

Scaling properties of correlated random walks

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Abstract

Many stochastic time series can be modelled by discrete random walks in which a step of random sign but constant length δx is performed after each time interval δt . In correlated discrete time random walks (CDTRWs), the probability q for two successive steps having the same sign is unequal $1/2$. The resulting probability distribution $P(\Delta x, \Delta t)$ that a displacement Δx is observed after a lagtime Δt is known analytically for arbitrary persistence parameters q . In this short note we show how a CDTRW with parameters $[\delta t, \delta x, q]$ can be mapped onto another CDTRW with rescaled parameters $[\delta t/s, \delta x \cdot g(q, s), q'(q, s)]$, for arbitrary scaling parameters s , so that both walks have the same displacement distributions $P(\Delta x, \Delta t)$ on long time scales. The nonlinear scaling functions $g(q, s)$ and $q'(q, s)$ are derived explicitly. This scaling method can be used to model time series measured at discrete sample intervals δt but actually corresponding to continuum processes with variations occurring on a much shorter time scale $\delta t/s$.

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I. THE DIMENSIONLESS CDTRW

Originally, a CDTRW does not have any physical scales δx or δt attached to its steps, but uses dimensionless integers k and n to count the discrete units of space and time. The trajectory of the fictuous random walker (the sequence of integer ‘positions’ X_n) is determined by the initial position X_0 (usually unimportant) and the sequence of steps, or differences $\nu_n = X_n - X_{n-1}$. These steps ν_n only take two possible values -1 and $+1$ and can thus be called sign factors. Since the probability of each sign depends only on the directly preceding one, a discrete correlated random walk is a Markov chain. It is therefore easy to show that for a given persistence probability q , the autocorrelation function of ν_n is given by

$$C_{\nu\nu}(n | q) = \frac{\langle \nu_m \nu_{m+n} \rangle_m}{\langle \nu_m^2 \rangle_m} = (2q - 1)^n. \quad (1)$$

Since the resulting statistical properties of the CDTRW model have been derived analytically in Ref.[1], we just repeat the main results here. The displacement distribution for a DTCRW with persistence parameter q is given by

$$\begin{aligned} P(k, n | q) &= \\ &= \sum_{m=1}^{(n-|k|)/2} \binom{(n+k-2)/2}{m-1} \binom{(n-k-2)/2}{m-1} \\ &\cdot (1-q)^{2m-1} q^{n-1-2m} \left(\frac{n(1-q) + 2m(2q-1)}{2m} \right), \end{aligned} \quad (2)$$

where n and k must either both be even or both be odd. It has additionally been assumed that initially (at time step $t=0$) the probability for the particle to go left or right are equal. Note that $P(k, n) = 0$ for $|k| > n$. The corresponding mean squared displacement, defined as the second moment of $P(k, n | q)$, is given by

$$\overline{\Delta k^2}(n | q) = \frac{nq}{1-q} \left\{ 1 - \frac{(2q-1)[1 - (2q-1)^n]}{2nq(1-q)} \right\}. \quad (3)$$

II. THE CDTRW WITH PHYSICAL DIMENSIONS

In order to relate the CDTRW model to an actual measured time series, it is necessary to associate the integers k and n with physical quantities (having dimensions) by defining $x_k = k \delta x$ and $t_n = n \delta t$. Then, the ‘physical’ displacement distribution is given by

$$p(\Delta x, \Delta t | q, \delta x, \delta t) = P(k = \Delta x / \delta x, n = \Delta t / \delta t | q) / (2\delta x). \quad (4)$$

Displacement probabilities for Δx and Δt that are not integer multiples of δx and δt can be obtained by interpolation (compare Fig.1).

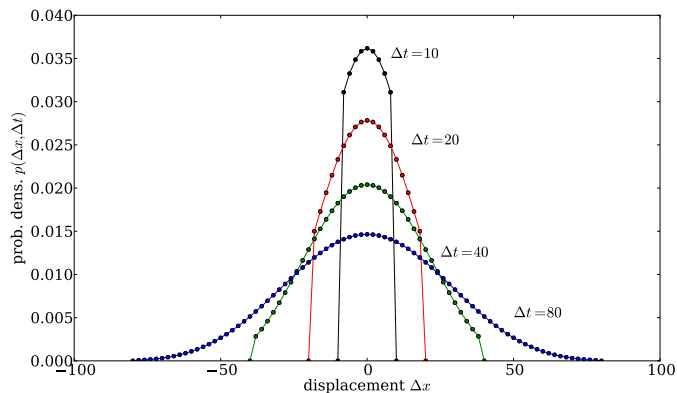


FIG. 1. *Displacement distributions for a CTRW model with parameters $\delta x = 1$, $\delta t = 1$ and $q = 0.9$. Note that the probability density is zero for $\Delta x > \Delta t$. The dots correspond to the discrete values of the model. The lines are interpolated.*

The physical mean squared is given by

$$\overline{\Delta x^2}(\Delta t | q, \delta x, \delta t) = \overline{\Delta k^2}(n = \Delta t / \delta t | q) \cdot \delta x^2. \quad (5)$$

Again, function values for non-integer multiples of δt can be interpolated.

III. SCALING OF THE PERSISTENCE PROBABILITY

Consider a correlated random walk A with a persistence parameter $q > \frac{1}{2}$. We seek to construct another walk B that after $s \cdot n$ steps has the same correlations as A after n steps. We therefore require

$$C_{\nu\nu}(n | q) = C_{\nu\nu}(s \cdot n | q'). \quad (6)$$

Inserting Eq.(1) and solving for the unknown persistence parameter q' of walk B yields $q' = \frac{1}{2} \left(1 + |2q - 1|^{1/s} \right)$. In the case of persistence parameters $q < \frac{1}{2}$ we can write $(2q - 1)^n$

as $(-1)^n |2q - 1|^n$ and perform an analogous calculation. As a result, we obtain a scaling rule for the persistence parameter:

$$q' = q'(q, s) = \frac{1}{2} \left(1 \pm |2q - 1|^{1/s} \right), \quad (7)$$

where the sign $+$ ($-$) has to be used if $q > 1/2$ ($q < 1/2$). This transformation has no effect on uncorrelated walks with $q = 0.5$. Persistent walks with $q > 0.5$ become even more persistent. Anti-persistent walks with $q < 0.5$ become even more anti-persistent (compare Fig.2).

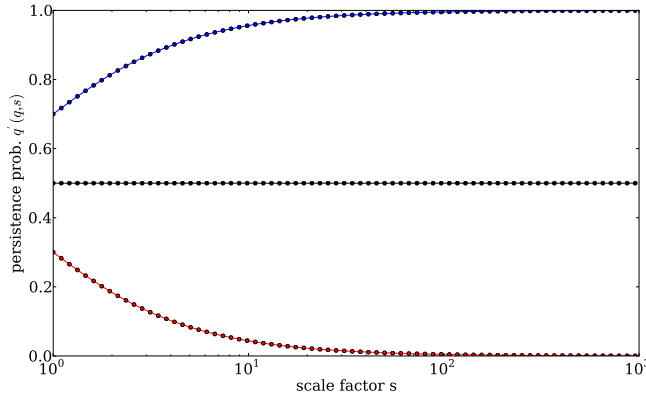


FIG. 2. Dependence of the rescaled persistence probability $q'(q, s)$ on the scaling parameter s for the original q -values 0.3 (red), 0.5 (black) and 0.7 (blue).

IV. SCALING OF THE STEP LENGTH

While the above transformation of the persistence probability ensures that the original walk A and the rescaled walk B have equivalent autocorrelation functions, this does not automatically guarantee equivalence of the mean squared displacements. In order to achieve the latter point, the step length has to be rescaled as well. For this purpose, note that for lagtimes Δt much longer than the correlation time, the mean squared displacement behaves like that of a diffusive process and is given by $\overline{\Delta x^2} \rightarrow \delta x^2 q(1 - q) \Delta t / \delta t$. To match the asymptotic values of the two walks, we therefore require

$$\overline{\Delta x^2}(s \cdot \Delta t | q', \delta x', \delta t) = \overline{\Delta x^2}(\Delta t | q, \delta x, \delta t). \quad (8)$$

As a result, we obtain an additional scaling law for the step length:

$$\boxed{\delta x' = \delta x \cdot g(q, s) \text{ with } g(q, s) = \sqrt{\frac{q(1-q)}{q'(1-q')} \frac{1}{s}}.} \quad (9)$$

V. SUMMARY AND EXAMPLES

We thereby have obtained two coupled scaling relations for the persistent probability and the step length: When the time step of a CDTRW is changed from δt to $\delta t/s$, the new walk will have equivalent statistical properties if its parameter pair $(q, \delta x)$ is changed to $(q' = q'(q, s), \delta x' = \delta x \cdot g(q, s))$, with the scaling functions $q'(q, s)$ and $g(q, s)$ as defined above (For an example, see Fig.3).

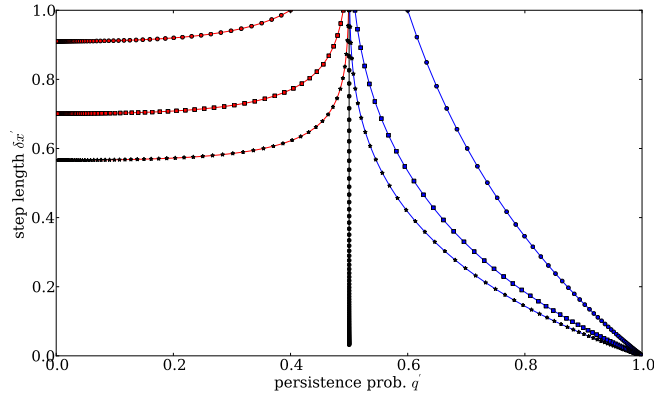


FIG. 3. Flux of parameter pairs $(q', \delta x')$ during the scaling transformation. The starting points, corresponding to scale parameter $s = 1$, all have identical step length $\delta x = 1$. With s increasing from 1 to 1000, the step lengths $\delta x'$ become smaller. For anti-persistent walks (red), the step lengths approach a constant asymptotic value, for persistent walks (blue), they decay to zero. The shown curves correspond to different initial q -values: red circles: $q = 0.4$, red squares: $q = 0.49$, red stars: $q = 0.499$, black circles: $q = 0.5$, blue stars: $q = 0.501$, blue squares: $q = 0.51$, blue circles: $q = 0.6$.

We have tested the scaling relations by directly comparing the displacement distributions and the mean squared displacements for an original walk with sampling time $\delta t = 1$ and a properly scaled walk with $\delta t = 1/10$. The results are shown in Fig.4.

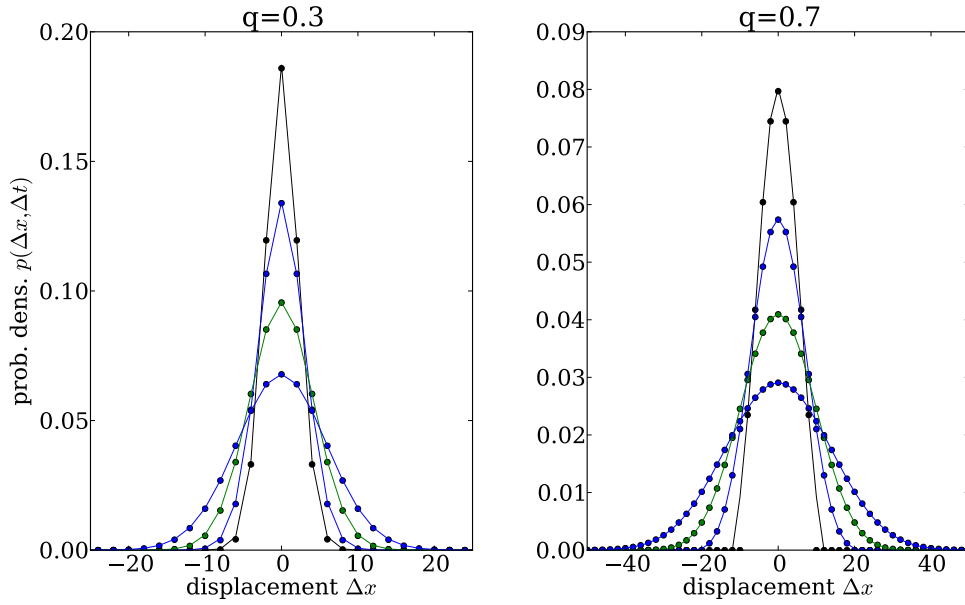


FIG. 4. Displacement distributions of CDTRWs for the persistence probabilities $q = 0.3$ (left) and $q = 0.7$ (right). Circles correspond to the ‘original’ random walk, using a time step $\delta t = 1$ and a step length $\delta x = 1$. Lines correspond to the rescaled random walk, using a ten times smaller time step $\delta t = 0.1$ and appropriately transformed values of the parameters q and δx . The results are identical, thus confirming the scaling transformation.

ACKNOWLEDGMENTS

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[1] J. Hanneken and D. Franceschetti, Journal of Chemical Physics **109**, 6533 (1998).