_m^A$, *then the equivalence ${}_1\theta_m^A$ (${}_2\theta_m^A$) is stable relative to middle quasigroup translations and its inverse;*

$\theta_l^A = \theta_r^A$, *then equivalences ${}_1\theta_l^A$, ${}_3\theta_l^A$, ${}_2\theta_r^A$, ${}_3\theta_r^A$ are normal congruences;*

$\theta_l^A = \theta_m^A$, *then equivalences ${}_1\theta_l^A$, ${}_3\theta_l^A$, ${}_1\theta_m^A$, ${}_2\theta_m^A$ are normal congruences;*

$\theta_r^A = \theta_m^A$, *then equivalences ${}_2\theta_r^A$, ${}_3\theta_r^A$, ${}_1\theta_m^A$, ${}_2\theta_m^A$ are normal congruences.*

Proof. We use Lemmas 4.23 and 4.12. $\square$

If in conditions of Lemma 4.23 $K = {}_1 N_l^A$, then corresponding equivalence we shall denote by ${}_1\Theta_l^A$ and so on.

Corollary 4.25.   *1. If $(Q, \circ)$ is a right loop, then ${}_1\theta_l^A = {}_3\theta_l^A = \theta_l^A$ and equivalence ${}_1\theta_l^A$ is normal from the right. Any subgroup of the left nucleus N_l (coset class $\theta_l^A(1)$) is normal from the right.*

   2. If $(Q, \circ)$ is a left loop, then ${}_2\theta_r^A = {}_3\theta_r^A = \theta_r^A$ and equivalence ${}_2\theta_r^A$ is normal from the left. Any subgroup of the right nucleus N_r (coset class $\theta_r^A(1)$) is normal from the left.

   3. If $(Q, \circ)$ is an unipotent quasigroup, then ${}_1\theta_m^A = {}_2\theta_m^A = \theta_m^A$ and equivalence ${}_1\theta_m^A$ is stable relative to middle quasigroup translations and its inverse. Any subgroup of the middle nucleus N_m (coset class $\theta_m^A(1)$) is stable relative to middle quasigroup translations and its inverse.

Proof. The proof follows from Corollary 2.27 and Corollary 4.24. □

Corollary 4.26.   *1. If $(Q, \circ)$ is a commutative loop, then any from equivalences θ_l^A and θ_r^A is a normal congruence. Left nucleus N_l (coset class $\Theta_l^A(1)$) and right nucleus N_r (coset class $\Theta_r^A(1)$) are equal and N_l is a normal subgroup of $(Q, \circ)$.*

2. If $(Q, \circ)$ is a unipotent right loop with identity $x \circ (x \circ y) = y$, then any from equivalences θ_l^A and θ_m^A is a normal congruence. Left nucleus N_l (coset class $\Theta_l^A(1)$) and middle nucleus N_m (coset class $\Theta_m^A(1)$) are equal and N_l is a normal subgroup of $(Q, \circ)$.

3. If $(Q, \circ)$ is a unipotent left loop with identity $(x \circ y) \circ y = x$, then any from equivalences θ_r^A and θ_m^A is a normal congruence. Right nucleus N_r (coset class $\Theta_r^A(1)$) and middle nucleus N_m (coset class $\Theta_m^A(1)$) are equal and N_r is a normal subgroup of $(Q, \circ)$.

Proof. Case 1. In any right loop first and third component of any element of the group N_l^A coincide, in any left loop second and third component of the group N_r^A coincide (Corollary 2.18). In a commutative loop in additional $L_a = R_a$ for any element $a \in Q$. From Corollary 4.24 it follows that in this case any equivalence θ_l^A is a normal congruence and therefore any subgroup of left nucleus N_l is normal as coset class $\theta_l^A(1)$ of normal congruence θ_l^A.

Cases 2 and 3 are proved using passage to parastrophy images ((23) and (13), respectively) of the loop from Case 1. □

Lemma 4.27. *In a loop $(Q, \cdot)$ any subgroup of group Z (Z_l^A, Z_r^A, Z_m^A, $_1Z_l^A$, $_3Z_l^A$, $_2Z_r^A$, $_3Z_r^A$, $_1Z_m^A$, $_2Z_m^A$) defines a normal congruence.*

Proof. We use Lemma 3.11 and Corollary 4.24. If H is a subgroup of the group Z, then corresponding to H subgroups of the groups $_1Z_l^A$, $_3Z_l^A$ are equal, since $(Q, \cdot)$ is a loop. Therefore corresponding equivalences $_1\zeta_l^A = {}_3\zeta_l^A$ are equal. Since $_1Z_l^A = {}_2Z_r^A = {}_3Z_r^A$ (Lemma 3.11) we obtain that $_1\zeta_l^A = {}_2\zeta_r^A = {}_3\zeta_r^A$ and equivalence $_1\zeta_l^A$ is normal congruence (Corollary 4.24) of loop $(Q, \cdot)$.

Other cases are proved in the similar way. □

Corollary 4.28. *In a loop $(Q, \cdot)$ A-center defines a normal congruence.*

Proof. We use Lemma 4.27. □

We give conditions of normality of A-nuclei of a loop. We also give conditions of normality of Garrison's nuclei, since Garrison's nuclei are coset classes by action of corresponding A-nuclei.

Theorem 4.29. *Let $(Q, \cdot)$ be a loop.*

1. Let H be a subgroup of the group $_1N_l^A$. Denote by $_1\theta_l^A$ corresponding equivalence to the group H.

(a) If equivalence $_1\theta_l^A$ is admissible relative to any translation L_x of loop $(Q, \cdot)$ and its inverse, then $_1\theta_l^A$ is normal congruence of $(Q, \cdot)$.

(b) If equivalence $_1\theta_l^A$ is admissible relative to any translation P_x of loop $(Q, \cdot)$ and its inverse, then $_1\theta_l^A$ is normal congruence of $(Q, \cdot)$.

2. Let H be a subgroup of the group $_2N_r^A$. Denote by $_2\theta_r^A$ corresponding equivalence to the group H.

(a) If equivalence $_2\theta_r^A$ is admissible relative to any translation R_x of loop $(Q, \cdot)$ and its inverse, then $_2\theta_r^A$ is normal congruence of $(Q, \cdot)$.

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(b) If equivalence $_2\theta_r^A$ is admissible relative to any translation P_x of loop $(Q, \cdot)$ and its inverse, then $_2\theta_r^A$ is normal congruence of $(Q, \cdot)$.

3. Let H be a subgroup of the group $_1N_m^A \cap {}_2N_m^A$. Denote by $_1\theta_m^A$ corresponding equivalence to the group H.

    (a) If equivalence $_1\theta_m^A$ is admissible relative to any translation R_x of loop $(Q, \cdot)$ and its inverse, then $_1\theta_m^A$ is normal congruence of $(Q, \cdot)$.

    (b) If equivalence $_1\theta_m^A$ is admissible relative to any translation L_x of loop $(Q, \cdot)$ and its inverse, then $_1\theta_m^A$ is normal congruence of $(Q, \cdot)$.

Proof. The proof follows from Lemma 4.12 and Corollary 4.25. $\qquad\square$

Theorem 4.30. *If in a quasigroup $(Q, \cdot)$ $H = {}_1N_l^A \cap {}_3N_l^A \cap {}_2N_r^A \cap {}_3N_r^A$, then H induces normal congruence of $(Q, \cdot)$.*

If in a quasigroup $(Q, \cdot)$ $H = {}_1N_l^A \cap {}_3N_l^A \cap {}_1N_m^A \cap {}_2N_m^A$, then H induces normal congruence of $(Q, \cdot)$.

If in a quasigroup $(Q, \cdot)$ $H = {}_2N_r^A \cap {}_3N_r^A \cap {}_1N_m^A \cap {}_2N_m^A$, then H induces normal congruence of $(Q, \cdot)$.

Proof. The proof follows from Lemma 4.23. $\qquad\square$

4.4   Normality of A-nuclei of inverse quasigroups

Mainly information on the coincidence of A-nuclei and nuclei of inverse quasigroups and loops are taken from [52]. Also we give results on normality of nuclei of some inverse quasigroups and loops.

Definition 4.31. *A quasigroup $(Q, \circ)$ has the λ-inverse-property if there exist permutations $\lambda_1, \lambda_2, \lambda_3$ of the set Q such that*

$$\lambda_1 x \circ \lambda_2(x \circ y) = \lambda_3 y \qquad (48)$$

for all $x, y \in Q$ [18].

Definition 4.32. *A quasigroup $(Q, \circ)$ has the ρ-inverse-property if there exist permutations ρ_1, ρ_2, ρ_3 of the set Q such that*

$$\rho_1(x \circ y) \circ \rho_2 y = \rho_3 x \qquad (49)$$

for all $x, y \in Q$ [18].

Definition 4.33. *A quasigroup $(Q, \circ)$ is an (α, β, γ)-inverse quasigroup if there exist permutations α, β, γ of the set Q such that*

$$\alpha(x \circ y) \circ \beta x = \gamma y \qquad (50)$$

for all $x, y \in Q$ [51, 7, 50, 96].

Definition 4.34. *A quasigroup $(Q, \circ)$ has the μ-inverse-property if there exist permutations μ_1, μ_2, μ_3 of the set Q such that*

$$\mu_1 y \circ \mu_2 x = \mu_3(x \circ y) \qquad (51)$$

for all $x, y \in Q$.

We give definition of main classes of inverse quasigroups using autostrophy [18, 52, 80].

Theorem 4.35. *A quasigroup $(Q, \cdot)$ is:*

1. *a λ-inverse quasigroup if and only if it has $[(2\,3), (\lambda_1, \lambda_2, \lambda_3)]$ autostrophy;*

2. *a ρ-inverse quasigroup if and only if it has $[(1\,3), (\rho_1, \rho_2, \rho_3)]$ autostrophy;*

3. *an (α, β, γ)-inverse quasigroup if and only if it has $[(1\,2\,3), (\alpha, \beta, \gamma)]$ autostrophy;*

4. *a μ-inverse quasigroup if and only if it has $[(12), (\mu_1, \mu_2, \mu_3)]$ autostrophy.*

Proof. If $(\sigma, (\alpha_1, \alpha_2, \alpha_3))$ is an autostrophism of a quasigroup (Q, A), then from formula (9) we obtain the following formula

$$A(x_1, x_2, x_3) = A(\alpha_1 x_{\sigma^{-1}1}, \alpha_2 x_{\sigma^{-1}2}, \alpha_3 x_{\sigma^{-1}3}) \tag{52}$$

Case 1. Using equality (52) we find values of σ and α_i. Since $\alpha_1 x_{\sigma^{-1}1} = \lambda_1 x_1$, we have $\alpha_1 = \lambda_1$, $x_{\sigma^{-1}1} = x_1$, $\sigma 1 = 1$. And so on.

Cases 2–4 are proved in similar way with Case 1. $\square$

Lemma 4.36.   1. *A λ-inverse quasigroup has $((2\,3), (\lambda_1^{-1}, \lambda_3^{-1}, \lambda_2^{-1}))$ autostrophy;*

2. *a ρ-inverse quasigroup has $((1\,3), (\rho_3^{-1}, \rho_2^{-1}, \rho_1^{-1}))$ autostrophy;*

3. *an (α, β, γ)-inverse quasigroup has $((1\ 3\ 2), (\beta^{-1}, \gamma^{-1}, \alpha^{-1}))$ autostrophy, i.e. in any (α, β, γ)-inverse quasigroup $(Q, \circ)$ the following equality is true $\beta^{-1}y \circ \gamma^{-1}(x \circ y) = \alpha^{-1}x$;*

4. *a μ-inverse quasigroup has $((12), (\mu_2^{-1}, \mu_1^{-1}, \mu_3^{-1}))$ autostrophy.*

Proof. From Lemma 1.35 we have. If $(\sigma, T) = (\sigma, (\alpha_1, \alpha_2, \alpha_3))$ is an autostrophism, then

$$(\sigma, T)^{-1} = (\sigma^{-1}, (T^{-1})^{\sigma^{-1}}) = (\sigma^{-1}, (\alpha_{\sigma 1}^{-1}, \alpha_{\sigma 2}^{-1}, \alpha_{\sigma 3}^{-1})) \tag{53}$$

$\square$

4.4.1 (α, β, γ)-inverse quasigroups

Lemma 4.37. *In (α, β, γ)-inverse quasigroup we have the following relations between components of A-nuclei:*

$$_1N_l^A = \alpha^{-1}{}_3N_r^A\alpha = \beta\,_2N_m^A\beta^{-1} \tag{54}$$

$$_3N_l^A = \gamma^{-1}{}_2N_r^A\gamma = \alpha\,_1N_m^A\alpha^{-1} \tag{55}$$

$$_2N_r^A = \beta^{-1}{}_1N_m^A\beta = \gamma\,_3N_l^A\gamma^{-1} \tag{56}$$

$$_3N_r^A = \gamma^{-1}{}_2N_m^A\gamma = \alpha\,_1N_l^A\alpha^{-1} \tag{57}$$

$$_1N_m^A = \alpha^{-1}{}_3N_l^A\alpha = \beta\,_2N_r^A\beta^{-1} \tag{58}$$

$$_2N_m^A = \beta^{-1}{}_1N_l^A\beta = \gamma\,_3N_r^A\gamma. \tag{59}$$

Proof. The proof follows from Theorem 4.35, Lemma 4.36 and Table 3. $\square$

Theorem 4.38.   1. *In $(\varepsilon; \beta; \gamma)$-inverse loop $(Q, \circ)$ $_1N_l^A = {}_3N_l^A = {}_2N_r^A = {}_3N_r^A = {}_1N_m^A$, $N_l = N_r = N_m \trianglelefteq Q$.*

2. *In $(\alpha; \varepsilon; \gamma)$-inverse loop $(Q, \circ)$ $_1N_l^A = {}_3N_l^A = {}_2N_m^A$, $_2N_r^A = {}_3N_r^A = {}_1N_m^A$, $N_l = N_r = N_m$.*

3. *In $(\alpha; \beta; \varepsilon)$-inverse loop $(Q, \circ)$ $_1N_l^A = {}_3N_l^A = {}_2N_r^A = {}_3N_r^A = {}_2N_m^A$, $N_l = N_r = N_m \trianglelefteq Q$.*

4. In $(\alpha; \alpha^{-1}; \gamma)$-inverse loop $(Q, \circ)$ $_1N_l^A = {_3}N_l^A = {_2}N_r^A = {_3}N_r^A = {_2}N_m^A$, $N_l = N_r = N_m \trianglelefteq Q$.

5. In $(\alpha; \beta; \alpha^{-1})$-inverse loop $(Q, \circ)$ $_1N_l^A = {_3}N_l^A = {_2}N_m^A$, $_2N_r^A = {_3}N_r^A = {_1}N_m^A$, $N_l = N_r = N_m$.

6. In $(\alpha; \beta; \beta^{-1})$-inverse loop $(Q, \circ)$ $_1N_l^A = {_3}N_l^A = {_2}N_r^A = {_3}N_r^A = {_1}N_m^A$, $N_l = N_r = N_m \trianglelefteq Q$.

Proof. The proof follows from Lemma 4.37 and the fact that in any loop $_1N_l^A = {_3}N_l^A$, $_2N_r^A = {_3}N_r^A$. Normality of Garrison's nuclei follows from Corollary 4.24. $\square$

Theorem 4.39. *If in $(\alpha; \beta; \gamma)$-inverse loop $(Q, \cdot)$ $\alpha \, {_3}N_r^A \alpha^{-1} \subseteq {_3}N_r^A$, $\beta \, {_1}N_m^A \beta^{-1} \subseteq {_1}N_m^A$, $\alpha^{-1} {_3}N_l^A \alpha \subseteq {_3}N_l^A$, then in this loop $N_l = N_m = N_r$.*

Proof. By proving this theorem we use standard way [49]. Let $L_a \in {_1}N_l^A$, i.e. $a \in N_l$. Then using (54) we have $\alpha^{-1} L_a \alpha \in {_3}N_r^A$. Therefore $L_a \in \alpha \, {_3}N_r^A \alpha^{-1} \subseteq {_3}N_r^A$. Thus $L_a 1 = a \in {_3}N_r^A 1 = N_r$.

If $a \in N_r$, then $(\varepsilon, R_a, R_a) \in N_r^A$ and using (56) we see that $\beta^{-1} R_a \beta \in {_1}N_m^A$. Then $R_a \in \beta \, {_1}N_m^A \beta^{-1} \subseteq {_1}N_m^A$. Therefore $a = R_a 1 \in {_1}N_m^A 1 = N_m$.

If $a \in N_m$, then $(R_a, L_a^{-1}, \varepsilon) \in N_m^A$ and using (58) we see that $\alpha R_a \alpha^{-1} \in {_3}N_l^A$. Then $R_a \in \alpha^{-1} {_3}N_l^A \alpha \subseteq {_3}N_l^A$. Therefore $a = R_a 1 \in {_3}N_l^A 1 = N_l$.

We have obtained $N_l \subseteq N_r \subseteq N_m \subseteq N_l$, therefore $N_l = N_r = N_m$. $\square$

Definition 4.40. *A quasigroup $(Q, \circ)$ is*

1. *an (r, s, t)-inverse quasigroup if there exist a permutation J of the set Q such that $J^r(x \circ y) \circ J^s x = J^t y$ for all $x, y \in Q$ [51, 50].*

2. *an m-inverse quasigroup if there exist a permutation J of the set Q such that $J^m(x \circ y) \circ J^{m+1} x = J^m y$ for all $x, y \in Q$ [49, 51, 50].*

3. *a WIP-inverse quasigroup if there exist a permutation J of the set Q such that $J(x \circ y) \circ x = Jy$ for all $x, y \in Q$ [14, 7, 96, 50].*

4. *a CI-inverse quasigroup if there exist a permutation J of the set Q such that $(x \circ y) \circ Jx = y$ for all $x, y \in Q$ [4, 5, 20, 14].*

It is easy to see, that classes of (r, s, t)-inverse, m-inverse, WIP-inverse, CI-inverse quasigroups are included in the class of (α, β, γ)-inverse quasigroups.

Lemma 4.41. *An (r, s, t)-inverse quasigroup has $((1\,2\,3), (J^r, J^s, J^t))$ autostrophy.*
An m-inverse quasigroup has $((1\,2\,3), (J^m, J^{m+1}, J^m))$ autostrophy.
A WIP-inverse quasigroup has $((1\,2\,3), (J, \varepsilon, J))$ autostrophy.
A CI-inverse quasigroup has $((1\,2\,3), (\varepsilon, J, \varepsilon))$ autostrophy.

Proof. The proof follows from Theorem 4.35. $\square$

Theorem 4.42. 1. *If in m-inverse loop $(Q, \circ)$ $J = I_r$, where $x \circ I_r x = 1$, then $N_l = N_r = N_m$.*

2. *In WIP-loop $_1N_l^A = {_3}N_l^A = {_2}N_m^A$, $_2N_r^A = {_3}N_r^A = {_1}N_m^A$, $N_l = N_r = N_m$.*

3. *In CI-loop $_1N_l^A = {_3}N_l^A = {_2}N_m^A = {_2}N_r^A = {_3}N_r^A = {_1}N_m^A$, $N_l = N_r = N_m \trianglelefteq Q$.*

Proof. Case 1. The proof follows from Theorem 4.39.
Cases 2 and 3. The proof follows from Theorem 4.38 and Corollary 4.25. $\square$

4.4.2 λ-, ρ-, and μ-inverse quasigroups

Lemma 4.43.   *1. In λ-inverse quasigroup*

  (a) $_1N_l^A = \lambda_1^{-1}{}_1N_m^A\lambda_1 = \lambda_1\ _1N_m^A\lambda_1^{-1}$;
  (b) $_3N_l^A = \lambda_3^{-1}{}_2N_m^A\lambda_3 = \lambda_2\ _2N_m^A\lambda_2^{-1}$;
  (c) $_2N_r^A = \lambda_2^{-1}{}_3N_r^A\lambda_2 = \lambda_3\ _3N_r^A\lambda_3^{-1}$.

 2. In ρ-inverse quasigroup

  (a) $_1N_l^A = \rho_1^{-1}{}_3N_l^A\rho_1 = \rho_3\ _3N_l^A\rho_3^{-1}$;
  (b) $_2N_r^A = \rho_2^{-1}{}_2N_m^A\rho_2 = \rho_2\ _2N_m^A\rho_2^{-1}$;
  (c) $_3N_r^A = \rho_3^{-1}{}_1N_m^A\rho_3 = \rho_1\ _1N_m^A\rho_1^{-1}$.

 3. In μ-inverse quasigroup

  (a) $_1N_l^A = \mu_1^{-1}{}_2N_r^A\mu_1 = \mu_2\ _2N_r^A\mu_2^{-1}$;
  (b) $_3N_l^A = \mu_3^{-1}{}_3N_r^A\mu_3 = \mu_3\ _3N_r^A\mu_3^{-1}$;
  (c) $_1N_m^A = \mu_1^{-1}{}_2N_m^A\mu_1 = \mu_2\ _2N_m^A\mu_2^{-1}$.

Proof. We can use Theorem 4.35 Lemma 4.36 and Table 3. $\square$

 A quasigroup $(Q,\cdot)$ with identities $xy = yx$, $x \cdot xy = y$ is called TS-quasigroup. In TS-quasigroup any parastrophy is an autostrophy [14], i.e. $(Q,\cdot)^\sigma = (Q,\cdot)$ for all parastrophies $\sigma \in S_3$.

Corollary 4.44.   *1. In LIP-quasigroup $_1N_l^A = \lambda\,(_1N_m^A)\lambda$, $\lambda^2 = \varepsilon$, $_3N_l^A = {}_2N_m^A$, $_2N_r^A = {}_3N_r^A$.*

 2. In RIP-quasigroup $_2N_r^A = \rho\,(_2N_m^A)\rho$, $\rho^2 = \varepsilon$, $_1N_l^A = {}_3N_l^A$, $_1N_m^A = {}_3N_r^A$.

 3. In IP-quasigroup $\lambda^2 = \varepsilon$, $\rho^2 = \varepsilon$, $_1N_l^A = {}_3N_l^A = {}_2N_m^A \cong {}_1N_m^A = {}_2N_r^A = {}_3N_r^A$.

 4. In TS-quasigroup $_1N_l^A = {}_3N_l^A = {}_1N_m^A = {}_2N_m^A = {}_2N_r^A = {}_3N_r^A$.

Proof.   The proof follows from Lemma 4.43. In *LIP*-quasigroup $\lambda_2 = \lambda_3 = \varepsilon$. In *RIP*-quasigroup $\rho_1 = \rho_3 = \varepsilon$. $\square$

Theorem 4.45.   *1. In LIP-quasigroup $_2\Theta_r^A = {}_3\Theta_r^A$ and these equivalences are normal from the left.*

 2. In RIP-quasigroup $_1\Theta_l^A = {}_3\Theta_l^A$, these equivalences are normal from the right.

 3. In TS-quasigroup we have $_1\Theta_l^A = {}_3\Theta_l^A = {}_2\Theta_r^A = {}_3\Theta_r^A = {}_1\Theta_m^A = {}_2\Theta_m^A$ and all these equivalences are normal congruences.

Proof. Cases 1 and 2. The proof follows from Lemma 4.43, Corollaries 4.24 and 4.26.
 By proving of Case 3 it is possible to use Table 3 and Corollary 4.24. $\square$

Corollary 4.46. *In commutative LIP-loop, commutative RIP-loop $N_l = N_r = N_m \trianglelefteq Q$. In $(\varepsilon; \mu_2; \mu_3)$-, $(\mu_1; \varepsilon; \mu_3)$-, and $(\mu_1; \mu_2; \varepsilon)$-inverse loop $N_l = N_r \trianglelefteq Q$.*

Proof. The proof follows from Corollary 4.44. $\square$

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