The thermal Casimir effect due to the blackbody photons

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Abstract

The photons within the box with the different edges generates pressure as the analogue of the Casimir zero-energy vacuum photons, or, quantum mechanical pressure of particles within such box. However, with regard to the fact that the photon gas has the temperature T, it is necessary to perform the transformation to the thermodynamical situation in the box. Then, the so called finite-temperature Casimir pressure on the wall of the thermal box is derived. The submitted approach can be easily generalized to phonon thermal bath, magnon thermal bath and so on.

1 Introduction

The Casimir effect and the Casimir-Polder force are physical forces arising from a quantized field. They are named after the Dutch physicist Hendrik Casimir who predicted it in 1948.

The Casimir effect is an interaction between disjoint neutral bodies caused by the fluctuations of the electrodynamic vacuum. It can be explained by considering the normal modes of electromagnetic fields, which explicitly depend on the boundary (or matching) conditions on the interacting bodies surfaces. Since electromagnetic field interaction is strong for a one-atom-thick material, the Casimir effect is of interest for graphene too.

At the most basic level, the field at each point in space is a simple quantum harmonic oscillator. Excitations of the field (oscillator) correspond to the elementary particles of particle physics. However, even the vacuum has a complex structure, all calculations must be made in relation to such model of the vacuum.

The Casimir effect at finite temperature is the integral part of the finite-temperature ($T \neq 0$) QED, QFT and also quantum chromodynamics (QCD) which usually deal with the specific processes in the heat bath of photons or other particles (Donoghue et al., 1985). The heat bath can be formed by different kinds of elementary particles and so such different hot media have a different influence on the same specific physical process developing in the media. We consider here the influence of the heat bath photons on the energy shift inside of the thermal box, leading to the attraction of the capacitor plates with a separation a .

The photons at the temperature T form so called blackbody, which has the distribution law of photons derived in 1900 by Planck (1900, 1901), (Schöpf, 1978). The derivation was based on the investigation of the statistics of the system of oscillators inside of the blackbody. Later Einstein (1917) derived the Planck formula from the Bohr model of atom where electrons have the discrete energies and the energy of the emitted photons are given by the Bohr formula $\hbar\omega = E_i - E_f$, E_i, E_f are the initial and final energies of electrons.

2 The Casimir effect at zero temperature

In order to understand the Casimir effect, we follow Holstein (1992) and imagine two capacitor plates with a separation a. The field modes permitted by the boundary condition have the electrical intensity vanishing on the surface on the plates. If the normal to the surface defines the z-direction, then for the propagation in this direction wavelength varies from zero to a. If the zero point energy of the oscillators representing the quantum field is $\hbar\omega_k/2$ (Berestetskii et al., 1999), then then the total energy between the plates is given by the formula

$$U(a) = \sum_k \frac{1}{2}\hbar\omega_k. \tag{1}$$

When the plate separation is increased, more modes are permitted so the energy is increasing function of separation a. In case that the separation a is lowered, then the energy is also lowered which means that the change of energy is force of the form:

$$F = -\frac{\partial U(a)}{\partial a}. \tag{2}$$

The force has been detected for instance by Sparnay (1958) and represents the macroscopic manifestation of the validity of quantum field theory.

The quantitative evaluation of the Casimir force is as follows. Let be wave numbers k_x, k_z in the x, y direction. Then the density of states is given by the formula

$$A\int \frac{d^2k}{(2\pi)^2}, \tag{3}$$

where A is the area of the plates.

In the z-direction, on the other hand, the boundary conditions $\mathbf{E}(0) = \mathbf{E}(a) = 0$ requires

$$E \sim \sin(k_z z) \tag{4}$$

with

$$k_z = \frac{n\pi}{a}\; n = 1, 2, ... \tag{5}$$

The frequencies are

$$\omega_k = \sqrt{k_x^2 + k_y^2 + \left(\frac{n\pi}{a}\right)^2}. \tag{6}$$

The total vacuum energy of photons (with two polarizations) between plates is evidently as follows:

$$U(a) = 2\sum_{n=1}^{\infty} A\int \frac{d^2k}{(2\pi)^2}\frac{1}{2}\omega_k. \tag{7}$$

Defining

$$k = \sqrt{k_x^2 + k_y^2}, \tag{8}$$

we have from eq. (5)

$$kdk = \omega d\omega \tag{9}$$

and the new mathematical form of the total intermediate vacuum energy is

$$U(a) = A \sum_{n=1}^{\infty} \frac{1}{2\pi} \int_{\frac{n\pi}{a}}^{\infty} d\omega\omega^2. \tag{10}$$

Using the cutoff operation with $\exp(-\varepsilon\omega)$, we get the following formulas:

$$U(a) = \frac{A}{2\pi} \sum_{n=1}^{\infty} \int_{\frac{n\pi}{a}}^{\infty} d\omega\omega^2 e^{-\varepsilon\omega} = \frac{A}{2\pi}\frac{d^2}{d\varepsilon^2} \sum_{n=1}^{\infty} \int_{\frac{n\pi}{a}}^{\infty} d\omega e^{-\varepsilon\omega} =$$

$$\frac{A}{2\pi}\frac{d^2}{d\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{\varepsilon} e^{-\frac{n\pi\varepsilon}{a}} = \frac{A}{2\pi}\frac{d^2}{d\varepsilon^2}\frac{1}{\varepsilon}\left(\frac{1}{1-e^{\frac{\varepsilon\pi}{a}}} - 1\right). \tag{11}$$

After application the formula with the Bernoulli numbers B_n (Prudnikov et al., 1984)

$$\frac{1}{1-e^{-t}} = -\sum_{n=1}^{\infty} B_n \frac{t^{n-1}}{n!}, \tag{12}$$

we get for $\varepsilon \to 0$ the final formula for the attraction of two plates immersed in the quantum vacuum (Holstein, 1992):

$$\frac{1}{A}F = -\frac{\partial}{\partial a}\frac{1}{A}U(a) = -\frac{\pi^2}{240a^4}. \tag{13}$$

Now, we can approach the calculation of the attractive force due to the photons of the blackbody sea.

3 The Casimir effect at finite temperature due to blackbody photons

The blackbody photons are supposed in the box with the edges l_l, l_2, l_3 and the situation is the analogue of the quantum mechanical particle inside such box. However with regard to the fact that the photon gas has the temperature T, it is necessary to perform the following transformation to the thermodynamical system in the box:

$$U(a) = \sum_k \frac{1}{2}\hbar\omega_k \to \sum_k \left(\frac{\omega_k^2}{\pi c^3}\right) \frac{\hbar\omega_k}{e^{\frac{\hbar\omega_k}{k_B T}} - 1} \tag{14}$$

with

$$\omega_k = \omega_{n_1,n_2,n_3} = \sqrt{\left(\frac{n_1\pi}{l_1}\right)^2 + \left(\frac{n_2\pi}{l_2}\right)^2 + \left(\frac{n_3\pi}{l_3}\right)^2}. \tag{15}$$

So, the energy of photons in the photon sea is

$$U(a) = \sum_{n_1,n_2,n_3} \left(\frac{\omega_{n_1,n_2,n_3}^2}{\pi c^3}\right) \frac{\hbar\omega_{n_1,n_2,n_3}}{e^{\frac{\hbar\omega_{n_1,n_2,n_3}}{k_B T}} - 1}. \tag{16}$$

It is elementary statement that if $l_1 \to \infty, l_2 \to \infty, l_3 \to \infty$, we get the classical Planck distribution

$$\varrho(\omega) \to \left(\frac{\omega^2}{\pi c^3}\right) \frac{\hbar\omega}{e^{\frac{\hbar\omega}{k_B T}} - 1} \tag{17}$$

with (Feynman, 1972; Isihara, 1971)

$$U(blackbody) = \int_0^\infty \varrho(\omega)d\omega = \sigma T^4; \quad \sigma = \frac{\pi^2(k_B T)^4}{15\hbar^3 c^3}. \tag{18}$$

The force in the x-direction is

$$F_x = -\frac{\partial U(l_1, l_2, l_3}{\partial l_1} = \sum_{n_1,n_2,n_3} \left(\frac{\hbar}{\pi c^3}\right)\left(\frac{n_1\pi}{l_1}\right)^2 \frac{1}{l_1} \quad \times$$

$$\left[\frac{3\omega}{e^{\frac{\hbar\omega}{k_B T}} - 1} - \frac{\omega^2 e^{\frac{\hbar\omega}{k_B T}}}{\left(e^{\frac{\hbar\omega}{k_B T}} - 1\right)^2} \frac{\hbar}{k_B T}\right]. \tag{19}$$

The force in the y-direction is

$$F_y = -\frac{\partial U(l_1, l_2, l_3)}{\partial l_2} = \sum_{n_1,n_2,n_3} \left(\frac{\hbar}{\pi c^3}\right)\left(\frac{n_2\pi}{l_2}\right)^2 \frac{1}{l_2} \quad \times$$

$$\left[\frac{3\omega}{e^{\frac{\hbar\omega}{k_B T}} - 1} - \frac{\omega^2 e^{\frac{\hbar\omega}{k_B T}}}{\left(e^{\frac{\hbar\omega}{k_B T}} - 1\right)^2} \frac{\hbar}{k_B T}\right] \tag{20}$$

and the force in the z-direction is

$$F_z = -\frac{\partial U(l_1, l_2, l_3)}{\partial l_3} = \sum_{n_1,n_2,n_3} \left(\frac{\hbar}{\pi c^3}\right) \left(\frac{n_3 \pi}{l_3}\right)^2 \frac{1}{l_3} \quad \times$$

$$\left[\frac{3\omega}{e^{\frac{\hbar\omega}{k_B T}} - 1} - \frac{\omega^2 e^{\frac{\hbar\omega}{k_B T}}}{\left(e^{\frac{\hbar\omega}{k_B T}} - 1\right)^2} \frac{\hbar}{k_B T}\right]. \tag{21}$$

The specific pressure on the unit area $l_2 l_3$, $l_1 l_3$, $l_1 l_2$. is

$$p_{23} = \frac{1}{l_2 l_3} F_x = -\frac{1}{l_2 l_3} \frac{\partial U(l_1, l_2, l_3)}{\partial l_1}, \tag{22}$$

$$p_{13} = \frac{1}{l_1 l_3} F_y = -\frac{1}{l_1 l_3} \frac{\partial U(l_1, l_2, l_3)}{\partial l_2}, \tag{23}$$

$$p_{12} = \frac{1}{l_1 l_2} F_z = -\frac{1}{l_1 l_2} \frac{\partial U(l_1, l_2, l_3)}{\partial l_3}. \tag{24}$$

In case of the equal edges of the thermal bath i.e. $l_1 = l_2 = l_3 = l$, the specific pressures are equal and it means that

$$p = \frac{1}{3 l^5} \sum_{n_1,n_2,n_3} \left(\frac{\hbar}{\pi c^3}\right) \left[\left(\frac{n_1 \pi}{l}\right)^2 + \left(\frac{n_2 \pi}{l}\right)^2 + \left(\frac{n_3 \pi}{l}\right)^2\right] \quad \times$$

$$\left[\frac{3\omega}{e^{\frac{\hbar\omega}{k_B T}} - 1} - \frac{\omega^2 e^{\frac{\hbar\omega}{k_B T}}}{\left(e^{\frac{\hbar\omega}{k_B T}} - 1\right)^2} \frac{\hbar}{k_B T}\right]. \tag{25}$$

Let us remark that the three-dimensional sums in eqs. (16), (19–22), (23–25) is not easy to calculate because they are not considered as the integral part of the standard mathematics. So, we can simplify the calculation by the so called continual limit. In other words, we perform replacing of the the sum by the ω-integral and for eq. (25) we get:

$$p = \frac{1}{3 l^5} \left(\frac{\hbar}{\pi c^3}\right) \int_0^\infty d\omega \omega^2 \left[\frac{3\omega}{e^{\frac{\hbar\omega}{k_B T}} - 1} - \frac{\omega^2 e^{\frac{\hbar\omega}{k_B T}}}{\left(e^{\frac{\hbar\omega}{k_B T}} - 1\right)^2} \frac{\hbar}{k_B T}\right]. \tag{26}$$

Now, we are prepared to evaluate the ω-integral in the last formula. Putting

$$x = \frac{\hbar\omega}{k_B T}; \quad \omega = \frac{x k_B T}{\hbar}; \quad d\omega = dx \frac{k_B T}{\hbar}; \quad C = \frac{k_B T}{\hbar}, \tag{27}$$

we get equation in the following form:

$$p = \frac{1}{3l^5}\left(\frac{\hbar}{\pi c^3}\right)\int_0^\infty dx C^5 \left[\frac{3x^3}{e^x - 1} - \frac{x^4 e^x}{(e^x - 1)^2}\right]. \tag{28}$$

According to textbook (Rumer et al., 1977)

$$\int_0^\infty dx \frac{x^n}{e^x - 1} = \Gamma(n+1)\zeta(n+1). \tag{29}$$

and (Prudnikov et al., 1984)

$$\int_0^\infty dx \frac{x^{2n} e^x}{(e^x - 1)^2} = 2^{2n-1}\pi^4 |B_{2n}|. \tag{30}$$

In case of the specification of n, we get (Rumer et al., 1977)

$$\int_0^\infty dx \frac{x^3}{e^x - 1} = \Gamma(4)\zeta(4) = 3!\left(\frac{\pi^4}{90}\right) \tag{31}$$

and (Prudnikov et al., 1984)

$$\int_0^\infty dx \frac{x^4 e^x}{(e^x - 1)^2} = 2^3\pi^4\left|-\frac{1}{30}\right| = 2^3\pi^4\frac{1}{30}, \tag{32}$$

where

$$|B_4| = \left|-\frac{1}{30}\right| = 1/30 \tag{33}$$

follows from the general formula (12).

So, the final formula for the so called Casimir effect at finite temperature is the numerical form of the formula (28). Or,

$$p = \frac{1}{3l^5}\left(\frac{\hbar}{\pi c^3}\right)\left(\frac{k_B T}{\hbar}\right)^5 \left[3.3!\left(\frac{\pi^4}{90}\right) - 2^3\left(\frac{\pi^4}{30}\right)\right]. \tag{34}$$

The last author formula is the original one and it was not published in the scientific physical research journals. The submitted approach can be easily generalized to phonon thermal bath, magnon thermal bath and and so on, or astrophysical thermal bath.

4 Discussion

We have seen how the thermal photons with the Planck blackbody statistics generated the Casimir effect at finite temperature. The motivation for considering such problem can be seen in quantum mechanics with the electron confined in the box with the infinite barriers at point 0 and l. Then, the energy levels of electron inside the box is (Sokolov et al. 1962)

$$E_n = \frac{\pi^2\hbar^2 n^2}{2ml^2} \tag{35}$$

and the corresponding wave function is

$$\psi_n = \sqrt{\frac{2}{l}}\sin\left(\pi n\frac{x}{l}\right). \tag{36}$$

The quantum pressure caused by the quantum mechanical motion of particle is obtained by the same operation as in the Casimir effect. Or,

$$F = -\frac{\partial E_n}{\partial l} = \frac{\pi^2\hbar^2 n^2}{ml^3}. \tag{37}$$

In case that the thermal box is three dimensional, we get (Sokolov et al., 1962)

$$E_{n_1,n_2,n_3} = \frac{\pi^2\hbar^2}{2m}\left[\left(\frac{n_1}{l_1}\right)^2 + \left(\frac{n_2}{l_2}\right)^2 + \left(\frac{n_3}{l_3}\right)\right] \tag{38}$$

and the corresponding wave function is

$$\psi_{n_1,n_2,n_3} = \sqrt{\frac{8}{l_1 l_2 l_3}}\sin\left(\pi n_1\frac{x}{l_1}\right)\sin\left(\pi n_2\frac{x}{l_2}\right)\sin\left(\pi n_3\frac{x}{l_3}\right). \tag{39}$$

The corresponding pressures are

$$p_{23} = -\frac{1}{l_2 l_3}\frac{\partial E_{n_1,n_2,n_3}}{\partial l_1} \tag{40}$$

$$p_{13} = -\frac{1}{l_1 l_3}\frac{\partial E_{n_1,n_2,n_3}}{\partial l_2} \tag{41}$$

$$p_{12} = -\frac{1}{l_1 l_2}\frac{\partial E_{n_1,n_2,n_3}}{\partial l_3}. \tag{42}$$

Let us only remark that the quantum pressure derived here is the perfect proof that the wave function in quantum mechanics is physical reality independent on the human mind, and not only mathematical object. The wave function is in such a way the objective form of matter, where matter is continuum which forms Universe.

The article is the continuation of the previous and related problems in the finite-temperature physics published by author (Pardy, 1989a; ibid., 1989b; ibid., 1994; ibid., 2013a; ibid., 2013b).

Information on the systematic examination of the finite temperature effects in quantum electrodynamics (QED) at one-loop order was given by Donoghue, Holstein and Robinett (1985). They have treated the calculation of mass, charge, wave function renormalization and so on, and demonstrated the running of the coupling constant at finite temperature and discussed the normalized vertex function and the energy momentum tensor.

Serge Haroche (2012) and his research group in the Paris microwave laboratory used a small cavity between two mirrors about three centimeter apart. During the long life-time of photons many quantum experiments were performed with the Rydberg atoms. We consider here the gas of photons (at temperature T) as the preamble for new experiments for the determination of the Casimir energy pressure of photon gas. It is not excluded, that the experiments performed by the well-educated physical experts will be the Nobelian ones.

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