The Uncertainty Principle, Space-Time Quantum Fluctuations, and Measurability Notion in Quantum Theory and Gravity

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PACS: 03.65, 05.20
Keywords: minimal length, measurability, quantum theory, gravity

Abstract

This paper is a continuation of the works earlier published by the author and deals with the theories involving a minimal length at all energy scales. The previously introduced notion of *measurability* is also used. As a new step, in this work at the initial stage possible contributions from the inclusion of the space-time quantum fluctuations into quantum theory are studied in the present formalism.

1 Introduction. Main Motivation and Aim

This paper directly continuous the earlier studies of the author [1]–[6] and in part [7]. The principal idea that has been put forward in [6] is further developed in this paper as follows: due to the *Uncertainty Principle*, in quantum theory there are solid grounds to consider as a background space not *continuous space-time* but *discrete space-time* involving the minimal length l_{min} and the minimal time t_{min}. Such a space represents a *lattice* but very *irregular lattice* in a sense that all its variations are determined by the existent energies.
At the present time all the fundamental theories at the well-known energy scales (gravity, quantum theory, statistical physics, and the like) are associated with *continuous space-time* and with the corresponding mathematical apparatus of infinitesimal space-time variations (increments) $dx_{\mu}, \delta x_{\mu}, ds, \delta s, ...$.

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Until very recently, this apparatus has been well applicable both in *classical* and *quantum* physics. However, in the first case the mathematical apparatus has been used without difficulties, whereas in the *quantum* case its use provokes various problems: (1)the problem of ultraviolet and infrared divergences; (2) the problem of the gravity non-renormalizability in a quantum approach, and (3)the problem, that is more general, of the adequate transition to the ultraviolet limit in quantum gravity.
The efforts to solve these problems in theoretical physics have generated numerous attractive and important approaches: supersymmetry, supergravity, superstrings, M-theories, and so on (for example, [8]).
But all the mentioned theories are actualized at high (Planck's) energies $E \approx E_P$.
At low energies $E \ll E_P$ they, to a high accuracy, should lead to the well-known Quantum Theory (QT) [9], [10] and General Relativity (GR) [11] defined for the *continuous space-time*.
In the majority of the above-mentioned approaches high (Planck's) energies $E \approx E_P$ are associated with the minimal length $l_{min} \propto l_P$ that disappears at low energies $E \ll E_P$, i.e. $l_{min} \rightarrow 0$.
But if l_{min} is *really present*, it must be present at all the *"Energy Levels"* of the theory, low energies including. Therefore, in this case the mathematical formalism of the theory should not involve any infinitesimal spatial-temporal quantities. Besides, some new parameters become involved, which are dependent on l_{min} [12]–[21]. But, on the other hand, these parameters could hardly disappear totally at low energies, *i.e.*, for QT and GR too. However, since the well-known canonical statement of QT [9], [10] and GR [11] has no such parameters, the inference is as follows: their influence at low energies is so small that it may be disregarded at the modern stage in evolution of the theory and of the experiment.

Still this does not imply that they should be ignored in future evolution of the theory, especially on going to its high-energy limit.

In this way this paper, similarly to the cited works [1]–[6], has been motivated by the need for *actualization* of the minimal length l_{min} and the minimal time t_{min}, and also of their associated parameters at all the *Energy Levels* of the theory, low energies ($E \ll E_P$) including. It should be noted that the inclusion of l_{min} (t_{min}) is dictated not only by selection of the corresponding model but by the application of the fundamental principles of quantum theory within the scope of quite natural assumptions ([6] and Section 2).

It is clear that, because of the actualization of l_{min} (t_{min}) at all the *Energy Levels*, the theory is specified for *discrete space-time* rather than for *continuous space-time*. By the main hypothesis set up by the author, the adequately resolved *discrete* theory should have the following properties:
a) at low energies, which are far from the Plank energies $E \ll E_P$, this theory is very close to the initial *continuous* theory;
b) the problems indicated in points (1)–(3) in this theory will be solved naturally (without the appearance of infinities) within the scope of the transition from low to high energies $E \approx E_P$ and vice versa.

The principal objective of the author is to suggest the adequate derivation of such *discrete* theory on the basis of his previous works [1]–[6]. As compared to [1]–[6], in this paper the author begins to analyze the inferences of the inclusion of *space-time quantum fluctuations* into quantum theory in the present formalism.

2 Uncertainty Principle and ≪Principle of Bounded Space-Time Variations (Increments)≫

In this Section the principal assumptions are introduced which have been implicitly used previously in [1]– [5] and explicitly in [6].

2.1 Principle of Bounded Space-Time Variations

It is well known that in a quantum study the key role is played by the measuring procedure, its fundamental principle being the Heisenberg Uncertainty Principle (HUP) [22, 9]:

$$\Delta x \geq \frac{\hbar}{\Delta p} \tag{1}$$

(Note that the normalization $\Delta x \Delta p \geq \hbar$ is used rather than $\triangle x \Delta p \geq \hbar/2$.)

Supposition 1. Any small variation (increment) $\widetilde{\Delta} x_\mu$ of any spatial coordinate x_μ of the arbitrary point $x_\mu, \mu = 1, ..., 3$ in some space-time system R may be realized in the form of the uncertainty (standard deviation) Δx_μ when this coordinate is measured within the scope of Heisenberg's Uncertainty Principle (HUP)

$$\widetilde{\Delta} x_\mu = \Delta x_\mu, \Delta x_\mu \simeq \frac{\hbar}{\Delta p_\mu}, \mu = 1, 2, 3 \qquad (2)$$

for some $\Delta p_\mu \neq 0$.

Similarly, for $\mu = 0$ for pair "time-energy" (t, E), any small variation (increment) in the value of time $\widetilde{\Delta} x_0 = \widetilde{\Delta} t_0$ may be realized in the form of the uncertainty (standard deviation) $\Delta x_0 = \Delta t$ and then

$$\widetilde{\Delta} t = \Delta t, \Delta t \simeq \frac{\hbar}{\Delta E} \qquad (3)$$

for some $\Delta E \neq 0$.

Here HUP is given for the nonrelativistic case. In the relativistic case HUP has the distinctive features [23] which, however, are of no significance for the general formulation of Supposition 1, being associated only with particular alterations in the right-hand side of the second relation Equation (2) as shown later.

It is clear that at low energies $E \ll E_P$ (momentums $P \ll P_{pl}$) Supposition 1 sets a lower bound for the variations (increments) $\widetilde{\Delta} x_\mu$ of any space-time coordinate x_μ.

At high energies E (momentums P) this is not the case if E (P) have no upper limit. But, according to the modern knowledge, E (P) are bounded by some maximal quantities E_{max}, (P_{max})

$$E \leq E_{max}, P \leq P_{max}, \qquad (4)$$

where in general E_{max}, P_{max} may be on the order of Planck quantities $E_{max} \propto E_P, P_{max} \propto P_{pl}$ and also may be the trans-Planck's quantities.

In any case the quantities P_{max} and E_{max} lead to the introduction of the minimal length l_{min} and of the minimal time t_{min}.

With this point of view, even at the ultimate (Planck) energies a minimal "detected" (*i.e.*, measurable) space-time volume is, within the known constants, restricted to

$$V_{min} \propto l_P^4. \tag{5}$$

Consequently, "detectability" of the infinitesimal space-time volume

$$V_{dx_\mu} = (dx_\mu)^4 \tag{6}$$

is impossible as this necessitates going to infinitely high energies

$$E \to \infty. \tag{7}$$

Because of this, it is natural to complete Supposition 1 with Supposition 2.

Supposition 2. There is the minimal length l_{min} as a minimal measurement unit for all quantities having the dimension of length, whereas the minimal time $t_{min} = l_{min}/c$ as a minimal measurement unit for all quantities having the dimension of time, where c is the speed of light.

l_{min} and t_{min} are naturally introduced as $\Delta x_\mu, \mu = 1, 2, 3$ and Δt in Equations (2) and (3) for $\Delta p_\mu = P_{max}$ and $\Delta E = E_{max}$.

For definiteness, we consider that E_{max} and P_{max} are the quantities on the order of the Planck quantities, then l_{min} and t_{min} are also on the order of Planck quantities $l_{min} \propto l_P$, $t_{min} \propto t_P$.

Suppositions 1 and 2 are quite natural in the sense that there are no physical principles with which these suppositions are inconsistent.

The combination of *Suppositions 1,2* will be called the **Principle of Bounded Space-Time Variations (Increments)**.

2.2 Minimal Length and Measurability

Now, from the start, we assume that the theory involves the minimal length l_{min} as a minimal measurement unit for all quantities having the dimension of length.

Then it is convenient to begin our study not with HUP Equation (1) but with its widely known high-energy generalization—the Generalized

Uncertainty Principle (GUP) that naturally leads to the minimal length l_{min} [24]–[35]:

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha' l_P^2 \frac{\Delta p}{\hbar}. \qquad (8)$$

Here α' is the model-dependent dimensionless numerical factor and l_P is the Planckian length.
Note also that initially GUP Equation (8) was derived within a string theory [24]–[26] and, subsequently, in a series of works far from this theory [27]–[33] it has been demonstrated that on going to high (Planck) energies in the right-hand side of HUP Equation (1) an additional "high-energy" term $\propto l_P^2 \frac{\triangle p}{\hbar}$ appears. Of particular interest is the work [27], where by means of a simple gedanken experiment it has been demonstrated that with regard to the gravitational interaction Equation (8) is the case.

As Equation (8) is a quadratic inequality, then it naturally leads to the minimal length $l_{min} = \xi l_P = 2\sqrt{\alpha'} l_P$.

This means that the theory for the quantities with a particular dimension has a minimal measurement unit. At least, all the quantities with such a dimension should be "quantized", *i.e.*, be measured by an integer number of these minimal units of measurement.

Specifically, if l_{min}—minimal unit of length, then for any length L we have the "Integrality Condition" (IC)

$$L = N_L l_{min}, \qquad (9)$$

where $N_L > 0$ is an integer number.

What are the consequences for GUP Equation (8) and HUP Equation (1)?

Assuming that HUP is to a high accuracy derived from GUP on going to low energies or that HUP is a special case of GUP at low values of the momentum, we have

$$(GUP, \Delta p \rightarrow 0) = (HUP). \qquad (10)$$

By the language of N_L from Equations (9) and (10) is nothing else but a change-over to the following:

$$(N_{\Delta x} \approx 1) \rightarrow (N_{\Delta x} \gg 1). \qquad (11)$$

The assumed equalities in Equations (1) and (8) may be conveniently rewritten in terms of l_{min} with the use of the deformation parameter α_a. This parameter has been introduced earlier in the papers [36]–[43] as a deformation parameter on going from the canonical quantum mechanics to the quantum mechanics at Planck's scales (early Universe) that is considered to be the quantum mechanics with the minimal length (QMML):

$$\alpha_a = l_{min}^2/a^2, \qquad (12)$$

where a is the measuring scale.

Definition 1
Deformation is understood as an extension of a particular theory by inclusion of one or several additional parameters in such a way that the initial theory appears in the limiting transition [44].

Then with the equality ($\Delta p \Delta x = \hbar$) Equation (8) is of the form

$$\Delta x = \frac{\hbar}{\Delta p} + \frac{\alpha_{\Delta x}}{4}\Delta x. \qquad (13)$$

In this case due to Equations (9), (11) and (13) takes the following form:

$$N_{\Delta x} l_{min} = \frac{\hbar}{\Delta p} + \frac{1}{4N_{\Delta}x} l_{min} \qquad (14)$$

or

$$(N_{\Delta x} - \frac{1}{4N_{\Delta}x}) l_{min} = \frac{\hbar}{\Delta p}. \qquad (15)$$

That is

$$\Delta p = \frac{\hbar}{(N_{\Delta x} - \frac{1}{4N_{\Delta}x}) l_{min}}. \qquad (16)$$

From Equations (14)–(16) it is clear that HUP Equation (1) in the case of the equality appears to a high accuracy in the limit $N_{\Delta}x \gg 1$ in conformity with Equation (11).

It is easily seen that the parameter α_a from Equation (12) is discrete as it is nothing else but

$$\alpha_a = l_{min}^2/a^2 = \frac{l_{min}^2}{N_a^2 l_{min}^2} = \frac{1}{N_a^2}. \qquad (17)$$

At the same time, from Equation (17) it is evident that α_a is irregularly discrete.

It is clear that from Equation (16) at low energies ($N_{\Delta}x \gg 1$), up to a constant

$$\frac{\hbar^2}{l_{min}^2} = \frac{\hbar c^3}{4\alpha' G} \tag{18}$$

we have

$$\alpha_{\Delta x} = (\Delta p)^2. \tag{19}$$

But all the above computations are associated with the nonrelativistic case. As known, in the relativistic case, when the total energy of a particle with the mass m and with the momentum p equals [45]:

$$E = \sqrt{p^2 c^2 + m^2 c^4}, \tag{20}$$

a minimal value for Δx takes the form [23]:

$$\Delta x \approx \frac{c\hbar}{E}. \tag{21}$$

And in the ultrarelativistic case

$$E \approx pc \tag{22}$$

this means simply that

$$\Delta x \approx \frac{\hbar}{p}. \tag{23}$$

Provided the minimal length l_{min} is involved and considering the "Integrality Condition" (IC) Equation (9), in the general case for Equation (21) at the energies considerably lower than the Planck energies $E \ll E_P$ we obtain the following:

$$\Delta x = N_{\Delta x} l_{min} \approx \frac{c\hbar}{E}, \quad or \quad E \approx \frac{c\hbar}{N_{\Delta x}}. \tag{24}$$

Similarly, at the same energy scale in the ultrarelativistic case we have

$$p \approx \hbar/N_{\Delta x}. \tag{25}$$

Next under Supposition 2, we assume that there is a minimal measuring unit of time

$$t_{min} = l_{min}/v_{max} = l_{min}/c. \tag{26}$$

Then the foregoing Equations (1)–(15) are rewritten by substitution as follows:

$$\Delta x \to \Delta t; \Delta p \to \Delta E; l_{min} \to t_{min}; N_L \to N_{t=L/c} \tag{27}$$

Specifically, Equation (15) takes the form

$$(N_{\Delta t} - \frac{1}{4N_{\Delta}t})t_{min} = \frac{\hbar}{\Delta E}. \tag{28}$$

And similar to Equation (9), we get the "Integrality Condition" (IC) for any time t:

$$t \equiv t(N_t) = N_t t_{min}, \tag{29}$$

for certain an integer $|N_t| \geq 0$.

According to Equation (28), let us define the corresponding energy E

$$E \equiv E(N_t) = \frac{\hbar}{|N_t - \frac{1}{4N_t}|t_{min}}. \tag{30}$$

Note that at low energies $E \ll E_P$, that is for $|N_t| \gg 1$, the formula Equation (30) naturally takes the following form:

$$E \equiv E(N_t) = \frac{\hbar}{|N_t|t_{min}} = \frac{\hbar}{|t(N_t)|}. \tag{31}$$

Definition 2 (**Measurability**)
(1) *Let us define the quantity having the dimensions of length L or time t measurable, when it satisfies the relation Equation (9 (and respectively Equation (29)).*
(2) Let us define any physical quantity measurable, when its value is consistent with points (1) of this Definition.

Thus, infinitesimal changes in length (and hence in time) are impossible and any such changes are dependent on the existing energies.

In particular, a minimal possible *measurable* change of length is l_{min}. It corresponds to some maximal value of the energy E_{max} or momentum P_{max}, If $l_{min} \propto l_P$, then $E_{max} \propto E_P$,$P_{max} \propto P_{Pl}$, where $P_{max} \propto P_{Pl}$, where P_{Pl} is where the Planck momentum. Then denoting in *nonrelativistic* case with $\triangle_p(w)$ a minimal measurable change every spatial coordinate w corresponding to the energy E we obtain

$$\triangle_{P_{max}}(w) = \triangle_{E_{max}}(w) = l_{min}. \tag{32}$$

Evidently, for lower energies (momenta) the corresponding values of $\triangle_p(w)$ are higher and, as the quantities having the dimensions of length are quantized Equation (9), for $p \equiv p(N_p) < p_{max}$, $\triangle_p(w)$ is transformed to

$$|\triangle_{p(N_p)}(w)| = |N_p| l_{min}. \tag{33}$$

where $|N_p| > 1$ is an integer number so that we have

$$|N_p - \frac{1}{4N_p}| l_{min} = \frac{\hbar}{|p(N_p)|}. \tag{34}$$

In the relativistic case the Equation (32) holds, whereas Equations (33) and (34) for $E \equiv E(N_E) < E_{max}$ are replaced by

$$|\triangle_{E(N_E)}(w)| = |N_E| l_{min}, \tag{35}$$

where $|N_E| > 1$ is an integer.

Next we assume that at high energies $E \propto E_P$ there is a possibility only for the nonrelativistic case or ultrarelativistic case.

Then for the ultrarelativistic case, with regard to Equations (22)–(28), Formula (34) takes the form

$$|N_E - \frac{1}{4N_E}| l_{min} = \frac{\hbar c}{E(N_E)} = \frac{\hbar}{|p(N_p)|}, \tag{36}$$

where $N_E = N_p$.

In the relativistic case at low energies we have

$$E \ll E_{max} \propto E_P. \tag{37}$$

In accordance with Equations (20) and (21) and Formula (33) is of the form

$$|\triangle_{E(N_E)}(w)| = |N_E| l_{min} = \frac{\hbar c}{E(N_E)}, |N_E| \gg \ 1 \ - \ integer. \tag{38}$$

In the nonrelativistic case at low energies Equation (37) due to Equation (34) we get

$$|\triangle_{p(N_p)}(w)| = |N_p| l_{min} = \frac{\hbar}{|p(N_p)|}, |N_p| \gg 1 - integer. \tag{39}$$

In a similar way for the time coordinate t, by virtue of Equations (29)–(31), at the same conditions we have similar Equations (32)–(34)

$$\triangle_{E_{max}}(t) = t_{min}. \tag{40}$$

For $E \equiv E(N_t) < E_{max}$

$$|\triangle_{E(N_t)}(t)| = |N_t| t_{min}, \tag{41}$$

where $|N_E| > 1$ is an integer, so that we obtain

$$|N_t - \frac{1}{4N_t}| t_{min} = \frac{\hbar c}{E(N_t)}. \tag{42}$$

In the relativistic case at low energies

$$E \ll E_{max} \propto E_P, \tag{43}$$

in accordance with Equations (20) and (21), Equation (33) takes the form

$$|\triangle_{E(N_t)}(w)| = |N_t| l_{min} = \frac{\hbar c}{E(N_t)}, |N_t| \gg 1 - integer. \tag{44}$$

Remark 1.

1.1. It should be noted that the lattice is usually understood as a uniform discrete structure with one and the same constant parameter a (lattice pitch). But in this case we have a nonuniform discrete structure (lattice in its nature), where the analogous parameter is variable, is a multiple of l_{min}, *i.e.*, $a = N_a l_{min}$, and also is dependent on the energies. Only in the limit of high (Planck's) energies we get a (nearly) uniform lattice with (nearly) constant pitch $a \approx l_{min}$ or $a = \kappa l_{min}$ where κ is on the order of 1.

1.2. Obviously, when l_{min} is involved, the foregoing formulas for the momenta $p(N_p)$ and for the energies $E(N_E), E(N_t)$ may certainly give the highly accurate result that is close to the experimental one only at the verified low energies: $|N_p| \gg 1, |N_E| \gg 1, |N_t| \gg 1$. In the case of high energies $E \propto E_{max} \propto E_P$ or, what is the same $|N_p| \to 1, |N_E| \to 1, |N_t| \to 1$, we have a certain, experimentally unverified, model with a correct low-energy limit.

1.3. It should be noted that dispersion relations Equation (20) are valid only at low energies $E \ll E_P$. In the last few years in a series of works [46]–[49] it has been demonstrated that within the scope of GUP the high-energy generalization of Equation (20)—Modified Dispersion Relations (MDRs)—is valid.

Specifically, in its most general form the Modified Dispersion Relation (Formula (9) in [49]) may be given as follows:

$$p^2 = f(E, m; l_p) \simeq E^2 - \mu^2 + \alpha_1 l_p E^3 + \alpha_2 l_p^2 E^4 + O\left(l_p^3 E^5\right), \qquad (45)$$

where in the notation of [49] the fundamental constants are $c = \hbar = k_B = 1$, f is the function that gives the exact dispersion relation, and in the right-hand side the applicability of the Taylor-series expansion for $E \ll 1/l_P$ is assumed. The coefficients α_i can take different values in different quantum-gravity proposals. m is the rest energy of a particle, and the mass parameter μ in the right-hand side is directly related to the rest energy but $\mu \neq m$ if not all the coefficients α_i are vanishing.

The general case of (MDRs) Equation (45) in terms of the considerations given in this section is yet beyond the scope of this paper and necessitates further studies of the transition from low $E \ll E_P$ to high $E \approx E_P$ energies.

For now it is assumed that at low energies Equation (20) is valid to within a high accuracy, whereas at high energies, *i.e.*, for $|N_p| \rightarrow 1, |N_E| \rightarrow 1, |N_t| \rightarrow 1$, Equation (20) should be replaced by Equation (45). Besides, it is important that in this paper, as distinct from [46]–[49], the author uses the simplest (earlier) variant of GUP [24]–[33], involving a minimal length but not a minimal momentum.

Also note that references [46]–[49] give not nearly so complete a list of the publications devoted to GUP (and, in particular, MDR)—a very complete and interesting survey may be found in [46].

1.4. The papers [1]–[6] point to the fact that the resolved discrete theory is very close to the initial continuous one ($l_{min} = 0$) at low energies $E \ll E_P$, *i.e.*, at $|N_p| \gg 1, |N_E| \gg 1$.

In what follows all the considerations are given in terms of "measurable quantities" in the sense of Definition 2 given in this Section.

3 Space-Time Lattice of Measurable Quantities and Dual Lattice

So, provided the minimal length l_{min} exists, two lattices are naturally arising [6].

I. Lattice of the space-time variation—Lat_{S-T} representing, to within the known multiplicative constants, the sets of nonzero integers $N_w \neq 0$ and $N_t \neq 0$ in the corresponding formulas from the set Equations (33) and (44) for each of the three space variables $w \doteq x; y; z$ and the time variable t

$$Lat_{S-T} \doteq (N_w, N_t). \tag{46}$$

Which restrictions should be initially imposed on these sets of nonzero integers?

It is clear that in every such set all the integers (N_w, N_t) should be sufficiently "close", because otherwise, for one and the same space-time point, variations in the values of its different coordinates are associated with principally different values of the energy E which are "far" from each

other. Note that the words "close" and "far" will be elucidated further in this text.
Thus, at the admittedly low energies (Low Energies) $E \ll E_{max} \propto E_P$ the low-energy part (sublattice) $Lat_{S-T}[LE]$ of Lat_{S-T} is as follows:

$$Lat_{S-T}[LE] = (N_w, N_t) \equiv (|N_x| \gg 1, |N_y| \gg, |N_z| \gg 1, |N_t| \gg 1). \quad (47)$$

At high energies (High Energies) $E \to E_{max} \propto E_P$ we, on the contrary, have the sublattice $Lat_{S-T}[HE]$ of Lat_{S-T}

$$Lat_{S-T}[HE] = (N_w, N_t) \equiv (|N_x| \approx 1, |N_y| \approx 1, |N_z| \approx 1, |N_t| \approx 1). \quad (48)$$

II. Next let us define the lattice momenta-energies variation Lat_{P-E} as a set to obtain $(p_x(N_{x,p}), p_y(N_{y,p}), p_z(N_{z,p}), E(N_t))$ in the nonrelativistic and ultrarelativistic cases for all energies, and as a set to obtain
$(E_x(N_{x,E}), E_y(N_{y,E}), E_z(N_{z,E}), E(N_t))$ in the relativistic
(but not ultrarelativistic) case for low energies $E \ll E_P$, where all the components of the above sets conform to the space coordinates (x, y, z) and time coordinate t and are given by the corresponding Formulas (32)–(44) from the previous Section.

Note that, because of the suggestion made after formula Equation (37) in the previous Section, we can state that the foregoing sets exhaust all the collections of momentums and energies possible for the lattice Lat_{S-T}.
From this it is inferred that, in analogy with point I of this Section, within the known multiplicative constants, we have

$$Lat_{P-E} \doteq (\frac{1}{N_w - \frac{1}{1/4N_w}}, \frac{1}{N_t - \frac{1}{1/4N_t}}), \quad (49)$$

where $N_w \neq 0, N_t \neq 0$ are integer numbers from Equation (46). Similar to Equation (47), we obtain the low-energy (Low Energy) part or the sublattice $Lat_{P-E}[LE]$ of Lat_{P-E}

$$Lat_{P-E}[LE] \approx (\frac{1}{N_w}, \frac{1}{N_t}), |N_w| \gg 1, |N_t| \gg 1. \quad (50)$$

In accordance with Equation (48), the high-energy (High Energy) part (sublattice) $Lat_{P-E}[HE]$ of Lat_{P-E} takes the form

$$Lat_{P-E}[HE] \approx (\frac{1}{N_w - \frac{1}{1/4N_w}}, \frac{1}{N_t - \frac{1}{1/4N_t}}), |N_w| \to 1, |N_t| \to 1. \quad (51)$$

Considering **Remark 1** from the previous Section, it should be noted that, as currently the low energies $E \ll E_{max} \propto E_P$ are verified by numerous experiments and thoroughly studied, the sublattice $Lat_{P-E}[LE]$ Equation (50) is correctly defined and rigorously determined by the sublattice $Lat_{S-T}[LE]$ Equation (47).

However, at high energies $E \rightarrow E_{max} \propto E_P$ we can not be so confident the sublattice $Lat_{P-E}[HE]$ may be defined more exactly.

Specifically, α_a is obviously a small parameter. And, as demonstrated in [50, 51], in the case of GUP we have the following:

$$[\vec{x}, \vec{p}] = i\hbar(1 + a_1\alpha + a_2\alpha^2 + ...). \tag{52}$$

But, according to Equation (17), $|1/N_a| = \sqrt{\alpha_a}$, then, due to Equation (52), the denominators in the right-hand side of Equation (51) may be also varied by adding the terms $\propto 1/N_w^2, \propto 1/N_w^3 ..., \propto 1/N_t^2, \propto 1/N_t^3 ...$, that is liable to influence the final result for $|N_w| \rightarrow 1, |N_t| \rightarrow 1$.
The notions "close" and "far" for Lat_{P-E} will be completely determined by the dual lattice $Lat_{S-T}[LE]$ and by Formulas (33) and (44).

It is important to note the following.

In the low-energy sublattice $Lat_{P-E}[LE]$ all elements are varying very smoothly enabling the approximation of a continuous theory.

4 A Gravitational Model, Which Can Be Considered as Universal

The Sections 4,5 are based on the results of [6].
In his work [52] M.A. Markov has considered the gravitational model that, at low energies far from Planck's energies, in fact included General Relativity. In [52], M.A.Markov has suggested that "by the universal decree of nature a quantity of the material density ϱ is always bounded by its upper value given by the expression that is composed of fundamental constants" ([52], p. 214):

$$\varrho \leq \varrho_p = \frac{c^5}{G^2\hbar}, \tag{53}$$

with ϱ_p as "Planck's density".

Then the quantity

$$\wp_\varrho = \varrho/\varrho_p \leq 1 \tag{54}$$

is the deformation parameter as it is used in [52] to construct the following of Einstein equations deformation or $\wp_\varrho$-deformation (Formula (2) in [52]):

$$R^\nu_\mu - \frac{1}{2}R\delta^\nu_\mu = \frac{8\pi G}{c^4}T^\nu_\mu(1-\wp^2_\varrho)^n - \Lambda\wp^{2n}_\varrho\delta^\nu_\mu, \tag{55}$$

where $n \geq 1/2$, T^ν_μ–energy-momentum tensor, Λ—cosmological constant.

The case of the parameter $\wp_\varrho \ll 1$ or $\varrho \ll \varrho_p$ correlates with the classical Einstein equations, and the case when $\wp_\varrho = 1$—with the de Sitter Universe. In this way Equation (55) may be considered as $\wp_\varrho$-deformation of the General Relativity in the meaning of **Definition 1** in Section 3.
Besides, the gravitational model (55) may be considered *universal* as it includes the principal cases covered by the General Relativity: at low energies – GR in the most general form and at high energies – de Sitter Universe.

As shown in [51], $\wp_\varrho$-of Einstein equations deformation Equation (55) is nothing else but α-deformation of GR for the parameter α_l at $a = l$ from Equation (12).
If $\varrho = \varrho_l$ is the average material density for the Universe of the characteristic linear dimension l, *i.e.*, of the volume $V \propto l^3$, we have

$$\wp_{l,\varrho} = \frac{\varrho_l}{\varrho_p} \propto \alpha^2_l = \omega\alpha^2_l, \tag{56}$$

where ω is some computable factor.

Then it is clear that α_l-representation Equation (55) is of the form

$$R^\nu_\mu - \frac{1}{2}R\delta^\nu_\mu = \frac{8\pi G}{c^4}T^\nu_\mu(1-\omega^2\alpha^4_l)^n - \Lambda\omega^{2n}\alpha^{4n}_l\delta^\nu_\mu, \tag{57}$$

or in the general form we have

$$R^\nu_\mu - \frac{1}{2}R\delta^\nu_\mu = \frac{8\pi G}{c^4}T^\nu_\mu(\alpha_l) - \Lambda(\alpha_l)\delta^\nu_\mu. \tag{58}$$

But, as r.h.s. of Equation (58) is dependent on α_l of any value and particularly in the case $\alpha_l \ll 1$, *i.e.*, at $l \gg \ell$, l.h.s of Equation (58) is also dependent on α_l of any value and Equation (58) may be written as

$$R^\nu_\mu(\alpha_l) - \frac{1}{2}R(\alpha_l)\delta^\nu_\mu = \frac{8\pi G}{c^4}T^\nu_\mu(\alpha_l) - \Lambda(\alpha_l)\delta^\nu_\mu. \tag{59}$$

Thus, in this specific case we obtain the explicit dependence of GR on the available energies $E \sim \frac{1}{l}$, that is insignificant at low energies or for $l \gg \ell$ and, on the contrary, significant at high energies, $l \to \ell$.

4.1 Low Energies, $E \ll E_P$

1. Low energies. Nonmeasurable case. In this case at low energies, using Formula (12) in the limit $\ell = 0$ for $a = l$, we get a continuous theory coincident with the General Relativity.

2. Low energies. Measurable case. In this case at low energies, using Formulas (12) and (17) for $\ell \neq 0$, for $a = l$ (and hence for $N_l \gg 1$), we get a discrete theory which is a "nearly continuous theory", practically similar to the General Relativity with the slowly (smoothly) varying parameter $\alpha_{l(t)}$, where t—time.
So, due to low energies and momentums ($E \ll E_P, p \ll P_{Pl}$), the "continuous case" **1** (General Relativity) and the "discrete case" **2** that is actually a "nearly continuous case".

4.2 High Energies, $E \approx E_P$

At high energies we consider the measurable case only. Then it is clear that at high energies the parameter $\alpha_{l(t)}$ is discrete and for the limiting value of $\alpha_{l(t)} = 1$ we get a discrete series of equations of the form Equation (59) (or a single equation of this form met by a discrete series of solutions) corresponding to $\alpha_{l(t)} = 1; 1/4; 1/9; ...$

As this takes place, $T_{\mu}^{\nu}(\alpha_l) \approx 0$, and in both cases as **2** in 6.1 as well as 4.2 $\Lambda(\alpha_l)$ is not longer a cosmological constant, being a dynamical cosmological term.

Note that because of Formula (19) given in Section 2.2, $\sqrt{\alpha_{l(t)}}$ in cases **2** in 6.1 and 4.2 is an element of the lattice Lat_{P-E} from Section 3. And in case **2** it is an element of the sublattice $Lat_{P-E}[LE]$, whereas case 4.2 is associated with the sublattice $Lat_{P-E}[HE]$.

It seems expedient to make some important remarks:
(1) Generally speaking, as 4.2 and case **2** in 4.1 are associated with measur-

able cases for low energies and high energies, respectively, all the terms of the Equation (59): $R^{\nu}_{\mu}(\alpha_l), R(\alpha_l), T^{\nu}_{\mu}(\alpha_l), \Lambda(\alpha_l)$ must be expressed in terms of measurable quantities in view of Definition 2 from Section 2.2. But this problem still remains to be solved. In fact, it is reduced to the construction of the following *"measurable"* deformations in the sense of Definition 2 in Section 2.2 as follows:

$$\lim_{\ell\to 0}(R^{\nu}_{\mu}(\alpha_l \ll 1), R(\alpha_l \ll 1), T^{\nu}_{\mu}(\alpha_l \ll 1), \Lambda(\alpha_l \ll 1)) \rightarrow \\ \rightarrow (R^{\nu}_{\mu}, R, T^{\nu}_{\mu}, \Lambda) \quad (60)$$

and

$$\lim_{(\alpha_l\approx 1)\to(\alpha_l\ll 1)}(R^{\nu}_{\mu}(\alpha_l \approx 1), R(\alpha_l \approx 1), T^{\nu}_{\mu}(\alpha_l \approx 1), \Lambda(\alpha_l \approx 1)) \rightarrow \\ \rightarrow \lim_{l_{min}\to 0}(R^{\nu}_{\mu}(\alpha_l \ll 1), R(\alpha_l \ll 1)\delta^{\nu}_{\mu}, T^{\nu}_{\mu}(\alpha_l \ll 1), \Lambda(\alpha_l \ll 1)) \rightarrow \\ \rightarrow (R^{\nu}_{\mu}, R, T^{\nu}_{\mu}, \Lambda). \quad (61)$$

Here the first Equation (60) is a pure low-energy limiting transition from the measurable variant of gravity to the nonmeasurable one (or from a discrete theory to a continuous theory), whereas the second Equation (61) from the beginning is associated with the measurable transition from high energies to low energies and then is coincident with the first one.

(2) It should be noted that in [1, 2] in terms of measurable quantities, as an example, we have studied gravity for the static spherically-symmetric horizon space. It has been shown that, "...despite the absence of infinitesimal spatial-temporal increments owing to the existence of l_{min} and the essential 'discreteness' of a theory, this discreteness at low energies is not 'felt', the theory in fact being close to the original continuum theory. The indicated discreteness is significant only in the case of high (Planck) energies " [1]. The Markov model considered in this section represents the generalization of the above-mentioned example. Of course, this model requires further thorough investigation in terms of measurable quantities.

5 Measurable Quantities in Momentum Representation. Start

For convenience, we denote the minimal length $l_{min} \neq 0$ by ℓ.

Let us consider the above calculations using the formalism of the well-known work [34]. Then GUP (Section 3.2 in [34]) has the following form:

$$[\mathbf{x},\mathbf{p}] = i\hbar(1+\beta\mathbf{p}^2), \tag{62}$$

where ($\beta > 0$) and

$$\beta = \frac{\ell^2}{\hbar^2}. \tag{63}$$

In the formalism of Section 2.2 of the present work, formula (7) from [34]

$$\Delta x \Delta p \geq \hbar(1+\beta(\Delta p)^2+\beta\langle,\mathbf{p}\rangle^2) \tag{64}$$

with regard to Equations (9), (14), (16), and (63) may be written as

$$\frac{\hbar N_{\Delta x}}{(N_{\Delta x}-\frac{1}{4N_{\Delta x}})} \geq \hbar(1+\frac{1}{(N_{\Delta x}-\frac{1}{4N_{\Delta x}})^2}+\frac{\ell^2}{\hbar^2}\langle\mathbf{p}\rangle^2). \tag{65}$$

In the equality case this results in the following expression:

$$\frac{-\hbar^2(12N_{\Delta x}^2+1)}{(4N_{\Delta x}^2-1)^2\ell^2} = \frac{-\hbar^2}{\ell^2}(3+\frac{4}{(4N_{\Delta x}^2-1)^2}) = \langle\mathbf{p}\rangle^2. \tag{66}$$

In this way at low energies $E \ll E_P$, *i.e.*, at $|N_{\Delta x}| \gg 1$, $\langle\mathbf{p}\rangle^2$ is varying practically continuously.
Next, hereinafter we use the Formula (34) with the replacement of $l_{min} = \ell$, *i.e.*, we have $N_{\Delta x} = N_p$ and

$$|p_N| = \frac{\hbar}{|N_p-\frac{1}{4N_p}|\ell}. \tag{67}$$

We can write

$$\imath\hbar(1+\beta p^2) = \imath\hbar(1+\frac{\ell^2}{\hbar^2}\frac{\hbar^2}{(N_p-\frac{1}{4N_p})^2\ell^2}) = \imath\hbar(1+\frac{1}{(N_p-\frac{1}{4N_p})^2}). \tag{68}$$

Let us introduce the following symbols:

$$\Delta_p p_N = p_N - p_{N+1}; \Delta_p^{-1}\psi(p_N) = \frac{\psi(p_N) - \psi(p_{N+1})}{p_N - p_{N+1}} =$$
$$= \frac{\psi(p_{N+1} + \Delta_p p_N) - \psi(p_{N+1})}{\Delta_p p_N}. \qquad (69)$$

Then we suppose that only in the classical dynamics variations of momenta (energies) have no lower bounds and we have dp. At the same time, in a quantum dynamics, due to the limited spatial domains, these variations have both upper and lower bounds.

In this case, as distinct from [34], in the theory there is a minimum variation of the momentum Δp_{min} that within the scope of the measurability (Definition 2 in Section 2.2) takes the form

$$\Delta p_{min} \equiv \mathrm{p} = \frac{\hbar}{\ell}\frac{1}{(\mathbf{N} - \frac{1}{4\mathbf{N}})} \approx \frac{\hbar}{\ell\mathbf{N}}. \qquad (70)$$

As in Equation (69) at high $|N_p|, (|N_p| \gg 1)$, $\Delta_p p_N = p_N - p_{N+1} \propto (\frac{1}{N_p} - \frac{1}{N_p+1}) = \frac{1}{N_p(N_p+1)}$, it is clear that

$$N_p(N_p+1) \leq \mathbf{N} \;\; or \;\; -\frac{1}{2} - \sqrt{(\frac{1}{4} + \mathbf{N})} \leq N_p \leq -\frac{1}{2} + \sqrt{(\frac{1}{4} + \mathbf{N})}. \qquad (71)$$

Considering that N_p is an integer number and $\mathbf{N} \gg 1$, it follows that

$$|N_p| \leq [\sqrt{\mathbf{N}}] - 1 \equiv \widetilde{\mathbf{N}}, \qquad (72)$$

where the square brackets [ ] in the right-hand side of Equation (72) denote an integer part of the number.

Next, due to Equations (68) and (69), an analog of Formulae (11) and (12) from [34] in the case under study at low energies will be of the form

$$\mathbf{p}.\psi(p) \Rightarrow p_N\psi(p_N) = \frac{\hbar}{(N_p - \frac{1}{4N_p})\ell}\psi(p_N) \approx \frac{\hbar}{N_p\ell}\psi(p_N),$$
$$\mathbf{x}.\psi(p) \Rightarrow \mathbf{x}.\psi(p_N) = \imath\hbar(1 + \frac{1}{(N_p - \frac{1}{4N_p})^2})\Delta_p^{-1}\psi(p_N) \approx$$
$$\approx \imath\hbar(1 + \frac{1}{N_p^2})\Delta_p^{-1}\psi(p_N). \qquad (73)$$

The scalar product $\langle\psi|\phi\rangle$ from [34]

$$\langle\psi|\phi\rangle = \int_{-\infty}^{+\infty} \frac{dp}{1+\beta p^2}\psi^*(p)\phi(p) \tag{74}$$

in the case of low energies $1 \ll |N_{\Delta p}| \leq \widetilde{\mathbf{N}} < \infty$ is replaced by the sum

$$\begin{aligned}
\langle\psi|\phi\rangle = \int_{-\infty}^{+\infty} \frac{dp}{1+\beta p^2}\psi^*(p)\phi(p) \Rightarrow \\
\Rightarrow \langle\psi|\phi\rangle_{1\ll|N_p|\leq\widetilde{\mathbf{N}}} = \sum_{1\ll|N_p|\leq\widetilde{\mathbf{N}}} \frac{\Delta_p(p_N)\psi^*(p_N)\phi(p_N)}{(1+\frac{1}{(N_p-\frac{1}{4N_p})^2})} \approx \\
\approx \sum_{1\ll|N_p|\leq\widetilde{\mathbf{N}}} \frac{\Delta_p(p_N)\psi^*(p_N)\phi(p_N)}{(1+\frac{1}{N_p^2})}.
\end{aligned} \tag{75}$$

And since $|N_p| \gg 1$ is a variable, in fact p_N is continuously varying and, proceeding from the above formulae, we can assume that to a high accuracy the function $\phi(p_N)$,$(\psi^*(p_N))$ is differentiable in terms of this variable.

On the other hand, at high energies, when for $|N_p| \approx 1$ the presentation is fairly discrete, the scalar product Equation (74) is replaced by the sum

$$\begin{aligned}
\langle\psi|\phi\rangle = \int_{-\infty}^{+\infty} \frac{dp}{1+\beta p^2}\psi^*(p)\phi(p) \Rightarrow \\
\Rightarrow \langle\psi|\phi\rangle_{|N_p|\approx 1} = \sum_{|N_p|\approx 1} \frac{\Delta_p(p_N)\psi^*(p_N)\phi(p_N)}{(1+\frac{1}{(N_p-\frac{1}{4N_p})^2})}.
\end{aligned} \tag{76}$$

We consider only two cases: (a) $1 \ll |N_p| \leq \widetilde{\mathbf{N}}$, "Quantum Consideration, Low Energies" and (b) $|N_p| \approx 1$,"Quantum Consideration, High Energies". The case (c)

$$\widetilde{\mathbf{N}} \ll |N_p| < \infty \tag{77}$$

is omitted in this Section as it is associated with the "Classical Picture".

Then at all the energy scales $\langle\psi|\phi\rangle_{N_p}$ may be formally represented as follows:

$$\langle\psi|\phi\rangle_{N_p} = \langle\psi|\phi\rangle_{1\ll|N_p|\leq\widetilde{\mathbf{N}}} + \langle\psi|\phi\rangle_{|N_p|\approx 1}. \tag{78}$$

However, with the formalism and terms proposed in this work, and also with the use of the Formula (11) that in this case takes the form

$$(|N_p| \approx 1) \to (1 \ll |N_p| \leq \widetilde{\mathbf{N}}), \tag{79}$$

it seems more logical to consider the two components in Equation (78) separately, the first component originating in the process of the low-energy transition from the second component as follows:

$$\langle \psi|\phi\rangle_{|N_p|\approx 1} \stackrel{|N_p|\gg 1}{\Rightarrow} \langle \psi|\phi\rangle_{1\ll |N_p|\leq \widetilde{\mathbf{N}}}. \tag{80}$$

We will return to the substantiation (80) in the next Section
Clearly, the first part of formula (13) from [34] holds as well in the general case for each of the components in Equation (78)

$$\langle (\psi|\mathbf{p})|\phi\rangle = \langle \psi|(\mathbf{p}|\phi)\rangle \tag{81}$$

The second part of formula (13) from [34]

$$\langle (\psi|\mathbf{x})|\phi\rangle = \langle \psi|(\mathbf{x}|\phi)\rangle \tag{82}$$

takes place (to a high accuracy) for the low-energy case $1 \ll |N_p| \leq \widetilde{\mathbf{N}} < \infty$, *i.e.*, for the first component in Equation (78).

Indeed, in this case, due to the condition $|N_p| \gg 1$, we have

$$\Delta_p p_N \approx dp; \Delta_p^{-1}\psi(p_N) \approx \partial_p \psi(p_N)$$

$$or$$

$$\lim_{|N_p|\to\infty,(\widetilde{\mathbf{N}}\to\infty)} \Delta_p p_N = dp; \lim_{|N_p|\to\infty,(\widetilde{\mathbf{N}}\to\infty)} \Delta_p^{-1}\psi(p_N) = \partial_p \psi(p_N). \tag{83}$$

Then in this (low-energy) case there exists the analog of formula (15) from [34]

$$\langle \psi|(\mathbf{x}|\phi)\rangle = \sum_{1\ll |N_p|\leq \widetilde{\mathbf{N}}-1} \frac{\Delta_p(p_N)}{(1+\frac{1}{N_p^2})}\psi^*(p_N)i\hbar(1+\frac{1}{N_p^2})\Delta_p^{-1}(\phi(p_N)) =$$

$$= \sum_{1\ll |N_p|\leq \widetilde{\mathbf{N}}-1} \Delta_p(p_N)\psi^*(p_N)i\hbar\Delta_p^{-1}(\phi(p_N)) \approx$$

$$\approx \langle (\psi|\mathbf{x})|\phi\rangle = \sum_{1\ll |N_p|\leq \widetilde{\mathbf{N}}-1} \Delta_p(p_N)(i\hbar\Delta_p^{-1}\psi(p_N))^*\phi(p_N). \tag{84}$$

It is important to note the following remarks:
(1) The operator $\mathbf{x}$ is defined in the case of low energies only for the functional space $\phi(p_N)_{1\ll|N_p|\leq\widetilde{\mathbf{N}}-1}$. Really, because of the existence of the Formula (69), the extreme point N_p, (such that $(N_p+1)(N_p+2)>\mathbf{N}$) "moves" this operator beyond the domain under study $\Delta p_{min}=$ p. Therefore, replacing $N_p \mapsto N_p+1, N_p+1 \mapsto N_p+2$ in Formula (71), one can easily get the estimate of $\widetilde{\mathbf{N}}-1$ instead of $\widetilde{\mathbf{N}}$ as seen in Equation (84).

(2) Despite the fact that the operator $\mathbf{x}$ is also defined at high energies, *i.e.*, for $\phi(p_N)_{|N_p|\approx 1}$,in general the property Equation (82) in this case has no place for lack of Formulae (83).

(3) In all the cases when we consider $|N_p|\gg 1$ (low energies) the "cut-off" for some upper bound p_{max},$(p_{max}\ll P_{pl})$,$1\ll N_{p_{max}}<|N_p|, p\neq p_{max}$ is determined by the initial conditions of the solved problem.

(4) It is clear that in the relativistic case $\Delta p_{min}=$ p leads to a minimal variation in the energy

$$|\Delta E_{min}|=(\Delta p)_{min}c=\frac{\mathrm{p}}{\mathbf{N}}c. \tag{85}$$

(5) In this work a minimal variation of the momentum Δp_{min} has been introduced from the additional assumptions but, as shown in [53], a minimal variation of the momentum may arise from the Extended Uncertainty Principle (EUP) as follows:

$$\Delta x_i \Delta p_j \geq \hbar\delta_{ij}[1+\beta^2\frac{(\Delta x_i)^2}{l^2}], \tag{86}$$

where l is the characteristic, large length scale $l\gg l_p$ and β is a dimensionless real constant on the order of unity [53]. From Equation (86) we get an absolute minimum in the momentum uncertainty

$$\Delta p_i \geq \frac{2\hbar\beta}{l}. \tag{87}$$

In [54] GUP and EUP are combined by the principle called the Symmetric Generalized Uncertainty Principle (SGUP):

$$\Delta x\Delta p \geq \hbar\left(1+\frac{(\Delta x)^2}{L^2}+l^2\frac{(\Delta p)^2}{\hbar^2}\right), \qquad (88)$$

where $l \ll L$ and l defines the limit of the UV-cutoff (not being such up to a constant factor as in the case of GUP).Then

$$\Delta x_{\min} = 2l/\sqrt{1-4l^2/L^2} = \ell,$$

whereas L defines the limit for IR-cutoff *i.e.*, we have a

$$\Delta p_{\min} = 2\hbar/(L\sqrt{1-4l^2/L^2}).$$

6 Quantum Theory in Terms of Measurable Quantities. Curvature and Space-Time Quantum Fluctuations

Now we consider the components (78)separately (as this seems more correct) and try to substantiate the low-energy transition (80).
In the previous Section (similar to [34]) the background space curvature has not been treated – implicitly it has been implied that quantum theory is considered in flat space-time. This assumption is sufficiently correct at low energies which are far from the Planck energies $E \ll E_P$. In this case the space-time curvature may be disregarded as the modern experimental cosmology demonstrates that the space-time geometry at the well-known energies $E \ll E_P$ is a geometry of flat space to a high accuracy [55].
However, at high energies $E \approx E_P$ this space is different from the flat space and there is no possibility to disregard this fact. According to the present-day knowledge, at Planck's scales the space exhibits high Space-Time Quantum Fluctuations (STQF) of the fundamental quantities: length, time, metric, and so on [56]–[78].
Let us briefly revert to STQF. The definition (STQF) is closely associated with the notion of ≪space-time foam≫. The notion ≪space-time foam≫,

introduced by J. A. Wheeler about 60 years ago for the description and investigation of physics at Planck's scales (Early Universe) [56],[57], is fairly settled. Despite the fact that in the last decade numerous works have been devoted to physics at Planck's scales within the scope of this notion, for example [58]–[77], by this time still their no clear understanding of the ≪space-time foam≫ as it is.

In accordance with the modern concepts, the space-time foam [57] notion forms the basis for space-time at Planck's scales (Big Bang). This object is associated with the quantum fluctuations generated by uncertainties in measurements of the fundamental quantities, inducing uncertainties in any distance measurement. A precise description of the space-time foam is still lacking along with an adequate quantum gravity theory. But for the description of quantum fluctuations we have a number of interesting methods (for example, [78],[67]–[77]).

In what follows, we use the terms and symbols from [69]. Then for the fluctuations $\widetilde{\delta l}$ of the distance l we have the following estimate:

$$(\widetilde{\delta l})_{\gamma} \gtrsim l_P^{\gamma} l^{1-\gamma} = l_P(\frac{l}{l_P})^{1-\gamma} = l(\frac{l_P}{l})^{\gamma} = l\lambda_l^{\gamma}, \tag{89}$$

or that same

$$|(\widetilde{\delta l})_{\gamma}|_{min} = \beta l_P^{\gamma} l^{1-\gamma} = \beta l_P(\frac{l}{l_P})^{1-\gamma} = \beta l\lambda_l^{\gamma}, \tag{90}$$

where $0 < \gamma \leq 1$, coefficient β is of order 1 and $\lambda_l \equiv l_P/l$.

From (89),(90), we can derive the quantum fluctuations for all the primary characteristics, specifically for the time $(\widetilde{\delta t})_{\gamma}$, energy $(\widetilde{\delta E})_{\gamma}$, and the metrics $(\widetilde{\delta g}_{\mu\nu})_{\gamma}$. In particular, for $(\widetilde{\delta g}_{\mu\nu})_{\gamma}$ we can use formula (10) in [69]

$$(\widetilde{\delta g}_{\mu\nu})_{\gamma} \gtrsim \lambda^{\gamma}. \tag{91}$$

But due to GUP (8), in the case under consideration the theory involves a minimal length on the order of the Planck length

$$\ell \propto l_P$$

or that is the same

$$\ell = \xi l_P, \tag{92}$$

where the coefficient ξ is on the order of unity too.
Evidently,that in this case replacement of Planck's length by the minimal length in all the above formulae is absolutely correct and is used without detriment to the generality [7],[1]

$$l_P \to \ell. \tag{93}$$

Thus, $\lambda_l \equiv l_{min}/l$ and then (89)– (91) upon the replacement (93) are read unchanged.
So, (90) may be written as

$$|(\widetilde{\delta l})_{\gamma}|_{min} = \beta l \lambda_l^{\gamma} = \beta N_l (N_l^{-\gamma})\ell = \beta N_l^{1-\gamma}\ell. \tag{94}$$

Here one should take into account the following consideration: due to the (Integrality Condition) (9) in the right-hand side of (94) for the factor $\beta N_l^{1-\gamma}$ before ℓ its integer part is always meant

$$\beta N_l^{1-\gamma} \mapsto [\beta N_l^{1-\gamma}] \tag{95}$$

and this goes without special mentioning for the whole text.
As noted in the overview [69], the value $\gamma = 2/3$ derived in [78] is totally consistent with the Holographic Principle [79]–[82].
The following points of importance should be noted [7],[1]:
6.1)It is clear that *at Planck's scales, i.e. at the minimal length scales*

$$l \to \ell \tag{96}$$

models for different values of the parameter γ are coincident.

6.2) In fact, the parameter λ_l is nothing else but

$$\lambda_l = \sqrt{\alpha_l}, \tag{97}$$

where α_l is defined in formula (12) for $a = l$.
It is important that the parameter α_l initially introduced in [36]–[43] is not given at the limiting point $l = l_{min}$ due to the appearance of singularity [37] and hence we have

$$0 < \alpha_l \leq 1/4. \tag{98}$$

It is obvious that nothing precludes λ_l to be variable over the interval

$$0 < \lambda_l \leq 1. \tag{99}$$

At the same time, for complete conformity to the domain of definition (99) and to the formula of (97), at the limiting point $l = \ell$ the parameter α_l may be redefined (regularized).
It should be noted that the parameter α_l has the following clear physical meaning:

$$\alpha_l^{-1} \sim S^{BH}, \tag{100}$$

where

$$S^{BH} = \frac{A}{4l_p^2} \tag{101}$$

is the well-known Bekenstein-Hawking formula for the black hole entropy in the semiclassical approximation [83],[84] for the black-hole event horizon surface A, with the characteristics linear dimension («radius») $R = l$. This is especially obvious in the spherically-symmetric case.

Reverting to the beginning of this Section, we can state the following:
as background spaces of the first and of the second components in (78) have absolutely different curvatures (the space is nearly flat for the first component and has a higher curvature for the second component), it is better to consider these components separately, the transition (80) being absolutely natural.

Considering this, the transition (80) from high to low energies may be given *differently* – as a transition from the high-curvature background space $K \gg 0$ to the asymptotically flat space [11]

$$\langle\psi|\phi\rangle_{K\gg 0} \stackrel{K\to 0}{\Rightarrow} \langle\psi|\phi\rangle_{K\approx 0}. \tag{102}$$

As this takes place, the curvature K in (102) is understood as the Gaussian curvature *K(l)* [85] corresponding to the scale l:

$$K \equiv K(l) = \frac{1}{l^2} = \frac{1}{N_l^2 \ell^2} = \alpha_l \ell^{-2}. \tag{103}$$

Because of this, the transition (102) is in complete conformity with the formula (79) from the previous Section.
The problem is, which models for *space-time foam* at the Planck scale adequately agree with the transition (102).
It is clear that this feature is attributed to the models considered by Fabio Scardigli in [62]–[64] and based on *micro-black holes* with the radius r that equals several Planck's lengths l_P or, in much the same way, several minimal lengths ℓ (within the scope of this paper) $r = N_r\ell$, where N_r – integer number on the order of 1.
Then, due to (103), in fact the transition (102)for $l = r$ takes the form

$$\langle\psi|\phi\rangle_{\frac{1}{N_r^2}\gg 0} \stackrel{N_r\to\infty}{\Rightarrow} \langle\psi|\phi\rangle_{\frac{1}{N_r^2}\approx 0}. \qquad (104)$$

Is it possible to correct the results of the previous Section due to STQF?
6.3) At high (Planck's) energies, according to 6.1) and the formula of (96), all fluctuations of the length l have the characteristic dimension $\approx \ell$. Because of this, we should take into consideration all components of the sum

$$\langle\psi|\phi\rangle_{|N_p|\approx 1} = \sum_{|N_p|\approx 1} \frac{\Delta_p(p_N)\psi^*(p_N)\phi(p_N)}{(1+\frac{1}{(N_p-\frac{1}{4N_p})^2})}. \qquad (105)$$

from the right-hand side of (76).

6.4)The situation is cardinally different at low energies $E \ll E_P$. According to formulae (90),(94) and considering (95), in the sum from the right-hand side of the formula (75)

$$\langle\psi|\phi\rangle_{1\ll|N_p|\leq\widetilde{\mathbf{N}}} = \sum_{1\ll|N_p|\leq\widetilde{\mathbf{N}}} \frac{\Delta_p(p_N)\psi^*(p_N)\phi(p_N)}{(1+\frac{1}{(N_p-\frac{1}{4N_p})^2})} \approx$$
$$\approx \sum_{1\ll|N_p|\leq\widetilde{\mathbf{N}}} \frac{\Delta_p(p_N)\psi^*(p_N)\phi(p_N)}{(1+\frac{1}{N_p^2})}. \qquad (106)$$

we select not all the components but only those corresponding to the points *lattice "sites"* $Lat_{P-E}[LE]$ *from Section 3* defined by (94), (**??**) In this case the *lattice spacing* in the sum from the right-hand side of (106) is not single and equal to ℓ, being variable, determined by the scale l, and dependent on the energies present

$$\widetilde{N_l}\ell = [\beta N_l^{1-\gamma}]\ell, \tag{107}$$

where β, γ is taken from formulae (90),(94),(95).
The situation is quite natural. Indeed, the transition (106) from the *point* with the number N to the *point* with the number $N+1$ means that the length $l=\ell$ corresponding to the *"difference"* of these *points* is a *measurable quantity*. By **Definition 2**, the minimal length ℓ is actually a *measurable quantity* but only for the energies $E \approx E_P$, and in this point we consider the case $E \ll E_P$.

Of course, Sections 5,6 of this paper are only the first step to resolve the Quantum Theory in terms of *measurable quantities* using Definition 2. It is necessary to study thoroughly the low-energy case $E \ll E_P$ and the correct transition to high energies $E \propto E_P$ taking into account STQF. The author is planning to treat these problems in his further works.

7 Measurable and Nonmeasurable Transitions in Gravity and Quantum Theory [6]

7.1 Measurable and Nonmeasurable Transitions in Gravity

First, using the formalism of this work, it is required to construct a measurable deformation of the General Relativity (GR) at low energies (Formula (60)). This deformation is denoted in terms of $Grav[LE]^\ell$

$$Grav[LE]^\ell \stackrel{\ell \to 0}{\Rightarrow} GR. \tag{108}$$

Next, we should construct the high-energy deformation (denoted in terms of $Grav[HE]^\ell$), this time for $Grav[LE]^\ell$ (the first arrow in the Formula (61))

$$Grav[HE]^\ell \stackrel{\alpha_l \to 0}{\Rightarrow} Grav[LE]^\ell. \tag{109}$$

At the present time the majority of the proposed approaches to quantization of gravity are associated with the construction of the following transition:

$$GR \Rightarrow Grav[HE]^{\ell}. \tag{110}$$

But, by author's opinion, this is impossible. It seems that for correct quantization of gravity one needs reversal of the arrow from Equation (109)

$$Grav[LE]^{\ell}(\alpha_l \approx 0, \alpha_l \neq 0) \stackrel{\alpha_l \to 1}{\Rightarrow} Grav[HE]^{\ell}(\alpha_l \approx 1). \tag{111}$$

The above results indicate that the low-energy "measurable" gravity variant $Grav[LE]^{\ell}$ should be very close to GR but different at the same time.

The author is hopeful that the correct construction of a low-energy $Grav^{\ell}$ close to GR allows for a more natural transition to quantum (Planck) gravity. Besides, within the notion of measurability, gravity could be saved from some odd solutions, from wormholes in particular.

7.2 Measurable and Nonmeasurable Transitions in Quantum Theory

The situation is similar for a quantum theory too. In the general case, based on the parameter α_a (Formula (17) of Section 2.2), this means that there exists the following correct limiting high-energy transition:

$$\lim_{\ell \neq 0, |N_a| \gg 1} \alpha_a \stackrel{High\ Energy}{\Rightarrow} \lim_{\ell \neq 0, |N_a| \approx 1} \alpha_a \tag{112}$$

and there is no correct limiting high-energy transition

$$\lim_{\ell = 0} \alpha_a \stackrel{High\ Energy}{\Rightarrow} \lim_{\ell \neq 0, |N_a| \approx 1} \alpha_a. \tag{113}$$

The first of them corresponds to the transition from a **measurable** theory at low energies to a measurable theory at high energies

$$QT[LE]^{\ell} \stackrel{N_a \to 1}{\Rightarrow} QT[HE]^{\ell}. \tag{114}$$

Whereas the second

$$QT \overset{N_a \to 1}{\Rightarrow} QT[HE]^{\ell} \tag{115}$$

(here $QT[LE]^{\ell}, QT[HE]^{\ell}, QT$ are quantum theories with the minimal length $\ell \neq 0$ at low energies $E \ll E_p$, at high energies $E \approx E_p$, and the well-known (continuous) quantum theory with $l_{min} = 0$).

However, the whole theoretical physics, where presently at low energies $E \ll E_P$ the minimal length ℓ is not involved at all (*i.e.*, $l_{min} = 0$), is framed around a search for the nonexistent limits Equation (113) (correspondingly Equation (115)).

Of course, in this case the low-energy "measurable" variant $QT[LE]^{\ell}$ of QT by its results will be very close to the initial theory QT, as indicated in [1, 2], and Section 5 of the present work. But these theories are different by nature: the first of them is discrete and the second one is continuous. Nevertheless, it is clear that the main requirement in this case is associated with the "Compatibility Principe":

at low energies the resolved variant $QT[LE]^{\ell}$ must, to a high accuracy, represent the well-known approved results of the corresponding continuous theory QT.

These theories should be differing considerably at least on going to high energies $E \approx E_p$.

The hypothesis set by the author is that correct construction of the "measurable" transition to high energies (Formula (114)) should naturally lead to solution of the ultraviolet divergences problem (initially in terms of the finite measurable quantities).

8 Conclusion

In several works [2, 6] the main points have been formulated and the problems associated with the suggested approach have been indicated. They may be concluded as follows.

8.1 When in the theory the minimal length $l_{min} \neq 0$ is actualized (involved) at all the energy scales, a mathematical apparatus of this theory must be changed considerably: no infinitesimal space-time variations (increments) must be involved, the key role being played by the definition of measurability (Definition 2 from Section 2.2).

8.2 As this takes place, the theory becomes discrete at all the energy scales but at low energies (far from the Planck energies) the sought for theory must be very close in its results to the starting continuous theory (with $l_{min} = 0$). In the process a real discreteness is exhibited only at high energies which are close to the Planck energies.

8.3 By this approach the theory at low and high energies is associated with a common single set of the parameters (N_L from Formula (9)) or with the dimensionless small parameters ($1/N_L = \sqrt{\alpha_L}$) which are lacking if at low energies the theory is continuous, *i.e.*, when $l_{min} = 0$.

The principal objective of my further studies is to develop for quantum theory and gravity, within the scope of the considerations given in points **8.1**–**8.3**, the corresponding discrete models (with $l_{min} \neq 0$) for all the energy scales and to meet the following requirements:

8.4 At low energies the models must, to a high accuracy, represent the results of the corresponding continuous theories.

8.5 The models should not have the problems of transition from low to high energies and, specifically, the ultraviolet divergences problem. By author's opinion, the problem associated with points **8.4** and **8.5** is as follows.

8.6 It is interesting to know why, with the existing $l_{min} \neq 0, t_{min} \neq 0$ and discreteness of nature, at low energies $E \ll E_{max} \propto E_P$ the apparatus of mathematical analysis based on the use of infinitesimal space-time quantities ($dx_\mu, \frac{\partial\varphi}{\partial x_\mu}$, and so on) is very efficient giving excellent results. The answer is simple: in this case l_{min} and t_{min} are very far from the available scale of L and t, the corresponding $N_L \gg 1, N_t \gg 1$ being in general true but insufficient. There is a need for rigorous calculations.

Based on Section 6 of the present paper, the points **8.1**–**8.6** should be supplemented with points 8.7 – 8.9.

8.7 Is the exponent of γ in formulae (89),(90), and so on constant $\gamma \equiv const$.

or is it dependent on the existent energies $\gamma = \gamma(E)$?

8.8 The Gaussian curvature *K(r)* considered in Section 6 leads to the simplest geometry that, to a high accuracy, is flat at low energies. Besides, it is readily expressed in terms of *measurable quantities*.
The problem is to suggest a *constructive* description for a maximally wide class of the geometries which are asymptotically flat in the limit of low energies $E \ll E_P$. In this context the word *constructive* is used in terms of *measurable quantities*. Solving of this problem is directly associated with the *constructive* description of space-time at Planck's scale or of space-time foam.

8.9 And, finally, is there a direct relation between the preceding point and the expression of the main gravity components $R_\mu^\nu(\alpha_l), R(\alpha_l), T_\mu^\nu(\alpha_l), \Lambda(\alpha_l)$ from Section 4 in terms of *measurable quantities*?

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this article.

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